Lesson 05/11/2026

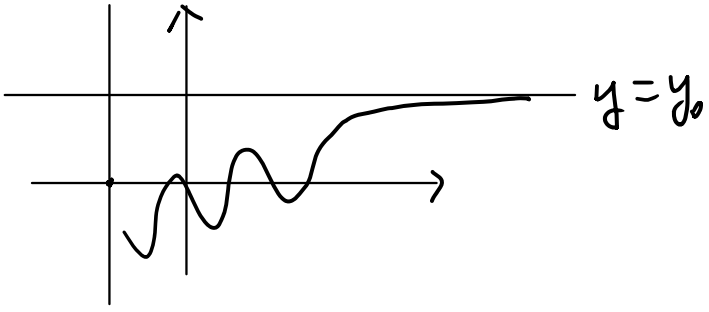
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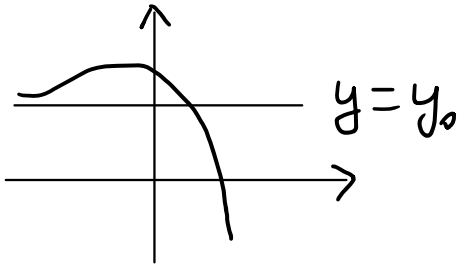
Horizontal asymptots:

Def: Let $f: X \subset \mathbb{R} \to \mathbb{R}$ be a function. Assume that either $+\infty$ or $-\infty$ is an accumulation point of X. We say that the straight line $y = y_0$ is a horizontal asymptot if

$$\lim_{x \to +\infty} f(x) = y_0 \quad \text{or} \quad \lim_{x \to -\infty} f(x) = y_0.$$

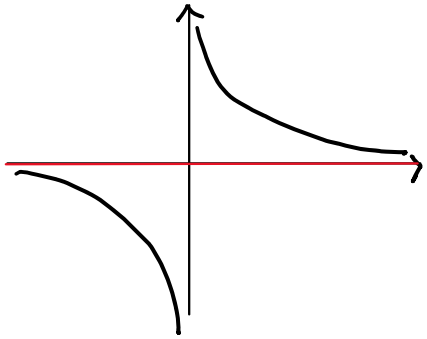
Graphical situation:





examples:

$f(x) = \frac{1}{x}$, $\forall x \in \mathbb{R} \setminus \{0\}$.



the straight line $y = 0$ is a horizontal asymptot.

we need to show that $\lim_{x \to +\infty} \frac{1}{x} = 0$.

$\left(\lim_{x \to -\infty} \frac{1}{x} = 0 \right)$

i.e. $\forall \varepsilon > 0$: solve the inequality $\frac{1}{x} < \varepsilon$

fix $\varepsilon > 0$: solve the inequality $\frac{1}{x} < \varepsilon$

we are interested in $x \to +\infty$, we can assume that $x > 0$: so $\frac{1}{x} < \varepsilon \iff x > \frac{1}{\varepsilon} =: M$

$x > M \Rightarrow \frac{1}{x} < \varepsilon$ ok.

. $\lim_{x \to -\infty} \frac{1}{x} = 0$

take $\varepsilon > 0$, $x < 0$, and solve the inequality

$\frac{1}{x} > -\varepsilon \iff 1 < -\varepsilon x \iff x < -\frac{1}{\varepsilon} =: -M$

$x < -M \Rightarrow \frac{1}{x} > -\frac{1}{\varepsilon}$.

Roughly: x large, $x > 0$ $\rightsquigarrow \frac{1}{x} \sim \frac{1}{+\infty} = 0^+$

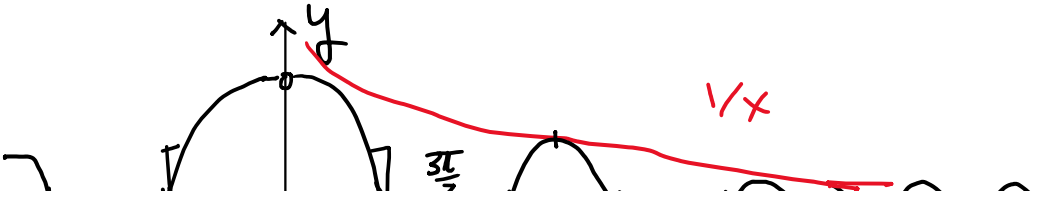
x small, $x < 0$ $\rightsquigarrow \frac{1}{x} \sim \frac{1}{-\infty} = 0^-$

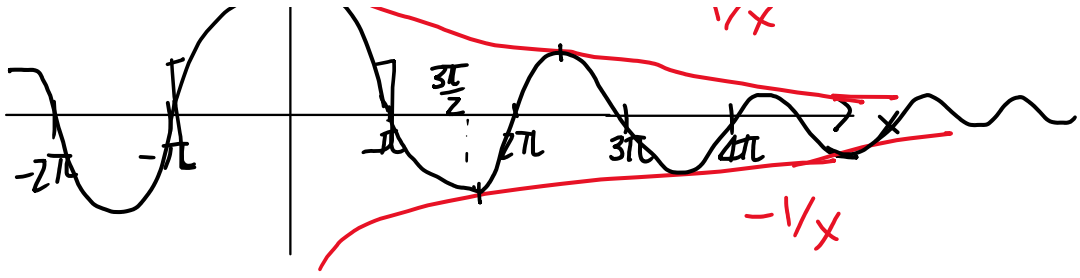
. Can the graph of a function intersect a horizontal asymptote? Yes.

example: $f(x) = \frac{\sin x}{x}$, $\forall x \in \mathbb{R} \setminus \{0\}$.

• $f(x) = 0 \iff \sin x = 0 \wedge x \neq 0$

$\iff x = k\pi : k \in \mathbb{Z} \setminus \{0\}$.





. $\lim_{x\to+\infty} f(x) = 0$ ✓

take $\varepsilon>0$, solve $-\varepsilon < \frac{\sin x}{x} < \varepsilon$, $x>0$.

$\Leftrightarrow \left|\frac{\sin x}{x}\right| < \varepsilon$, $x>0$

$\left|\frac{\sin x}{x}\right| \le \frac{1}{x} < \varepsilon$ $\quad \forall x \in \mathbb{R}\setminus\{0\}$ $\quad (|\sin x| \le 1)$

for $x > \frac{1}{\varepsilon} = M$.

In particular: $x>M \Rightarrow |f(x)| < \varepsilon$.

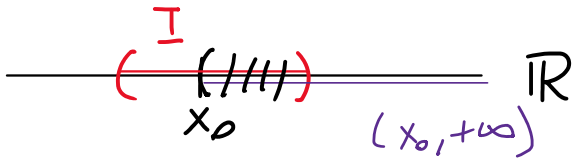
Vertical asymptotes:

Definition: . let $x_0 \in \mathbb{R}$. We say that an interval $J \subset \mathbb{R}$ is a right neighbourhood of x_0 if $\exists$ a neighbourhood $I \subset \mathbb{R}$ of x_0 such that $J = I \cap (x_0, +\infty)$.

. We say that an interval $J \subset \mathbb{R}$ is a left neighbourhood of x_0 if $\exists$ a neighbourhood $I \subset \mathbb{R}$ of x_0:

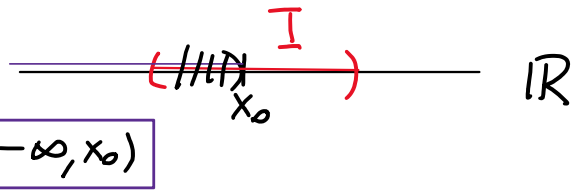
$J = (-\infty, x_0) \cap I$.

right neighbourhoods:



$I = (x_0 - \delta, x_0 + \delta) \Rightarrow J = (x_0, x_0 + \delta)$

left neighbourhoods:



$J = (x_0 - \delta, x_0)$.

Definition (limits from the right and from the left)
Let $f: X \subset \mathbb{R} \to \mathbb{R}$ be a function. Let $y_0 \in \overline{\mathbb{R}}$ and let $x_0 \in \mathcal{A}(X)$. We say that

$$\lim_{x \to x_0^+} f(x) = y_0 \quad \left(\text{or } \lim_{x \to x_0^-} f(x) = y_0 \right)$$

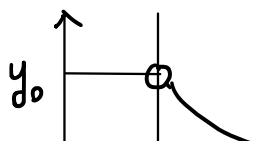
if $\forall$ neighbourhood $J \subset \mathbb{R}$ of y_0, $\exists$ a right (left) neighbourhood $I \subset \mathbb{R}$ of x_0 such that, $\forall x \in I$, $f(x) \in J$.

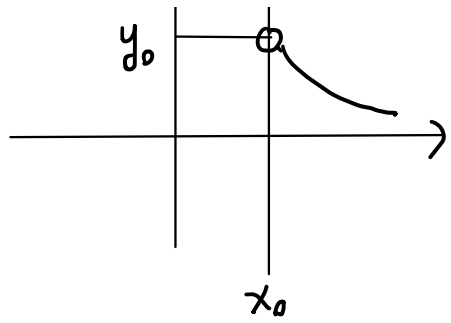
Equivalently:

$y_0 \in \overline{\mathbb{R}}$.

$\lim_{x \to x_0^+} f(x) = y_0$ if

$\forall \varepsilon > 0$, $\exists \delta > 0$: $\forall x \in \mathbb{R}$, $x_0 < x < x_0 + \delta$, $|f(x) - y_0| < \varepsilon$.





$y_0 = +\infty$. $\lim_{x \to x_0^+} f(x) = +\infty$ if

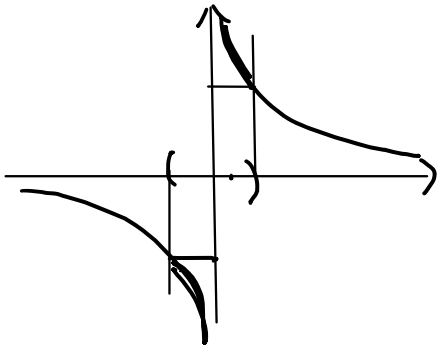
$\forall M > 0 \quad \exists \delta > 0 : \forall x \in \mathbb{R} : x_0 < x < x_0 + \delta, \quad f(x) > M.$

$\lim_{x \to x_0^+} f(x) = -\infty$ if

$\forall M > 0 \quad \exists \delta > 0 : \forall x \in \mathbb{R} : x_0 < x < x_0 + \delta, \quad f(x) < M.$

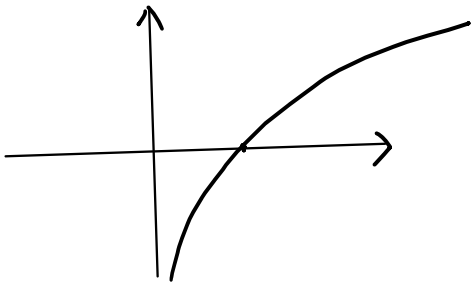
<u>examples</u>: $f(x) = \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$.

$\lim_{x \to 0^+} f(x) = +\infty$. $\wedge$ $\lim_{x \to 0^-} f(x) = -\infty$.



<u>Claim</u>: $\exists \lim_{x \to x_0} f(x) \iff$ the limit from the right and the limit from the left both exist and are equal.

$f(x) = \log x$, $\forall x > 0$. $D(f) = (0, \infty)$

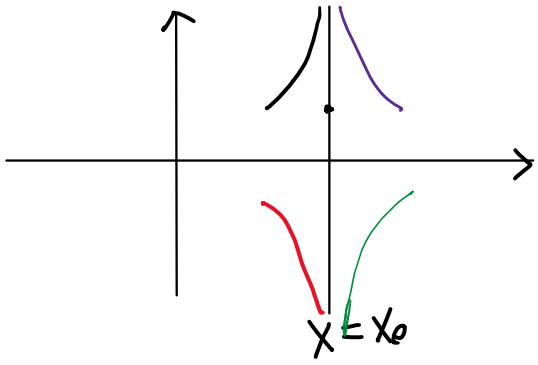


$\lim_{x \to 0^+} \log x = -\infty$.

Definition: (Vertical asymptote)

Let $f: X \subset \mathbb{R} \to \mathbb{R}$ be a function. Let $x_0 \in \mathcal{A}(X) \setminus X$. Then the straight line $\{x = x_0\}$ is said to be a vertical asymptote if either $\lim_{x \to x_0^+} f(x) \in \{\pm\infty\}$ or $\lim_{x \to x_0^-} f(x) \in \{\pm\infty\}$

Graphical situation:

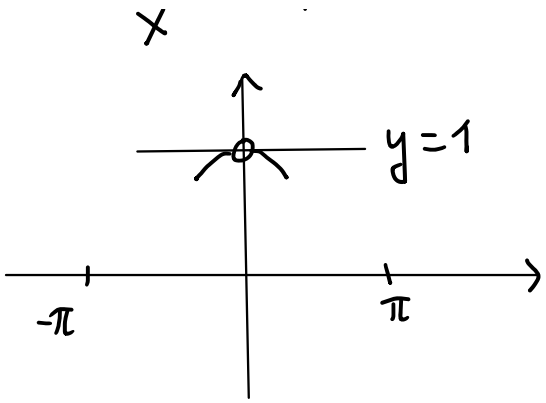


Roughly: an asymptote is a straight line which is "tangent to the graph at infinity".

Question: Can a graph intersect a vertical asymptote? No because $x_0 \notin X$ by definition.

$f(x) = \frac{\sin x}{x}$, $D(f) = \mathbb{R} \setminus \{0\}$

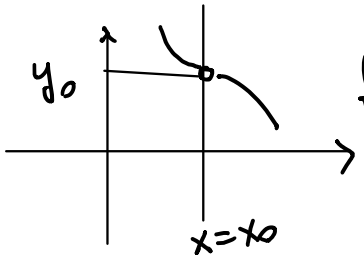
↑ Claim



Claim

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$x=0$ is not an asymptote.



$\{x=x_0\}$ this is not an asymptote.

· Limits from above and from below:

Def: Let $f: X \subset \mathbb{R} \to \mathbb{R}$ be a function.
Let $y_0 \in \mathbb{R}$, $x_0 \in \mathcal{A}(X)$. We say that

$$\lim_{x\to x_0} f(x) = y_0^{\pm}$$

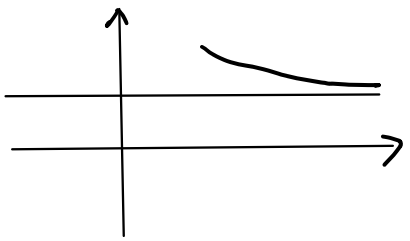
if $\forall$ right (left) neighbourhood $J \subset \mathbb{R}$ of y_0,
$\exists$ a neighbourhood $I \subset \mathbb{R}$ of x_0 s.t.

$$\forall x \in I \setminus \{x_0\},\ f(x) \in J.$$

· Equivalently: $(x_0 = +\infty)$ $\lim_{x\to+\infty} f(x) = y_0^+$

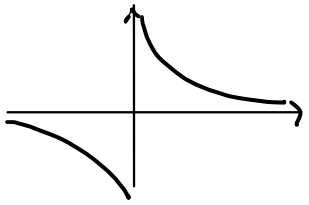
$\forall \varepsilon > 0\ \exists M > 0 : \forall x > M,\ y_0 < f(x) < y_0 + \varepsilon.$

↑ "the graph is approaching



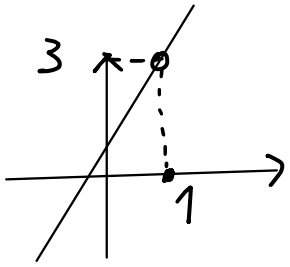
"the graph is approaching the straight line from above".

<u>example:</u> $f(x)=\frac{1}{x}$, $\lim_{x\to+\infty}\frac{1}{x}=0^+$



$\lim_{x\to-\infty}\frac{1}{x}=0^-$

$$f(x)=\begin{cases}2x+1 & x\neq 1\\ 0 & x=1\end{cases}$$



$\lim_{x\to 1^-} f(x)=3^-$, $\lim_{x\to 1^+} f(x)=3^+$

<u>Definition:</u>

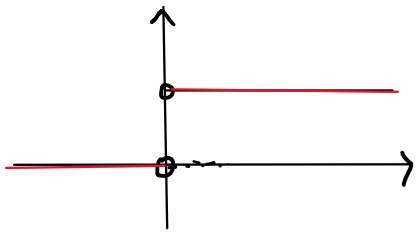
Let $f: X\subset\mathbb{R}\to\mathbb{R}$ be a function, $y_0\in\mathbb{R}$, $x_0\in d(X)\cap\mathbb{R}$.

We say that

$$\lim_{x\to x_0^\pm} f(x)=y_0^\pm$$

if $\forall$ right (left) neighborhood J of y_0, $\exists$ a right (left) neighbourhood I of x_0: $\forall x\in I$, $f(x)\in J$.

<u>Example:</u> $f(x)=\begin{cases}1 & x>0\\ 0 & x<0\end{cases}$ Heaviside function



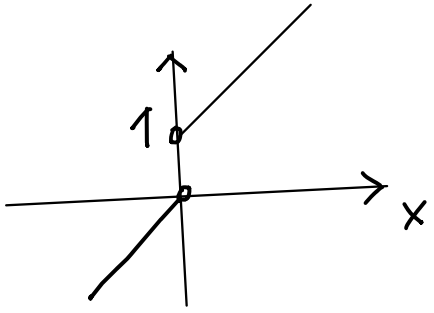
$\lim_{x\to 0^+} f(x) = 1$

$\varepsilon > 0$: take any $\delta > 0$: if $0 < x < \delta \Rightarrow |f(x)-1| =$

$= |1-1| = 0 < \varepsilon$

$\lim_{x\to 0^-} f(x) = 0$

$$f(x) = \begin{cases} x & x < 0 \\ 1+x & x > 0 \end{cases}$$



$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0^-$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = 1^+$

$\nexists \lim_{x\to 0} f(x)$.

Exercises:

$f(x) = \log(x^2-5x+6)$

Determine $D(f)$ and $f^{-1}(\log 2, \infty)$.

- $D(f) = \{x \in \mathbb{R} : x^2-5x+6 > 0\} = (-\infty, 2) \cup (3, +\infty)$.

$x^2-5x+6 > 0 \iff x^2-2x-3x+6 > 0$

$\iff x(x-2)-3(x-2) > 0 \iff (x-2)(x-3) > 0$

	2	3	
$x-2$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
	$(+)$	$-$	$(+)$

$\Leftrightarrow x>3 \ \vee \ x<2$

- $f^{-1}((\log 2, +\infty)) = ?$

here f^{-1} is <u>not</u> the inverse of f.

$$f^{-1}(Y) = \{x \in D(f) : f(x) \in Y\}.$$

$$Y = (\log 2, +\infty)$$

$$f^{-1}((\log 2, +\infty)) = \{x \in D(f) : f(x) \in (\log 2, +\infty)\}$$

$$= \{x \in D(f) : f(x) > \log 2\}$$

$$= \{x \in D(f) : \log(x^2 - 5x + 6) > \log 2\}$$

log is strictly increasing $\Rightarrow$

$$= \{x \in D(f) : x^2 - 5x + 6 > 2\}$$

$$= \{x \in D(f) : x^2 - 5x + 4 > 0\}$$

$$= (-\infty, 1) \cup (4, \infty)$$

$x^2 - 5x + 4 > 0$. $\quad x_{\pm} = \dfrac{5 \pm \sqrt{25 - 4 \cdot 1 \cdot 4}}{2} = \dfrac{5 \pm \sqrt{9}}{2}$

$$x_{\pm} = \frac{5 \pm 3}{2} = 4/1$$

$\Delta = \frac{\dots}{2} = 4/1$

$\Rightarrow x^2-5x+4 = (x-1)(x-4)$

	1		4	
$x-1$	−	+		+
$x-4$	−	−		+
	+	−		+

$x^2-5x+4 > 0 \iff \boxed{x < 1 \ \vee \ x > 4}$

Finally $\boxed{f^{-1}((\log 2, +\infty)) = (-\infty, 1) \cup (4, \infty)}$

. Determine the intersections of the following sets:

$A = [0,1] \cup \{3\}$ and $B = (2, +\infty)$

$\Rightarrow A \cap B = \{3\}$.

$A = (-\infty, -1) \cap [0, 2]$ and $B = [-3, 2) \cup \{4\}$.

$A = \phi \Rightarrow A \cap B = \phi$.

- $|x+1| + 2x - 3 = 2$

$|x+1| = 2+3-2x$

$|x+1| = 5-2x$

$(x+1 = 5-2x \ \vee \ x+1 = -(5-2x)) \ \wedge \ 5-2x \geq 0$

$(|a| = b \iff b \geq 0$ and $a = b \vee a = -b)$

$3x = 5 - 1 \quad \vee \quad x + 1 = -5 + 2x$

$3x = 4 \quad \vee \quad 6 = x$

$x = 4/3 \quad \vee \quad x = 6.$

- $x = 4/3 \Rightarrow 5 - 2x = 5 - 2 \cdot \frac{4}{3} = 5 - \frac{8}{3} = \frac{7}{3} > 0$ ok.
- $x = 6 \Rightarrow 5 - 2 \cdot x = 5 - 2 \cdot 6 = 5 - 12 = -7 < 0$ Not ok

$\boxed{x = 4/3}$

- $2^{-x} - 2^x + 6 = 0 \iff \frac{1}{2^x} - 2^x + 6 = 0$

$\iff 1 - (2^x)^2 + 6 \cdot 2^x = 0 \qquad 2^x = y > 0.$

$1 - y^2 + 6y = 0, \quad y > 0$

$\iff y^2 - 6y - 1 = 0 \qquad y_\pm = \frac{6 \pm \sqrt{40}}{2}$

$\Delta = 36 - 4 \cdot 1 (-1) = 40 \qquad y_\pm = \frac{6 \pm \sqrt{4 \cdot 10}}{2}$

$y_\pm = \frac{6 \pm 2\sqrt{10}}{2}$

$y_\pm = 3 \pm \sqrt{10}$

$y_- < 0 \quad [\sqrt{10} > \sqrt{9} = 3]$

$y_+ > 0.$

$2^x = y^-$ has no solution, $2^x = y^+ = 3 + \sqrt{10}$

has a unique solution $\boxed{x = \log_2(3+\sqrt{10})}$

$2^x = y \overset{\text{def log}}{\iff} x = \log_2 y$

$\forall y > 0$
$x \in \mathbb{R}$.

- $4x^4 + 11x^2 = 3 \iff 4x^4 + 11x^2 - 3 = 0$

$x^2 = y \geq 0 \iff 4y^2 + 11y - 3 = 0$

$$y_\pm = \frac{-11 \pm \sqrt{121 - 4 \cdot 4 (-3)}}{2 \cdot 4} = \frac{-11 \pm \sqrt{121 + 48}}{8}$$

$$y_\pm = \frac{-11 \pm \sqrt{169}}{8} = \frac{-11 \pm 13}{8} = \frac{1}{4} / -3$$

$x^2 = \frac{1}{4} \vee x^2 = -3$ (impossibile) $\iff x^2 = \frac{1}{4} \iff \boxed{x = \pm \frac{1}{2}}$

- $\left(\sqrt[3]{\frac{x^2+3x-1}{2x+1}}\right)^3 = 1^3 \iff \frac{x^2+3x-1}{2x+1} = 1$

$$\iff \frac{x^2+3x-1}{2x+1} - 1 = 0$$

$$\frac{x^2+3x-1-(2x+1)}{2x+1} = 0$$

$\iff N = 0, \ D \neq 0.$

$$x^2 + 3x - 1 - (2x+1) = 0 \ \wedge \ 2x+1 \neq 0$$

$$x^2 + 3x - 1 - 2x - 1 = 0 \ \wedge \ x \neq -\frac{1}{2}$$

$$x^2 + x - 2 = 0 \ \wedge \ x \neq -\frac{1}{2}$$

$$x_{\pm} = \frac{-1 \pm \sqrt{1 + 4 \cdot 2}}{2} = \frac{-1 \pm \sqrt{9}}{2}$$

$$x_{\pm} = \frac{-1 \pm 3}{2} = 1 \ / \ -2$$

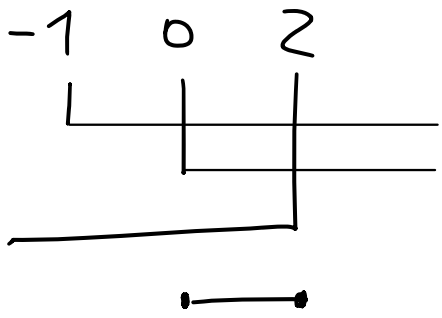
$$\boxed{x = 1 \quad \vee \quad x = -2}$$

- $|2x - 1| \leq x + 1$

$$\begin{cases} x + 1 \geq 0 \\ -(x+1) \leq 2x - 1 \leq x + 1 \end{cases} \iff \begin{cases} x \geq -1 \\ -x - 1 \leq 2x - 1 \leq x + 1 \end{cases}$$

$$\iff \begin{cases} x \geq -1 \\ -x - 1 \leq 2x - 1 \\ 2x - 1 \leq x + 1 \end{cases} \iff \begin{cases} x \geq -1 \\ -x \leq 2x \\ x \leq 2 \end{cases}$$

$$\iff \begin{cases} x \geq -1 \\ 0 \leq 3x \\ x \leq 2 \end{cases} \iff \begin{cases} x \geq -1 \\ x \geq 0 \\ x \leq 2 \end{cases} \iff \boxed{0 \leq x \leq 2}$$



- $|2x+1| \geq x^2 \iff 2x+1 \geq x^2 \vee 2x+1 \leq -x^2$

$x^2 \geq 0$:

$x^2-2x-1 \leq 0 \vee x^2+2x+1 \leq 0$

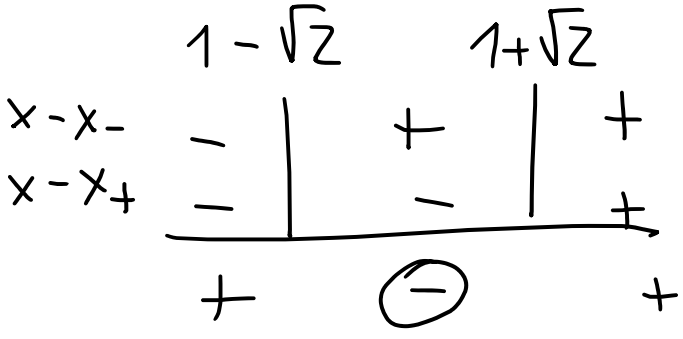
$x^2-2x-1 \leq 0 \vee (x+1)^2 \leq 0$

$x^2-2x-1 \leq 0 \vee x=-1$

$$x_\pm = \frac{2 \pm \sqrt{4-4(-1)\cdot 1}}{2} = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2}$$

$$x_\pm = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

$x^2-2x-1 = (x-x_+)(x-x_-) \leq 0 \iff 1-\sqrt{2} \leq x \leq 1+\sqrt{2}$



$$\boxed{1-\sqrt{2} \leq x \leq 1+\sqrt{2} \quad \vee \quad x=-1}$$

$(-1 < 1-\sqrt{2})$

- $\frac{|5-x|-3}{x^2-3} > 0$

$N>0$, $D>0$.

$N>0 \iff |5-x|-3>0 \iff |5-x|>3$

$\iff 5-x>3 \ \vee \ 5-x<-3$

$2>x \ \vee \ 8<x$

$\iff x<2 \ \vee \ x>8$

- $D>0 \iff x^2-3>0 \iff x^2>3 \iff x<-\sqrt{3} \vee x>\sqrt{3}$

	$-\sqrt{3}$		$\sqrt{3}$		2		8	
N	+	+		+		−		+
D	+	−		+		+		+
	(+)	−		(+)		−		(+)

$x<-\sqrt{3} \vee \sqrt{3}<x<2 \ \vee \ x>8$

- $e^{x^2}-4 \leq 0 \iff e^{x^2} \leq 4 \iff x^2 \leq \log 4$

$\iff \boxed{-\sqrt{\log 4} \leq x \leq \sqrt{\log 4}} \iff -(\log 4)^{\frac{1}{2}} \leq x \leq (\log 4)^{\frac{1}{2}}$

- $\log_{\frac{1}{3}}\left(\frac{x^2-x-2}{x+9}\right) \geq 1 \iff \left(\frac{1}{3}\right)^{\log_{\frac{1}{3}}\left(\frac{x^2-x-2}{x+9}\right)} \leq \left(\frac{1}{3}\right)^1$

$x^2-x-2 \ \ldots \ 1 \iff x^2 \ldots$

$$\frac{x^2-x-2}{x+9} \leq \frac{1}{3} \iff \frac{x^2-x-2}{x+9} - \frac{1}{3} \leq 0$$

$$\frac{3x^2-3x-6-(x+9)}{3(x+9)} \leq 0$$

$$\frac{3x^2-3x-6-x-9}{3(x+9)} \leq 0$$

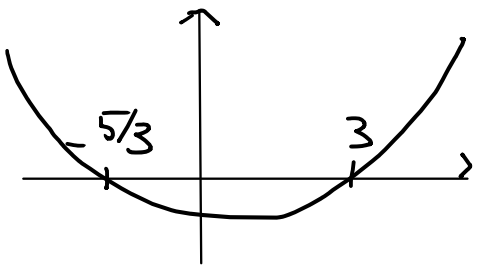
$$\frac{3x^2-4x-15}{3(x+9)} \leq 0$$

$N \geq 0$: $\quad x_{\pm} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$x_{\pm} = \frac{-b/2 \pm \sqrt{\left(\frac{b}{2}\right)^2-ac}}{a}$$

$$x_{\pm} = \frac{2 \pm \sqrt{4+15 \cdot 3}}{3} = \frac{2 \pm \sqrt{49}}{3} = \frac{2 \pm 7}{3} = 3 \,/\, -5/3$$

$3x^2-4x-15 \geq 0$



$N \geq 0 \iff x \leq -\frac{5}{3} \vee x \geq 3$

$D > 0 \iff x > -9$

	-9		$-\frac{5}{3}$		3	
N	+	+		−		+
D	−	+		+		+
	⊖	+		⊖		+

$$x < -9 \vee -\frac{5}{3} \leq x \leq 3$$

- $\left(\log_{\frac{1}{3}} x\right)^2 - 2\log_{\frac{1}{3}} x - 3 \leq 0$.

$y = \log_{\frac{1}{3}} x$, $\quad y^2 - 2y - 3 \leq 0$

$(y-3)(y+1) \leq 0$

	-1		3	
$y-3$	−	−		+
$y+1$	−	+		+
	+	−		+

$-1 \leq y \leq 3$

$-1 \leq \log_{\frac{1}{3}} x \leq 3$

$$\begin{cases} \log_{\frac{1}{3}} x \geq -1 \\ \log_{\frac{1}{3}} x \leq 3 \end{cases} \iff \begin{cases} x \leq \left(\frac{1}{3}\right)^{-1} = 3 \\ x \geq \left(\frac{1}{3}\right)^3 = \frac{1}{9} \end{cases}$$

$\iff \boxed{\frac{1}{9} \leq x \leq 3}$