Lesson 22/10/2026

giovedì 22 ottobre 2026 08:53

Definition: Let $A \subset \mathbb{R}$ be a set. We say that

(i) A is open if $\forall x \in A, \exists \delta > 0 : (x-\delta, x+\delta) \subset A$ or $A = \emptyset$

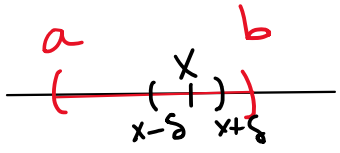
(ii) A is closed if $\mathbb{R} \setminus A$ is open

(iii) A is bounded if $\exists a, b \in \mathbb{R}, a < b : A \subset (a, b)$.

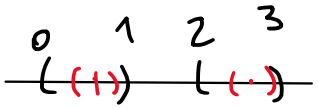
(iv) A is unbounded if it is not bounded.

Examples:

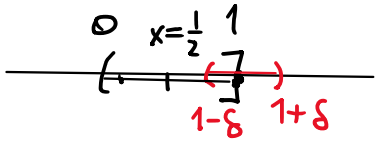
- the interval $(a,b) = \{x \in \mathbb{R} : a < x < b\}$ is open.



- the set $(0,1) \cup (2,3)$ is open.

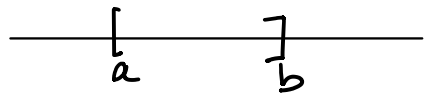


- $\mathbb{R}$ is open: $x \in \mathbb{R}, \delta > 0 \Rightarrow (x-\delta, x+\delta) \subset \mathbb{R}$.
- $\emptyset$ is open.
- $A = \{x \in \mathbb{R} : 0 < x \leq 1\}$



Is A open? No ($1 \in A$)

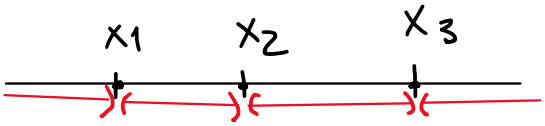
- $\mathbb{R}$ is closed (because $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is open).
- $\emptyset$ is closed (because $\mathbb{R}$ is open).
- $A = \{x \in \mathbb{R} : a \leq x \leq b\}$, $a < b$, is closed



$\mathbb{R} \setminus A = \{x \in \mathbb{R} : x < a \vee x > b\} = \underbrace{(-\infty, a) \cup (b, +\infty)}$ this set is open.

- Any finite set is closed.

$A = \{x_1, \dots, x_n\}$ is closed



$n = 3$

$\mathbb{R} \setminus A = \{x \in \mathbb{R} : x \notin A\} = \{x \in \mathbb{R} : x \neq x_1, x \neq x_2, x \neq x_3\}$

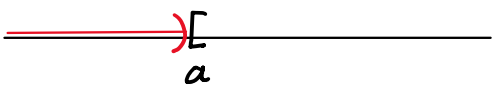
$= (-\infty, x_1) \cup (x_1, x_2) \cup (x_2, x_3) \cup (x_3, +\infty)$ = open

open open open open

General property: the union of open sets is open

- the interval $[a, +\infty) = \{x \in \mathbb{R} : x \geq a\}$ is closed.

$\mathbb{R} \setminus [a, +\infty) = (-\infty, a)$ is open



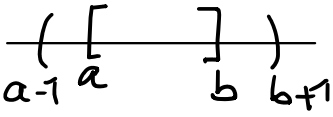
- the interval $(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$ is closed.

$\mathbb{R} \setminus (-\infty, a] = (a, +\infty)$ is open.

Boundedness:

- the interval (a, b) is bounded.
- the interval $[a, b]$ is bounded

$[a, b] \subset (a-1, b+1)$



The intervals $[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$ and

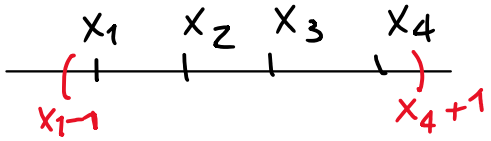
a-1 ⋯ b b+1

- the intervals $[a,b) := \{x \in \mathbb{R} : a \le x < b\}$ and $(a,b] := \{x \in \mathbb{R} : a < x \le b\}$ are bounded.

- Finite sets are bounded.

$A = \{x_1 \dots x_n\} \subset (x_1 - 1, x_n + 1)$

assume $x_1 < \dots < x_i < \dots < x_n$



$n = 4$

- A finite union of bounded set is bounded.

$\bigcup_{n \in \mathbb{Z}} (n, n+1) = \mathbb{R} \setminus \mathbb{Z}$ is unbounded

(bounded $\forall n \in \mathbb{Z}$)

- the interval $(a, +\infty) = \{x \in \mathbb{R} : x > a\}$ is unbounded
- the interval $[a, +\infty)$ is unbounded.
- the intervals $(-\infty, a)$ and $(-\infty, a]$ are unbounded.
- $\mathbb{R} = (-\infty, \infty)$ is unbounded, ϕ is bounded.

$\mathbb{R} \not\subset (a,b) \quad \forall a, b \in \mathbb{R}$.

Classification of the intervals:

(1) (a,b), $a, b \in \mathbb{R}$: open bounded intervals.

(i) (a,b), $a,b \in \mathbb{R}$: open bounded intervals.

(ii) $[a,b)$, $a,b \in \mathbb{R}$: bounded intervals, neither closed nor open.

(iii) $(a,b]$, $a,b \in \mathbb{R}$: bounded intervals, neither closed nor open

(iv) $[a,b]$, $a,b \in \mathbb{R}$: closed bounded intervals.

(v) (a,∞), $a \in \mathbb{R}$: open unbounded intervals.

(vi) $[a,\infty)$, $a \in \mathbb{R}$: closed unbounded intervals.

(vii) $(-\infty,a)$, $a \in \mathbb{R}$: open unbounded intervals.

(viii) $(-\infty,a]$, $a \in \mathbb{R}$: open unbounded intervals.

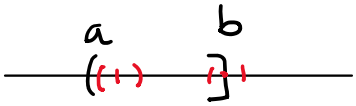
(ix) $\mathbb{R} = (-\infty,\infty)$ open and closed unbounded interval

Definition: Let $A \subset \mathbb{R}$ be a set. We say that a point $x \in A$ is an interior point if $\exists\, \delta > 0 : (x-\delta, x+\delta) \subset A$.

- $\mathring{A} := \{x \in A : x \text{ is an interior point of } A\}$.

($\mathring{A}$ is known as the interior of A).

$A = (a,b]$; $\mathring{A} = (a,b)$



property: $\mathring{A} \subset A$.

The extrema of a set

Let $A \subset \mathbb{R}$ be a set.

Definition: • we say that a point $x \in A$ is the maximum of A if $x \geq y, \forall y \in A$, and we write $x = \max A$.

• we say that a point $x \in A$ is the minimum of A if $x \leq y, \forall y \in A$, and we write $x = \min A$.

examples: $A = [0,1] \Rightarrow 1 = \max A$

• $1 \in A$

• $1 \geq y, \forall y \in A$.

$0 = \min A$

• $0 \in A$

• $0 \leq y, \forall y \in A$

In particular, $[0,1]$ has both a min and a max.

• $A = (0,1]$ • $1 = \max A$. $(1 \in A, 1 \geq y, \forall y \in A)$.

• $0 \leq y, \forall y \in A$; $0 \in A$? No

0 is not the min of A.

• A has no minimum.

take $x \in A \Rightarrow x > 0$; $\frac{x}{2} > 0 \Rightarrow \frac{x}{2} \in A$

$\frac{x}{2} < x \Rightarrow x$ is not the min of A.

• there are sets $A \subset \mathbb{R}$ which have no

maximum. $A = [0,1)$

- there are sets $A \subset \mathbb{R}$ which have neither a min nor a max. For example $(0,1)$ or $(0,\infty)$

- Given a set $A \subset \mathbb{R}$, we define the set

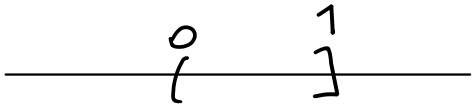
$$M(A) \doteq \{x \in \mathbb{R} : x \geq y, \ \forall y \in A\}$$

and

$$m(A) := \{x \in \mathbb{R} : x \leq y, \ \forall y \in A\}.$$

The set $M(A)$ is known as the set of majorants of A, while the set $m(A)$ is known as the set of minorants of A.

$A = (0,1]$



$M(A) = [1,\infty)$, $m(A) = (-\infty, 0]$

$1 \geq x, \forall x \in A$ $\qquad$ $0 \leq x, \forall x \in A$.

- $A = (0,\infty) \Rightarrow M(A) = \emptyset$.

Theorem Let $A \subset \mathbb{R}$ be a set.

Then $M(A)$ is empty or it has a min.

and $m(A)$ is empty or it has a max.

equivalently: if $M(A) \neq \emptyset \Rightarrow M(A)$ has a min

if $m(A) \neq \emptyset \Rightarrow m(A)$ has a max.

As a consequence, we are allowed to introduce the following def:

Definition: Let $A \subset \mathbb{R}$ be a set.

$$\sup(A) := \begin{cases} +\infty & \text{if } M(A) = \emptyset \\ \min M(A) & \text{if } M(A) \neq \emptyset \end{cases}$$

$$\inf(A) := \begin{cases} -\infty & \text{if } m(A) = \emptyset \\ \max m(A) & \text{if } m(A) \neq \emptyset \end{cases}$$

In particular: any subset of $\mathbb{R}$ has both an inf and a sup.

$A = (0,1]$, $0 = \inf A$.

$m(A) = (-\infty, 0] \Rightarrow 0 = \max m(A)$.

$A = [0,1) \Rightarrow 1 = \sup A$

$$\sup(a,b) = \sup(a,b] = \sup[a,b] = b$$
$$\| \sup[a,b).$$

Theorem: Let $A \subset \mathbb{R}$ be a set. Then

(i) A has a max $\Leftrightarrow \sup A \in A$ and, in this case, $\sup A = \max A$.

(ii) A has a min $\Leftrightarrow \inf A \in A$ and, in this

case, $\sup A = \max A$.

(ii) A has a min $\iff \inf A \in A$ and, in this case, $\inf A = \min A$.

Remark: . the sup of a set can be $+\infty$ while the max cannot.

. the inf of a set can be $-\infty$, while the min cannot.

Examples:

$\max(a,b] = \max[a,b] = b$: $b \in (a,b]$, $b \in [a,b]$

(a,b) has no max.

$\sup(a,b) = b$

$\sup(0,\infty) = +\infty$

$\inf(-\infty,0) \stackrel{?}{=} -\infty$

$\inf(a,b) = a$

Roughly speaking, if the extrema of A belong to A, we call them max and min. If they do not belong to A, we call them inf and sup.

By definition: $A \subset \mathbb{R}$ is bounded if and only

By definition: $A \subset \mathbb{R}$ is bounded if and only if $A \subset (a, b)$ for some $a, b \in \mathbb{R}$.

Remark: . $A \neq \emptyset$ is bounded $\Leftrightarrow -\infty < \inf A \leq \sup A < \infty$

$\Leftrightarrow \inf A, \sup A \in \mathbb{R}$.

Definition: $\emptyset \neq A \subset \mathbb{R}$ is bounded below if $\inf A > -\infty$
is bounded above if $\sup A < +\infty$.

Example: $(0, \infty)$ is bounded below.
$(-\infty, 1)$ is bounded above.

Remark: . if A is bounded $\Leftrightarrow$ A is both bounded below and bounded above

. A bounded below $\not\Rightarrow$ A is bounded.
. A bounded above $\not\Rightarrow$ A is bounded.

Powers and roots:

$2^2 = 4$, $\sqrt{25} = 5$ . . . .

Let $x \in \mathbb{R}$, let $n \in \mathbb{Z}$

$$x^n := \begin{cases} \underbrace{x \cdot \ldots \cdot x}_{n \text{ times}} & n > 0 \\ 1 & n = 0, \ x \neq 0 \\ \frac{1}{x^{-n}} & n < 0, \ x \neq 0 \end{cases}$$

$\frac{1}{x-n}$ $\quad n<0, \ x \neq 0$

$2^3 = 8, \ 2^{-3} = \frac{1}{2^3} = \frac{1}{8}, \ 3^2 = 9 \ldots$

· Roots:

· Let $y \geq 0$, let $n \in \mathbb{N}$.

$$\sqrt[2n]{y} = \sup \{ x \in \mathbb{R} : 0 \leq x^{2n} \leq y \} \geq 0$$

$n=1, \ y=4, \ \sqrt[2]{4} = 2 \quad (n=2, \sqrt{} = \sqrt[2]{})$

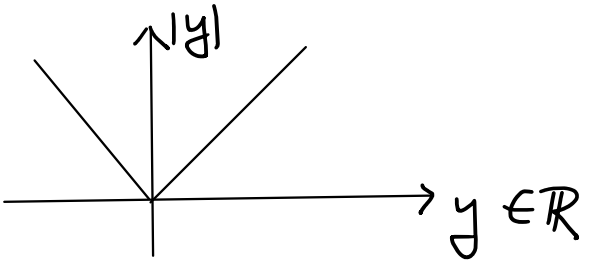
· Properties:

· $\left(\sqrt[2n]{y}\right)^{2n} = y, \quad \forall y \geq 0.$

· $\sqrt[2n]{0} = 0$

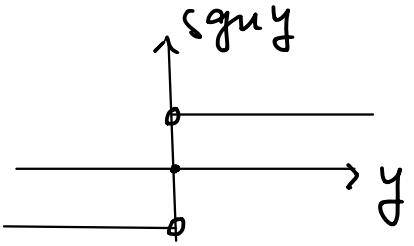
· Odd roots: we want to give a precise meaning, for example, to $\sqrt[3]{8} = 2$

· $\forall y \in \mathbb{R}$, we set $|y| = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases}$



$\forall y \in \mathbb{R}, \ |y| \geq 0.$

- $\forall y \in \mathbb{R}, \ \operatorname{sgn}(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -1 & \text{if } y < 0 \end{cases}$

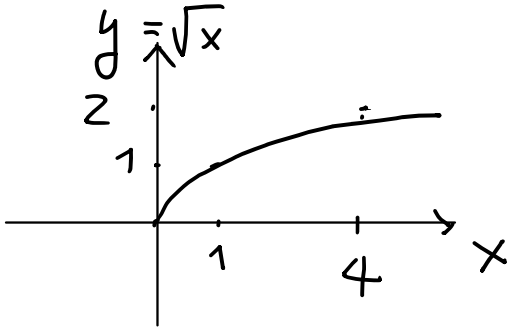


- In particular, $\forall y \in \mathbb{R}, \quad y = \operatorname{sgn}(y)\,|y|$

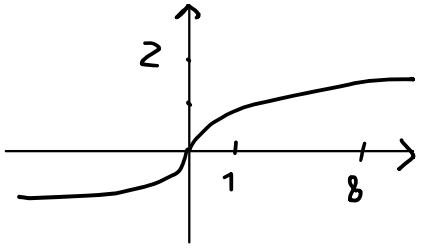
- Take $y \in \mathbb{R}$, $n \in \mathbb{N}$

$$\sqrt[2n+1]{y} := \operatorname{sgn}(y) \sup \{ x \in \mathbb{R} : 0 \leq x^{2n+1} \leq |y| \}$$

$\sqrt[3]{8} = 2$, $\sqrt[3]{-8} = -2$



even roots: always take +



odd roots: if $x \geq 0$ take +
if $x < 0$ take −

Remark:

- $\left(\sqrt[2n]{y}\right)^{2n} = y \quad \forall y \geq 0$

- $x \in \mathbb{R}$ $\quad \sqrt[2n]{x^{2n}} = |x|$.

$x=-2,\ n=1$. $(-2)^2 = 4 \rightsquigarrow \sqrt{(-2)^2} = \sqrt{4} = 2 \neq -2$

- $y \in \mathbb{R}$, $\quad \left(\sqrt[2n+1]{y}\right)^{2n+1} = y$

- $x \in \mathbb{R}$ $\quad \sqrt[2n+1]{x^{2n+1}} = x$

Equation:

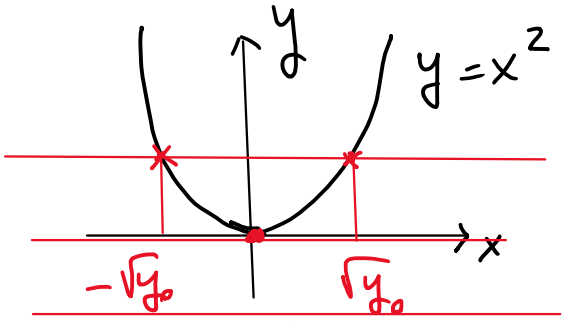
Let $y \in \mathbb{R}$, $n \in \mathbb{N}$. We want to solve equation $x^{2n} = y$

Which are the solution!

- $y < 0$: $\quad x^2 = -1$. There is no solution.
- $y \geq 0$: $\quad x^{2n} = y \iff x = \pm \sqrt[2n]{y}$

$$\left(-\sqrt[2n]{y}\right)^{2n} = (-1)^{2n}\left(\sqrt[2n]{y}\right)^{2n} = 1 \cdot y = y$$

$n=1$



we want to solve equation $\boxed{y_0 = x^2}$

$y_0 > 0 \Rightarrow \quad x = \pm\sqrt{y_0}$

$y_0 = 0 \Rightarrow \quad x = 0$

$y_0 < 0 \Rightarrow \quad \nexists$ intersection.

$x^2 = 4 \iff x = \pm 2$

$x^2 = 0 \iff x = 0$

$x^2 = -1$ · impossible.

Odd exponents:

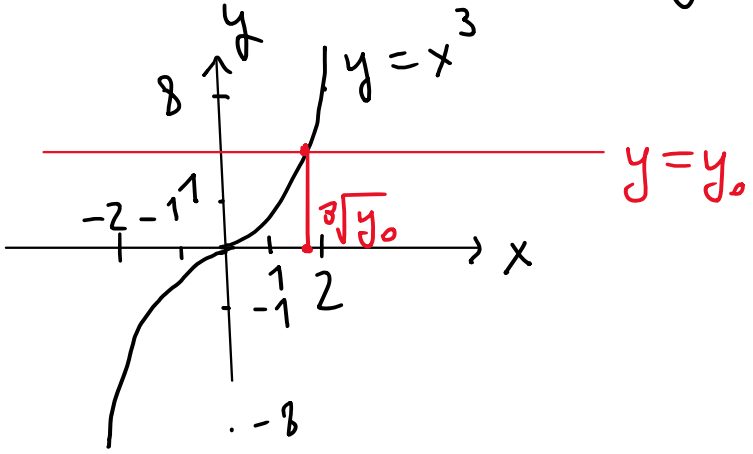
· we want to solve $x^{2n+1} = y$, $\forall y \in \mathbb{R}$.

$\iff \boxed{x = \sqrt[2n+1]{y}}$

$\forall y \in \mathbb{R}$, equation $x^{2n+1} = y$ has a unique solution, which is given by $x = \sqrt[2n+1]{y}$

$n = 1$, we want to solve $x^3 = y$

the solution is $x = \sqrt[3]{y}$.



$x^3 = y_0 \iff \boxed{x = \sqrt[3]{y_0}}$

Equations:

even powers: $x^2 = y$ $\begin{cases} y > 0 : x = \pm\sqrt{y} \text{ (2 solut)} \\ y = 0 : x = 0 \text{ (1 solut.)} \end{cases}$

even powers: $x^2 = y$ $\begin{cases} y = 0 : x = 0 \text{ (1 solut.)} \\ y < 0 : \text{no solution.} \end{cases}$

odd powers: $x^3 = y \iff x = \sqrt[3]{y}$ (unique sol)
$\forall y \in \mathbb{R}$.