Lesson 06/11/2026

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Operations with limits:

We consider two functions, f and g, and we want to compute $\lim_{x\to x_0} f+g$, $\lim_{x\to x_0} f\cdot g$, $\lim_{x\to x_0} f^g$.

operations in $\overline{\mathbb{R}}$:

- $\alpha+\infty=+\infty$, $\forall \alpha\in\mathbb{R}$
- $\alpha-\infty=-\infty$, $\forall \alpha\in\mathbb{R}$
- $+\infty+\infty=+\infty$, $-\infty-\infty=-\infty$.

Remark: $+\infty-\infty$ and $-\infty+\infty$ are not defined.

Theorem (The limit of a sum).
Let $f, g: X\subset\mathbb{R}\to\mathbb{R}$ be functions. Let $x_0\in$ cl(X). Assume that $\lim_{x\to x_0} f(x)=\alpha$ and $\lim_{x\to x_0} g(x)=\beta$, for some $\alpha,\beta\in\overline{\mathbb{R}}$. If $(\alpha,\beta)\notin\{(\pm\infty,\mp\infty)\}$, then $\lim_{x\to x_0}\big(f(x)+g(x)\big)=\alpha+\beta$.

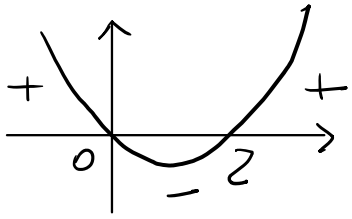
Roughly: the limit of a sum is the sum of the limits, provided we have no $+\infty-\infty$ or $-\infty+\infty$.

$f(x)=x^2$, $g(x)=-2x$

$f(x) = x^2$, $g(x) = -2x$

$\lim_{x \to +\infty} f(x) = +\infty$, $\lim_{x \to +\infty} g(x) = -\infty$.

$\lim_{x \to +\infty} (f(x) + g(x)) = \lim_{x \to +\infty} (x^2 - 2x) = +\infty$



$f(x) = \sqrt{x}$, $g(x) = -2x^2$

$\lim_{x \to +\infty} f(x) = +\infty$, $\lim_{x \to +\infty} g(x) = -\infty$.

$\lim_{x \to +\infty} (f(x) + g(x)) = \lim_{x \to +\infty} (\sqrt{x} - 2x^2)$

$= \lim_{x \to +\infty} -2x^2 \left(1 - \frac{\sqrt{x}}{2x^2}\right)$

$= \lim_{x \to +\infty} -2x^2 \left(1 - \frac{1}{2x^{3/2}}\right)$ ($\frac{1}{2x^{3/2}} \to 0$)

$= -\infty$.

Idea: larger powers dominate when x is large ($x \to +\infty$, or $x \to -\infty$).

Theorem (The limit of a product).
Let $f, g : X \subset \mathbb{R} \to \mathbb{R}$ be functions. Let $x_0 \in cA(X)$.
Assume that $\lim_{x \to x_0} f(x) = \alpha$, $\lim_{x \to x_0} g(x) = \beta$,

Assume $\lim_{x\to x_0} f(x) = \alpha$, $\lim_{x\to x_0} g(x) = \beta$

for some $\alpha, \beta \in \overline{\mathbb{R}}$. Assume $(\alpha,\beta) \notin \{(0,\pm\infty),(\pm\infty,0)\}$.

Then $\lim_{x\to x_0} f(x)\cdot g(x) = \alpha\cdot\beta$.

Roughly: the limit of a product is the product of the limits, provided there is no indefinite form $0\cdot\infty$.

Def: (Product on $\overline{\mathbb{R}}$): $\alpha(+\infty) = +\infty$, $\forall \alpha \in (0,\infty]$

the sign rules still holds true, so

$(-1)(+\infty) = -\infty \cdot \Rightarrow \quad \alpha(+\infty) = -\infty$, $\forall \alpha \in [-\infty, 0)$.

Examples: $f(x) = x^2$, $g(x) = \frac{1}{x}$

$\lim_{x\to\infty} f(x) = +\infty$, $\lim_{x\to+\infty} g(x) = 0$.

$\lim_{x\to\infty} f(x)\cdot g(x) = \lim_{x\to+\infty} x^2 \cdot \frac{1}{x} = \lim_{x\to+\infty} x = +\infty$.

$f(x) = x^2$, $g(x) = \frac{1}{x^2+1}$

$\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to+\infty} g(x) = 0$

$\lim_{x\to+\infty} \frac{x^2}{x^2+1} = 1$

$\lim_{x\to+\infty} \frac{x^2}{x^2+1} = \lim_{x\to+\infty} \frac{x^2}{x^2(1+(1/x^2))} = 1$

$x \to +\infty \quad x^2 + 1 \qquad x \to +\infty \quad x^2\left(1 + \frac{1}{x^2}\right)$, $\frac{1}{x^2} \to 0$

$f(x) = x^2, \quad g(x) = \dfrac{1}{x^4 + 5x^3}$

$\lim_{x\to\infty} f(x) = +\infty, \quad \lim_{x\to\infty} g(x) = 0$

$$\lim_{x\to\infty} f(x)\,g(x) = \lim_{x\to\infty} \frac{x^2}{x^4 + 5x^3} = \lim_{x\to\infty} \frac{x^2}{x^4\left(1 + \frac{5}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{x^2\left(1 + \frac{5}{x}\right)} = 0.$$

($\frac{5}{x} \to 0$)

Tricks:

Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial. Then

$$\lim_{x\to+\infty} \frac{a^x}{p(x)} = +\infty, \qquad \forall\, a > 1.$$

$a^x \to \infty, \quad p(x) \to \begin{cases} +\infty \\ -\infty \end{cases}$ as $x \to +\infty$.

$p(x) = x^2 - 3x \to +\infty$

$p(x) = -x^3 + y^2 \to -\infty.$

"The exponential wins against polynomials".

$$\lim_{x\to\infty} \frac{p(x)}{\log_a x} = +\infty \qquad a > 1$$

$p(x) \to \begin{cases} +\infty \\ -\infty \end{cases}$, $\log x \to +\infty$ $\quad x \to +\infty$

"Polynomials win against logarithms".

- exponentials > polynomials > logarithms.

A similar hierarchy holds when $x \to 0$.

$$\lim_{x \to 0^+} x^\alpha |\log x|^\beta = 0 \quad \text{if } \alpha > 0,\ \beta \in \mathbb{R}.$$

$x^\alpha \to 0$ $x \to 0$, $\alpha > 0$; $\log x \to -\infty$ as $x \to 0^+$

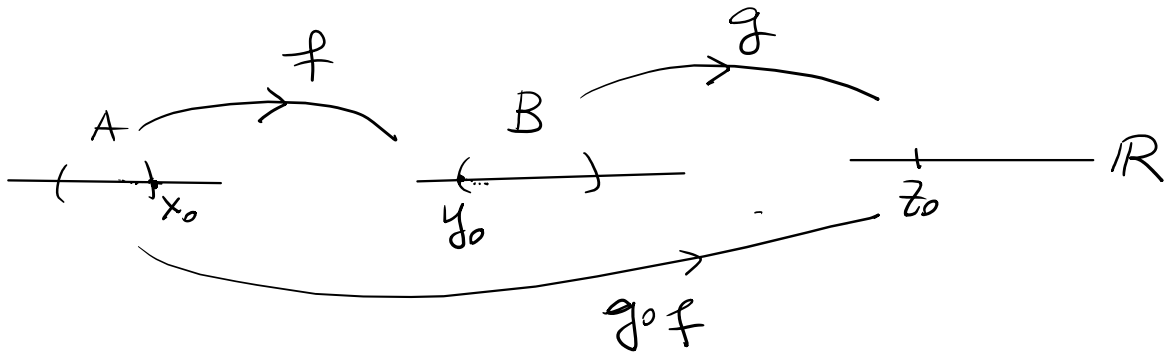
- if $\beta < 0 \Rightarrow |\log x|^\beta \to 0$ $x \to 0^+ \Rightarrow$ there is no indefinite form
- if $\beta = 0 \Rightarrow |\log x|^\beta = 1 \Rightarrow$ there is no indefinite form.
- if $\beta > 0 \Rightarrow |\log x|^\beta \to +\infty \Rightarrow$ there is an indefinite form $0 \cdot \infty$.

<u>Theorem</u> (The limit of a composition)

Let $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$ be functions, $A, B \subset \mathbb{R}$.

- Assume that $x_0 \in \mathcal{A}(A)$ and $\lim_{x \to x_0} f(x) = y_0$.
- Assume that $f(A) \subset B$, $y_0 \in \mathcal{A}(B)$ and $\lim_{y \to y_0} g(y) = z_0$.
- Assume that $\exists$ a neighbourhood $I \subset \mathbb{R}$ of x_0: $f(x) \neq y_0$, $\forall x \in I$.

Then $\lim_{x \to x_0} g \circ f(x) = z_0$.



Roughly: If you have to compute the lim of a composition, compute the interior limit first and then plug the value into the exterior limit.

Examples: $\lim_{x \to x_0} f(x) = +\infty \Rightarrow \lim_{x \to x_0} \frac{1}{f(x)} = 0^+$.

$g(y) = \frac{1}{y}$ $\quad \frac{1}{f(x)} = g(f(x))$

$f(x) \underset{x \to +\infty}{\to} \infty$, $g(y) \underset{y \to +\infty}{\to} 0^+ \Rightarrow g \circ f(x) \underset{x \to +\infty}{\to} 0^+$.

The limit of powers:

Theorem: Let $f, g : X \subset \mathbb{R} \to \mathbb{R}$. Assume that $x \in d(X)$

$\lim_{x \to x_0} f(x) = y_0$, $\lim_{x \to x_0} g(x) = z_0$, with $y_0, z_0 \in \overline{\mathbb{R}}$.

Then $\lim_{x \to x_0} f(x)^{g(x)} = y_0^{z_0}$, provided

$(y_0, z_0) \notin \{(1, \pm\infty), (0,0), (+\infty, 0)\}$.

$(y_0, z_0) \notin \{(1, \pm\infty), (0,0), (+\infty, 0)\}$.

Remark: 1^∞, 0^0, ∞^0 are undefinided forms.

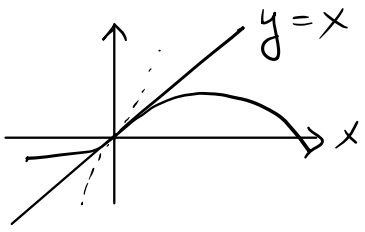
Important limits to remember:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{\arcsin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \qquad \lim_{x\to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x\to 0} \frac{\arctan x}{x} = 1 \qquad \lim_{x\to 0} \frac{a^x - 1}{x} = \log a$$

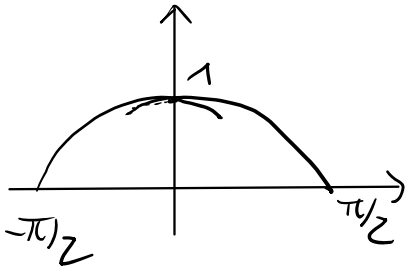
$$\lim_{x\to 0} \frac{\log_a(x+1)}{x} = \frac{1}{\log a} \qquad \lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \quad \forall \alpha \in \mathbb{R}.$$



$$1-\cos x \sim \frac{1}{2}x^2$$

$$\cos x \sim 1 - \frac{1}{2}x^2$$

$$x \sim 0$$



1^∞ is an indefinite form:

$$f(x) = 1+x, \quad g(x) = \frac{1}{x^2}, \quad x \to 0^+$$

$$\lim_{x\to 0} f(x) = 1, \quad \lim_{x\to 0^+} g(x) = +\infty$$

$$1/2 \qquad \log|(1+x)^{1/x^2}| \qquad \frac{1}{x}\log(1+x)$$

$x \to 0$ $\quad x \to 0^+$

- $(1+x)^{1/x^2} = e^{\log((1+x)^{1/x^2})} = e^{\frac{1}{x^2}\log(1+x)}$

$= e^{\frac{1}{x}\cdot\frac{\log(1+x)}{x}}$ ($\frac{\log(1+x)}{x} \to 1$) $\sim e^{1/x} \Rightarrow e^{+\infty} = +\infty$ $(x \to 0^+)$

- $f(x) = 1+x, \quad g(x) = \frac{1}{x}$

$(1+x)^{1/x} = e^{\log((1+x)^{1/x})} = e^{\frac{1}{x}\log(1+x)} \to e$ $(x \to 0^+)$

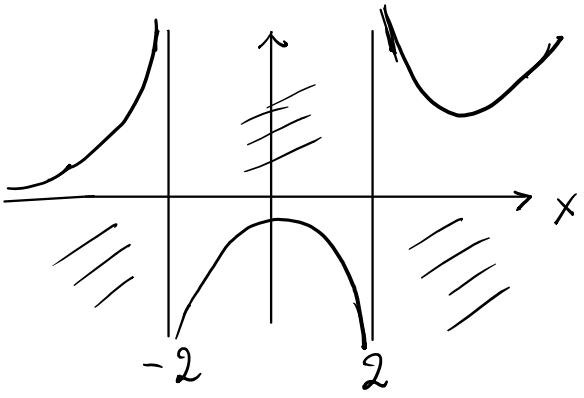
- $f(x) = (1+x), \quad g(x) = -1/x^2$

$(1+x)^{-1/x^2} = e^{\log((1+x)^{-\frac{1}{x^2}})} = e^{-\frac{\log(1+x)}{x^2}}$

$= e^{-\frac{\log(1+x)}{x}\cdot\frac{1}{x}}$ ($\frac{\log(1+x)}{x} \to 1$) $\simeq e^{-1/x} \to e^{-\infty} = 0.$

- $f(x) = \frac{e^x}{x^2-4}$ Determine the domain, the sign and the asymptots of f.

$D(f) = \{x \in \mathbb{R} : x^2 - 4 \neq 0\} = \{x \in \mathbb{R} : x^2 \neq 4\} = \mathbb{R} \setminus \{\pm 2\}$



· the sign:

$f(x) > 0 \iff x^2 - 4 > 0$

(because $e^x > 0$)

$f(x) > 0 \iff x < -2 \vee x > 2$

$f(x) < 0 \iff -2 < x < 2$

$\pm\infty, \pm 2 \in \mathcal{A}(D(f)) \setminus D(f).$

we need to compute $\lim_{x\to\pm\infty} f(x)$ and $\lim_{x\to\pm 2^{\pm}} f(x)$.

$$\lim_{x\to+\infty} \frac{e^x}{x^2-4} = +\infty \; ; \; \lim_{x\to-\infty} \frac{e^x}{x^2-4} = \frac{0^+}{+\infty} = 0^+$$

$\boxed{\{y=0\}}$ is a horizontal asymptote.

$$\lim_{x\to 2^+} \frac{e^x}{x^2-4} = \frac{e^2}{0^+} = +\infty$$

$$\lim_{x\to 2^-} \frac{e^x}{x^2-4} = \frac{e^2}{0^-} = -\infty$$

$\boxed{\{x=2\}}$ is a vertical asymptote (just one of the 2 above limits $x\to 2^{\pm}$ is enough to conclude that $\{x=2\}$ is a vertical asymptote).

$$\lim_{x\to -2^+} \frac{e^x}{x^2-4} = \frac{e^{-2}}{0^-} = -\infty \; ; \; \lim_{x\to -2} \frac{e^x}{x^2-4} = \frac{e^{-2}}{0^+} = +\infty$$

$\boxed{\{x=-2\}}$ is a vertical asymptote.