Lesson 03/11/2026

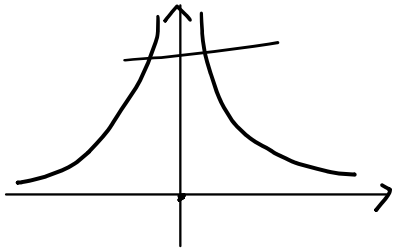
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Limits:

$f: X \subset \mathbb{R} \to \mathbb{R}$, $x \in X$, we can compute $f(x)$.

$f(x) = \frac{1}{x^2}$, $x \in \mathbb{R} \setminus \{0\}$

$f(1) = 1$ ...



what happens when x approaches 0 ?

$\lim_{x \to 0} f(x) = +\infty$.

Roughly: when x becomes very small, $f(x)$ becomes very large.

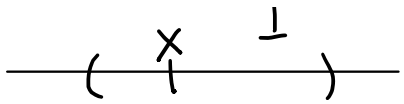
Neighbourhoods:

Definition: (i) given $x \in \mathbb{R}$, a neighbourhood of x is any open interval $I \subset \mathbb{R}$ s.t. $x \in I$.

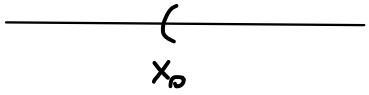
(ii) A neighbourhood of $+\infty$ is an open interval of the form $(x_0, +\infty)$, where $x_0 \in \mathbb{R}$.

(iii) A neighbourhood of $-\infty$ is an open interval $I \subset \mathbb{R}$ of the form $(-\infty, x_0)$, for some $x_0 \in \mathbb{R}$.

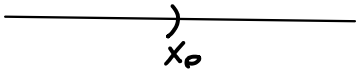
(i) $\overset{I}{\overbrace{\quad\quad}}$
$\quad\quad (\quad \overset{x}{\cdot} \quad)$; $x = \frac{1}{2}$, $I = (0,1)$ is a

(i) 

; $x=\frac{1}{2}$, $I=(0,1)$ is a neighbourhood of $\frac{1}{2}$.

(ii) 

$I=(x_0,+\infty)$

(iii) 

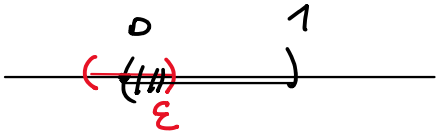
$I=(-\infty,x_0)$.

Definition: let $X\subset\mathbb{R}$ be a subset. A point $x_0\in\overline{\mathbb{R}}$, where $\overline{\mathbb{R}}=\mathbb{R}\cup\{\pm\infty\}$, is an accumulation point of X if, $\forall I\subset\mathbb{R}$ neighbourhood of x_0, we have $(I\cap X)\setminus\{x_0\}\neq\emptyset$.

Equivalently: a point $x_0\in\overline{\mathbb{R}}=\mathbb{R}\cup\{\pm\infty\}$ is an accumulation point of a set $X\subset\mathbb{R}$ if any neighbourhood $I\subset\mathbb{R}$ of x_0 contains at least one point $x\in X$, $x\neq x_0$.

$x_0=0$, $X=(0,1)$.

x_0 is an accumulation point of X.



$I=(-\varepsilon,\varepsilon)$

$I\cap X=(0,\varepsilon)$

$(I\cap X)\setminus\{0\}=(0,\varepsilon)\neq\emptyset$.

for example $\frac{\varepsilon}{2}\in(I\cap X)\setminus\{0\}=(0,\varepsilon)\neq\emptyset$.

1 is an accumulation point of X

$\overline{2} \sim (1 \wedge) \cdot 10 = (0,2) \neq \varphi$.

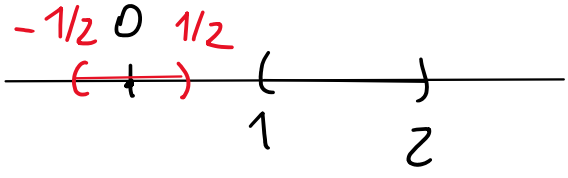
1 is an accumulation point of X.

Remark: • x_0 is an accumulation point of X $\not\Rightarrow$ $x_0 \in X$.

• $x_0 \in X$ $\not\Rightarrow$ x_0 is an accumulation point of X.

$X = \{0\} \cup (1,2) = \{x \in \mathbb{R} : x = 0 \vee 1 < x < 2\}$

Question: Is 0 an accumulation point of X? No



$I = \left(-\frac{1}{2}, \frac{1}{2}\right) = \{-\frac{1}{2} < x < \frac{1}{2}\}$

$I \cap X = \{0\}$

. $(I \cap X) \setminus \{0\} = \phi$. $\Rightarrow$ 0 is not an accumulation point of X.

. $0 \in X$.

Definition: A point $x_0 \in X \subset \mathbb{R}$ is said to be an isolated point if it is not an accumulation point.

Notation: let $X \subset \mathbb{R}$, we set

$$cA(X) = \{x_0 \in \overline{\mathbb{R}} : x_0 \text{ is an accumulation point of } X\}$$

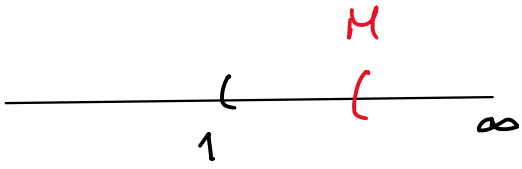
Remark: . $cA(X) \not\subset X$ in general

(It can be true or false,)

(It can be true or false, depending on X)

- $X \setminus cA(X)$ is the set of isolated points of X.
- $\forall x \in X$, x is either isolated or an accumulation point of X.

Example: • $X = (1, \infty)$; $x_0 = +\infty \in cA(X)$.



; take $M \in \mathbb{R}$,

$$((M, \infty) \cap (1, \infty)) \setminus \{+\infty\} \neq \emptyset.$$

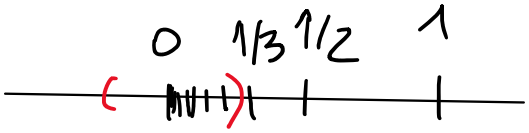
$$(M, \infty) \cap (1, \infty) = (\max\{1, M\}, \infty) \neq \emptyset$$

$$\left((M, \infty) \cap (1, \infty)\right) \setminus \{+\infty\} = (\max(1, M), \infty) \neq \emptyset.$$

• $X = (-\infty, x_0)$, $x_0 \in \mathbb{R}$, $-\infty \in cA(X)$.

• $X = \left\{ \frac{1}{n} : n \in \mathbb{N} \setminus \{0\} \right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots\right\}$

0 is an accumulation point of X, $0 \notin X$.

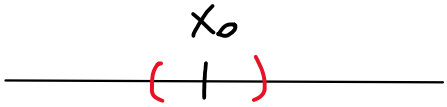


Take $\varepsilon > 0$, $I = (-\varepsilon, \varepsilon)$.

$\exists n_0 > 0 : \forall n \geq n_0, \ \frac{1}{n} < \varepsilon$

take $n_0 > \frac{1}{\varepsilon}$: if $n \geq n_0$, we have $\frac{1}{n} < \varepsilon$.

- $x_0 \in \mathbb{R}$, $X = \mathbb{R} \setminus \{x_0\} \Rightarrow x_0$ is an accumulation point of X.

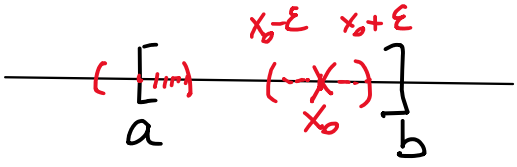


$I = (x_0 - \varepsilon, x_0 + \varepsilon)$

$X \cap I = (x_0 - \varepsilon, x_0) \cup (x_0, x_0 + \varepsilon)$

Roughly: x_0 is an accumulation of X if it is "surrounded" by points of X.
$x_0 \in X$ is isolated if "there are no other points of X too close to x_0".

- $X = [a, b]$, $a, b \in \mathbb{R}$, $a < b$
- $c\mathcal{A}(X) = X$



- I = interval $\Rightarrow c\mathcal{A}(I)$ = the smallest closed interval containing I.

- $I = (a, b), [a, b), (a, b], [a, b] \Rightarrow c\mathcal{A}(I) = [a, b]$

- $I = (a, +\infty), [a, +\infty) \Rightarrow c\mathcal{A}(I) = [a, +\infty] \subset \overline{\mathbb{R}}$

- $I = (-\infty, a), (-\infty, a] \Rightarrow c\mathcal{A}(I) = [-\infty, a] \subset \overline{\mathbb{R}}$

- $I = (-\infty, a), (-\infty, a] \Rightarrow \mathcal{A}(I) = [-\infty, a] \subset \overline{\mathbb{R}}$
- $I = \mathbb{R} \Rightarrow \mathcal{A}(I) = \overline{\mathbb{R}}$.
- $X = \mathbb{R} \setminus \{x_1 \dots x_n\} \Rightarrow \mathcal{A}(X) = \overline{\mathbb{R}}$.

Definition: Let $f: X \subset \mathbb{R} \to \mathbb{R}$ be a function. Let $x_0 \in \mathcal{A}(X)$ and $y_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

We say that

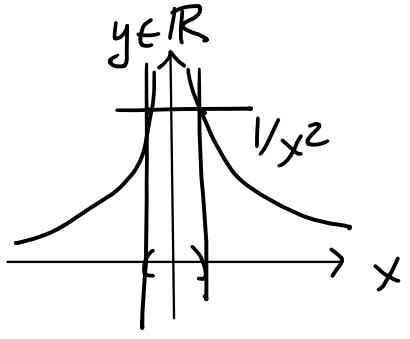
$$\lim_{x \to x_0} f(x) = y_0$$

if for any neighbourhood $J \subset \mathbb{R}$ of y_0, there exists a neighbourhood $I \subset \mathbb{R}$ of x_0:

$\forall x \in (I \cap X) \setminus \{x_0\}$, we have $f(x) \in J$.

Roughly: "If x is close enough to x_0 $\Rightarrow f(x)$ is close enough to y_0."

Example: $X = \mathbb{R} \setminus \{0\}$, $f(x) = \frac{1}{x^2}$



$$\lim_{x \to 0} \frac{1}{x^2} = +\infty$$

Remark: Let $f: X \subset \mathbb{R} \to \mathbb{R}$ be a function. Let $x_0 \in \mathcal{A}(X)$. The limit

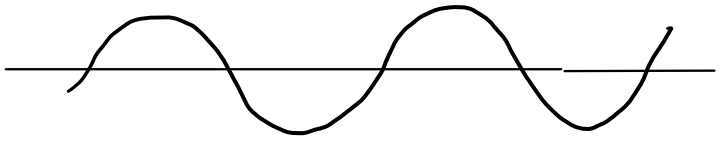
$\lim_{x \to x_0} f(x)$

does not always exist.

Examples: $X = \mathbb{R}$, $f(x) = \sin x$ (periodic function)

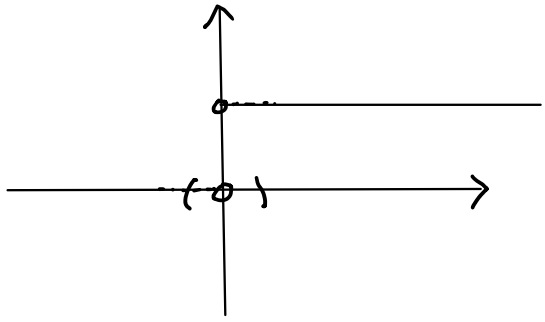
$\pm\infty$ are accumulation points of X

but f has no limit as $x \to \pm\infty$



$X = \mathbb{R} \setminus \{0\}$

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$



f has no limit as $x \to 0$.

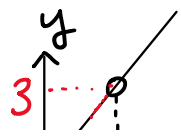
Take $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$. Assume $x_0 \in cA(X)$.

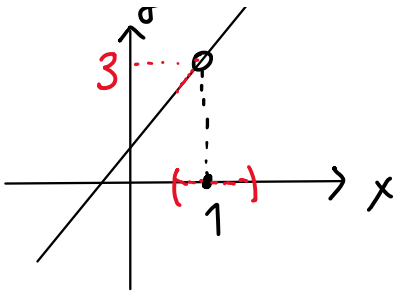
We say that $\lim_{x \to x_0} f(x) = y_0$ if

$\forall \varepsilon > 0$, $\exists \delta > 0$: $\forall x \in X$, $0 < |x - x_0| < \delta$, we have

$|f(x) - y_0| < \varepsilon$.

$$f(x) = \begin{cases} 2x + 1 & x \neq 1 \\ 0 & x = 1 \end{cases}$$





Claim: $\lim_{x\to 1} f(x) = 3 \ (= 2\cdot 1+1) \neq f(1) = 0$.

Take $\varepsilon > 0$ (small). We want to find $\delta > 0$:

if $|x-1| < \delta$, $x \neq 1 \Rightarrow |f(x) - 3| < \varepsilon$.

$\Leftrightarrow |2x+1-3| < \varepsilon$

we want to know where

$|2x+1-3| < \varepsilon$.

$\Leftrightarrow \quad -\varepsilon < 2x+1-3 < \varepsilon$

$\Leftrightarrow \quad \begin{cases} 2x+1-3 > -\varepsilon \\ 2x+1-3 < \varepsilon \end{cases} \Leftrightarrow \begin{cases} 2x-2 > -\varepsilon \\ 2x-2 < \varepsilon \end{cases}$

$\Leftrightarrow \quad \begin{cases} 2x > 2-\varepsilon \\ 2x < 2+\varepsilon \end{cases} \Leftrightarrow \begin{cases} x > 1-\varepsilon/2 \\ x < 1+\varepsilon/2 \end{cases}$

$\Leftrightarrow \quad \begin{cases} x-1 > -\varepsilon/2 \\ x-1 < \varepsilon/2 \end{cases} \Leftrightarrow -\frac{\varepsilon}{2} < x-1 < \frac{\varepsilon}{2}$

$\Leftrightarrow \quad |x-1| < \frac{\varepsilon}{2}$

we proved that, if $|x-1| < \frac{\varepsilon}{2} \Rightarrow |2x+1-3| < \varepsilon$.

we proved that, if $|x-1| < \frac{\varepsilon}{2} \Rightarrow |2x+1-3| < \varepsilon$.

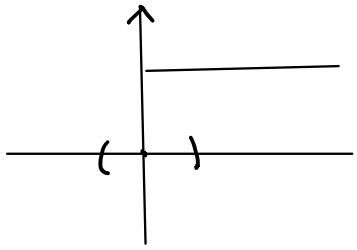
take $\boxed{\delta := \varepsilon/2 > 0}$

$|x-1| < \delta \Rightarrow |2x+1-3| < \varepsilon$.

if $|x-1| < \delta$, $x \neq 1 \Rightarrow |f(x)-3| < \varepsilon$

$[x \neq 1 \Rightarrow f(x) = 2x+1]$

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$



if $\lim_{x \to 0} f(x) = y_0 \in \mathbb{R}$.

$\varepsilon > 0$: $\exists\, \delta = \delta(\varepsilon) > 0$: if $|x| < \delta$, $x \neq 0$

$\Rightarrow |f(x) - y_0| < \varepsilon$.

$\forall x\, |x| < \delta$, $|f(x) - y_0| < \varepsilon$

take $0 < x < \delta \Rightarrow |1 - y_0| < \varepsilon$

take $-\delta < x < 0 \Rightarrow |0 - y_0| = |y_0| < \varepsilon$.

$\varepsilon = \frac{1}{4}$: $|1-y_0| < \frac{1}{4}$, $|y_0| < \frac{1}{4}$ impossible.

$$-\frac{1}{4} < 1 - y_0 < \frac{1}{4} \wedge -\frac{1}{4} < y_0 < \frac{1}{4}$$

$$\frac{3}{4} < y_0 < \frac{5}{4} \wedge -\frac{1}{4} < y_0 < \frac{1}{4}.$$

· Example: $f: X \subset \mathbb{R} \to \mathbb{R}$

· Example: $f: X \subset \mathbb{K} \to \mathbb{K}$

assume that $+\infty \in cA(X)$. We say that

$$\lim_{x \to +\infty} f(x) = +\infty$$

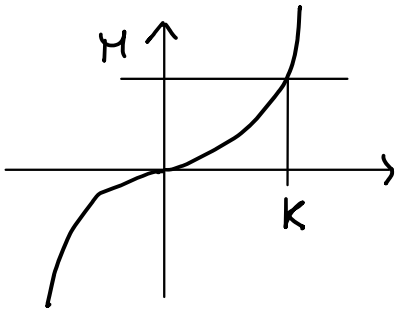
if $\forall M > 0 \ \exists K > 0 : x > K \Rightarrow f(x) > M$.

$f(x) = x^3$, take $M > 0$: we can see that

$x^3 > M \iff x > \sqrt[3]{M}$.

Take $K := \sqrt[3]{M}$: $x > K \Rightarrow f(x) > M$.

$\Rightarrow \lim_{x \to +\infty} x^3 = +\infty$.



Roughly: take the value $x_0 = +\infty$ and plug it into f.

What is $f(+\infty)$?

In this case "$(+\infty)^3 = +\infty$".

• $x_0 = +\infty$, $y_0 = -\infty$. Assume that $x_0 \in cA(X)$.

We say that $\lim_{x \to +\infty} f(x) = -\infty$ if

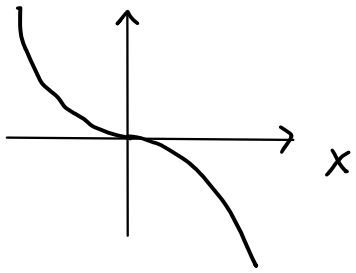
$\forall M > 0, \ \exists K > 0 : x > M \Rightarrow f(x) < -M$.

"If x is very large, then $f(x)$ is very small".

$f(x) = -x^3 \Rightarrow \lim_{x \to +\infty} (-x^3) = -\infty$.

Roughly:

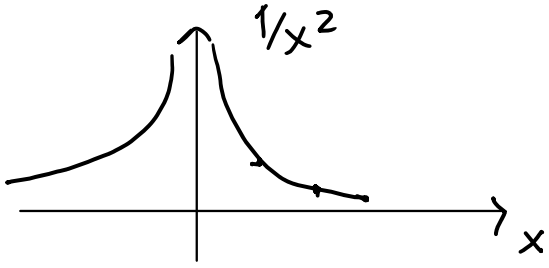
$$-(+\infty)^3 = -(+\infty) = -\infty$$



Similarly, we can define

$$\lim_{x \to -\infty} f(x) = \pm\infty .$$

Horizontal asymptotes:



$$f(x) = \frac{1}{x^2}$$

$1 \Rightarrow f(1) = 1$ $\quad$ $3 \Rightarrow f(3) = 1/9$

$2 \Rightarrow f(2) = \frac{1}{4}$

Definition: Let $f: X \subset \mathbb{R} \to \mathbb{R}$ be a function.

Assume that $+\infty \in \mathcal{A}(X)$, $y_0 \in \mathbb{R}$.

we say that $\lim_{x \to +\infty} f(x) = y_0$ if.

$\forall \varepsilon > 0 \ \exists M > 0 : \forall x > M, \ |f(x) - y_0| < \varepsilon .$

$$\lim_{x \to +\infty} \frac{1}{x^2} = 0 .$$

Take $\varepsilon > 0$: we need to solve $\frac{1}{x^2} < \varepsilon$.

$$\iff x^2 > \frac{1}{\varepsilon} \iff x < -\frac{1}{\sqrt{\varepsilon}} \ \vee \ x > \frac{1}{\sqrt{\varepsilon}}$$

$x^2 > 4 \iff x^2 - 4 > 0 \iff (x-2)(x+2) > 0$

$-2 \quad 2$

$\iff x < -2 \ \vee \ x > 2$

$$\begin{array}{c|ccccc} & & -2 & & 2 & \\ x+2 & - & | & + & | & + \\ x-2 & - & | & - & | & + \\ \hline & + & & - & & + \end{array}$$

$\Leftrightarrow \quad x < -2 \ \vee \ x > 2$

In particular: $x > \frac{1}{\sqrt{\varepsilon}} \Rightarrow \frac{1}{x^2} < \varepsilon$

Take $M = \frac{1}{\sqrt{\varepsilon}}$: $x > M \Rightarrow f(x) < \varepsilon$

This means that $\lim_{x \to +\infty} f(x) = 0$.

Definition: $f: X \subset \mathbb{R} \to \mathbb{R}$ be a function. Assume that $-\infty \in \mathcal{A}(X)$, $y_0 \in \mathbb{R}$.

$\lim_{x \to -\infty} f(x) = y_0$

$\Leftrightarrow \forall \varepsilon > 0, \exists M > 0 : \forall x < -M, \ |f(x) - y_0| < \varepsilon$.

Example: $\lim_{x \to -\infty} \frac{1}{x^2} = 0$.

Definition: Let $f: X \subset \mathbb{R} \to \mathbb{R}$ be a function. Assume that $+\infty \in \mathcal{A}(X)$ or $-\infty \in \mathcal{A}(X)$. We say that the straight line $y = y_0$ is a horizontal asymptot of f if

$\lim_{x \to +\infty} f(x) = y_0$ or $\lim_{x \to -\infty} f(x) = y_0$