Lesson 02/11/2026

lunedì 2 novembre 2026 08:59

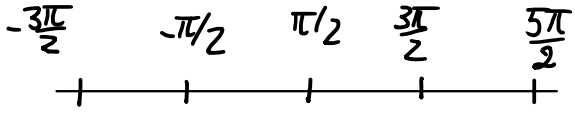
Tangent and cotangent:

Definition:

$$\tan : \bigcup_{k\in\mathbb{Z}} \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right) \longrightarrow \mathbb{R}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\bigcup_{k\in\mathbb{Z}} \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right) = \{x \in \mathbb{R} : x \neq k\pi, \forall k \in \mathbb{Z}\}$$



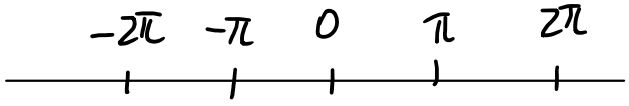
We need to exclude these points from the domain since cos vanishes at these points.

Definition:

$$\cot : \bigcup_{k\in\mathbb{Z}} (k\pi, (k+1)\pi) \longrightarrow \mathbb{R}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\bigcup_{k\in\mathbb{Z}} (k\pi, (k+1)\pi) = \{x \in \mathbb{R} : x \neq k\pi, \forall k \in \mathbb{Z}\}$$



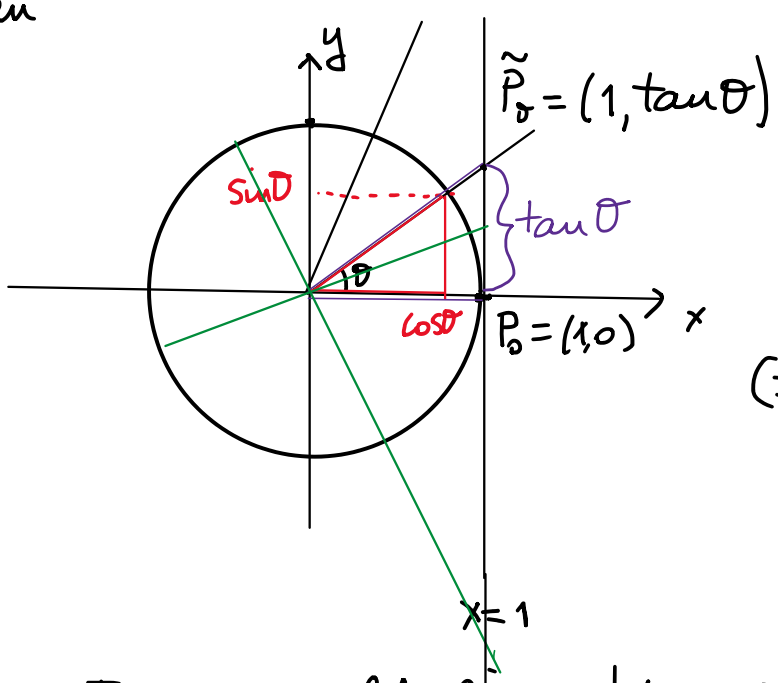
Remark: If $x \neq \frac{\pi}{2} + k\frac{\pi}{2}$, we have $\tan x = \frac{1}{\cot x}$.

$$\Leftrightarrow \quad x \neq k\frac{\pi}{2}, \forall k \in \mathbb{Z}$$

Geometrical interpretation:

Geometrical interpretation:

tan what is tanθ?

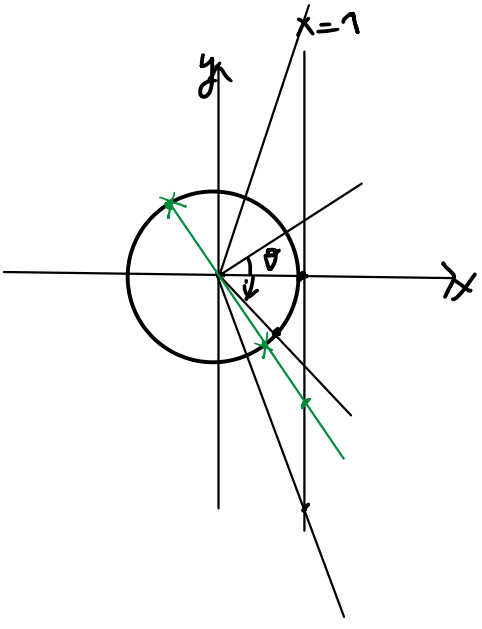


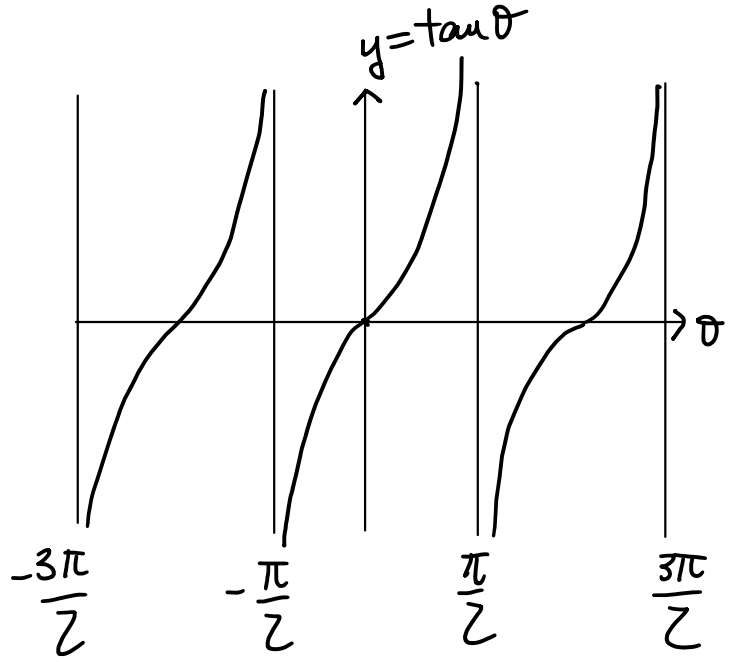
$$\frac{\sin\theta}{\cos\theta} = \frac{\ell}{1}$$

$$\Leftrightarrow \quad \ell = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$\theta = \frac{\pi}{2}$: the half line become $\{(0,y), y>0\}$, so that $\{(0,y): y>0\} \cap \{x=1\} = \phi$. This is the geometric reason why the tangent is not defined at $\theta = \pi/2$.

Graph of the function tan





$\tan\theta = \tan(\theta+\pi) \quad \forall \theta \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$

. tan is π-periodic.

once we know the graph of tan in $(-\frac{\pi}{2}, \frac{\pi}{2})$, we can extend it by periodicity to the whole domain

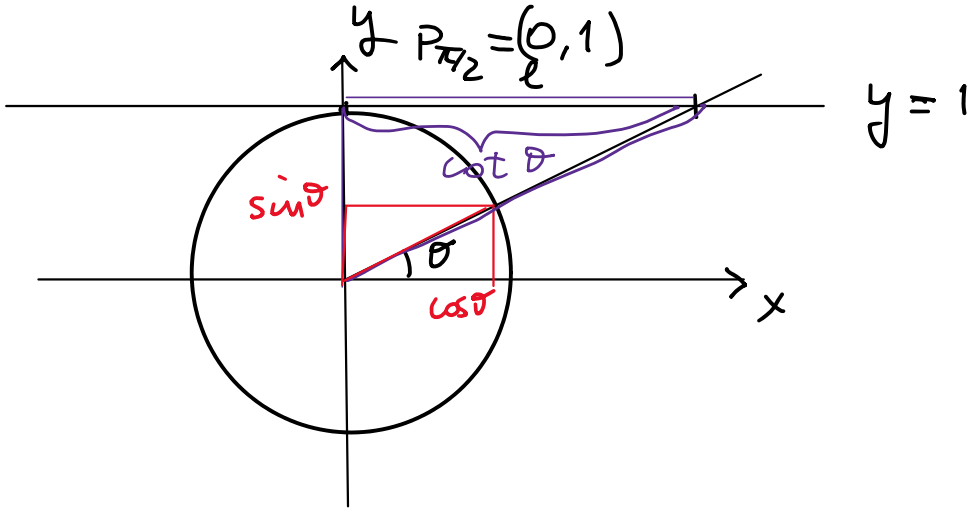
Remarks: . Is the tan bounded? No, the tan is unbounded.

• Is the tangent odd or even?

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

tan is odd.

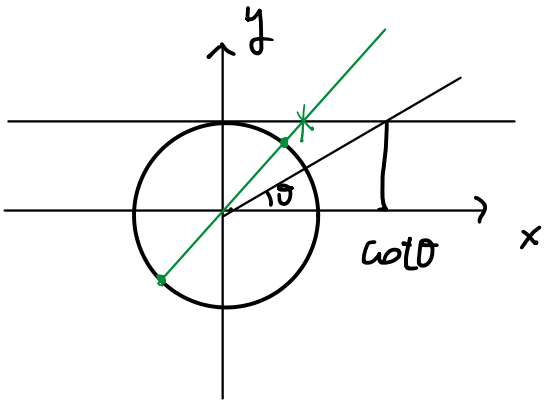
Cotangent: geometric interpretation.

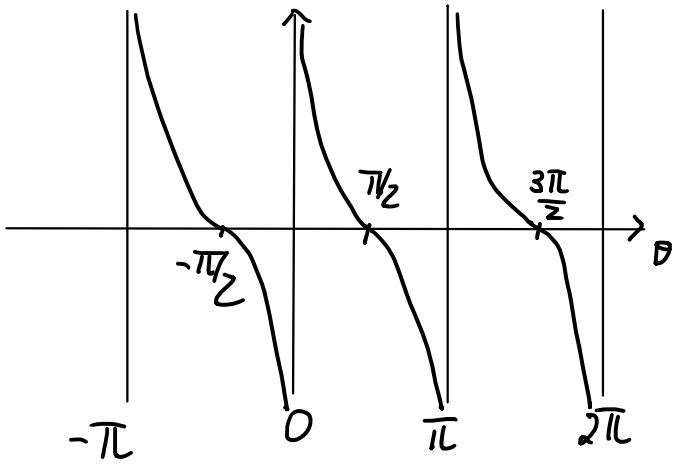


$$\cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \frac{\cos\theta}{\sin\theta} = \frac{\ell}{1}$$

$$\Rightarrow \ell = \cot\theta$$

The graph of the cotangent:





$\cot\theta = \cot(\theta+\pi)$, $\forall \theta = k\pi$

. The cotangent is π-periodic.

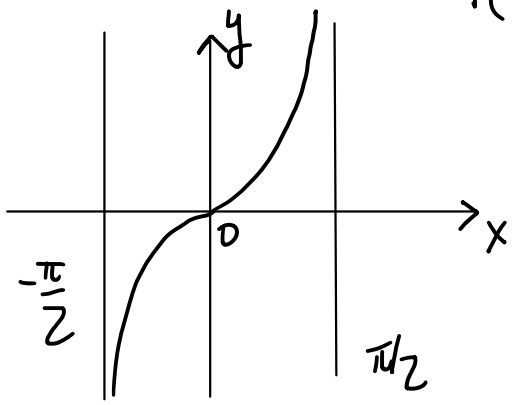
. Is the cotangent bounded? No, the cotangent is unbounded.

. Is the cotangent odd or even?

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta, \quad \forall \theta \neq k\pi.$$

cot is odd.

Monotonicity: $\tan\big|_{(-\frac{\pi}{2},\frac{\pi}{2})}$ is increasing



$\Rightarrow \tan\big|_{(-\frac{\pi}{2},\frac{\pi}{2})}$ is injective.

$\Rightarrow \tan\big|_{(-\frac{\pi}{2},\frac{\pi}{2})} : (-\frac{\pi}{2},\frac{\pi}{2}) \to \tan(-\frac{\pi}{2},\frac{\pi}{2})$

is bijective.

What is $\tan(-\frac{\pi}{2},\frac{\pi}{2}) = ?$ $\tan(\frac{\pi}{2},\frac{\pi}{2}) = \mathbb{R}$.

$(-\frac{\pi}{2}, \frac{\pi}{2})$ $(\frac{\pi}{2}, \frac{\pi}{2})$

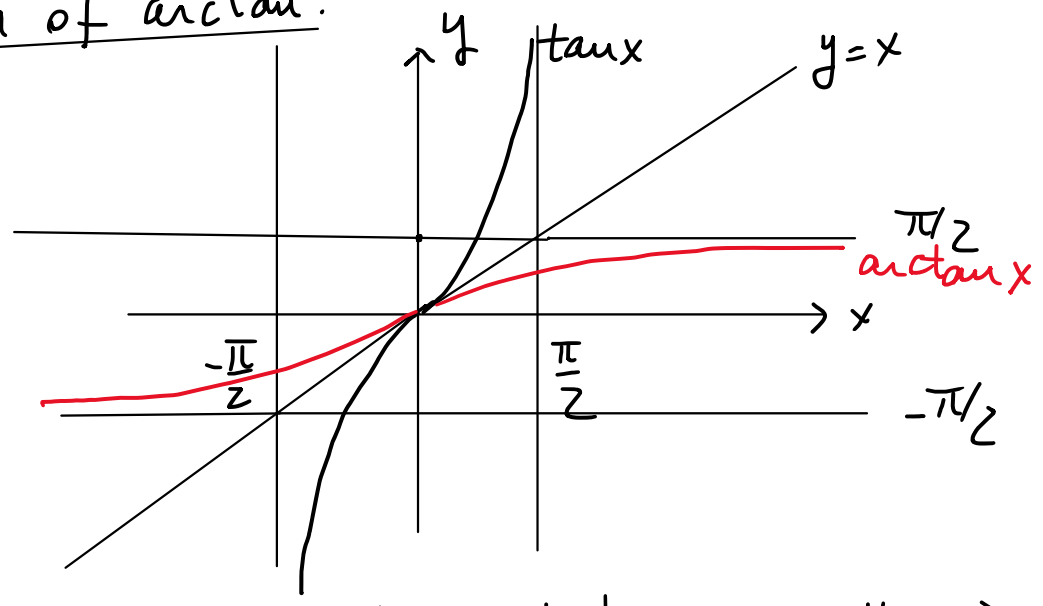
So $\tan\big|_{(-\frac{\pi}{2}, \frac{\pi}{2})} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is invertible.

The inverse is known as the arctangent and it is denoted by

$$\arctan: \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

If $f: f: X \to Y$ is invertible, then $f^{-1}: Y \to X$

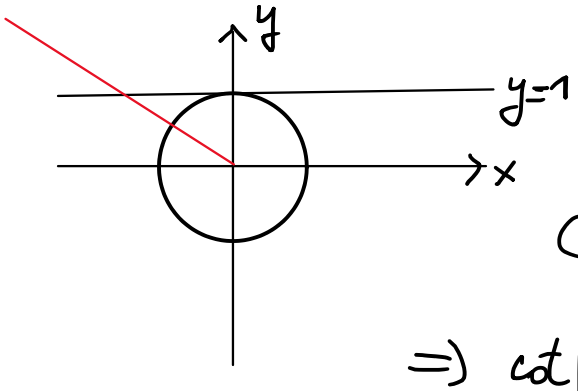
Graph of arctan:



arctan is odd, bounded because the image is $(-\frac{\pi}{2}, \frac{\pi}{2})$, strictly increasing.

general fact: If $f: I \subset \mathbb{R} \to \mathbb{R}$ is monotone (strictly) $\Rightarrow$ $f|_I : I \to f(I)$ is invertible and $f^{-1}: f(I) \to I$ has exactly the same monotonicity as f (I is an interval).

Monotonicity of the cotangent:



the cot is decreasing in $(0,\pi)$

$\Leftrightarrow \cot|_{(0,\pi)} : (0,\pi) \to \mathbb{R}$ is decreasing.

$\Rightarrow \cot|_{(0,\pi)}$ is injective

$\Rightarrow \cot|_{(0,\pi)} : (0,\pi) \to \cot(0,\pi)$ is bijective.

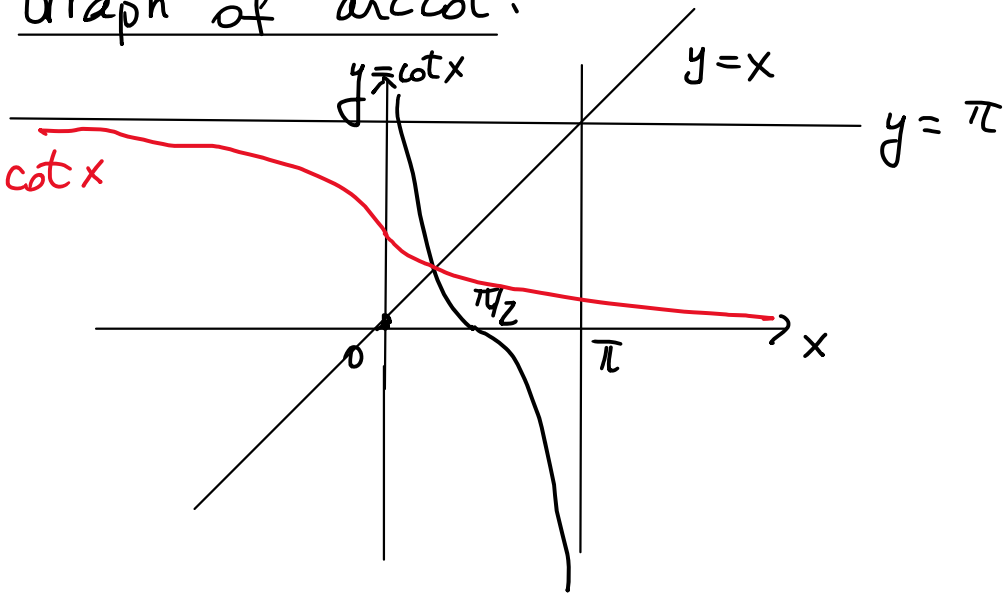
$\cot(0,\pi) = \mathbb{R}$

$\Rightarrow \cot|_{(0,\pi)} : (0,\pi) \to \mathbb{R}$ is invertible.

The inverse of $\cot|_{(0,\pi)}$ is known as the arccotangent and denoted by

$$\operatorname{arccot} : \mathbb{R} \to (0,\pi).$$

Graph of arccot:



arccot is bounded. decreasing.

arccot is bounded, decreasing.

Is arccot odd or even? No.

· arccot $x > 0$ $\forall x \in \mathbb{R}$ $\Rightarrow$ arccot is not odd. ($f(0) \neq 0$)

· arccot $\searrow$ (is decreasing) $\Rightarrow$ arccot is not even.

If $f: I \to \mathbb{R}$ is even $\Rightarrow f(x) = f(-x)$, $\forall x \in I$. If f is decreasing, then $f(x) < f(-x)$, $\forall x \in I$, $x > 0$.

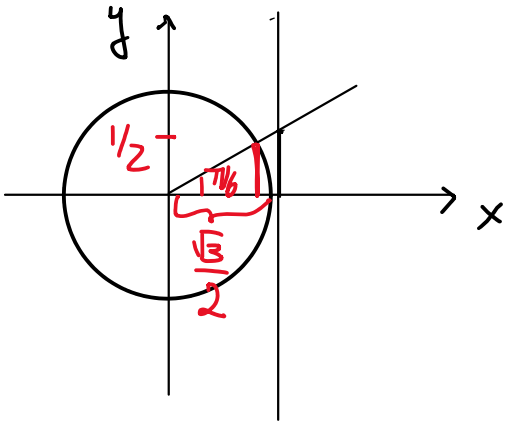
f decreasing $\Rightarrow$ f not even.

Tangent and cotangent of "known" angles:

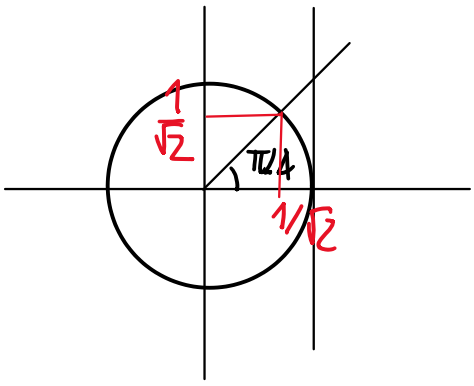
· $\theta = 0$: $\tan 0 = \dfrac{\sin 0}{\cos 0} = \dfrac{0}{1} = 0$

· $\theta = \pi/6$, $\tan(\pi/6) = \dfrac{\sin(\pi/6)}{\cos(\pi/6)} = \dfrac{1/2}{\sqrt{3}/2} = \dfrac{1}{2} \cdot \dfrac{2}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

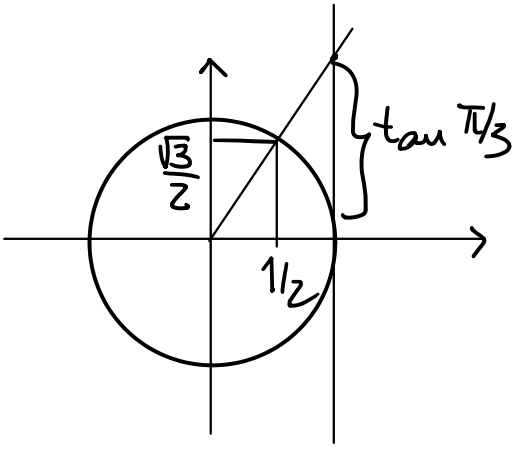
$\tan\left(\dfrac{\pi}{6}\right) = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$.



· $\theta = \dfrac{\pi}{4}$ $\qquad \tan\dfrac{\pi}{4} = \dfrac{\sin(\pi/4)}{\cos(\pi/4)} = \dfrac{1/\sqrt{2}}{1/\sqrt{2}} = 1$

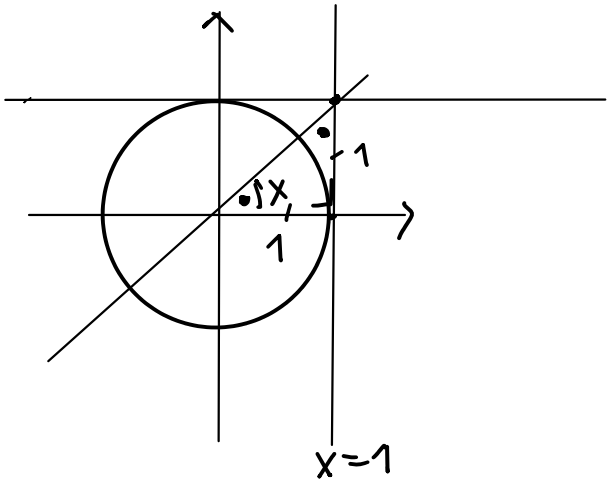


$\theta = \frac{\pi}{3}$, $\tan\frac{\pi}{3} = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \frac{\sqrt{3}}{2}\cdot 2 = \sqrt{3}$



Equations and inequalities:

$\tan x = 1$ $\qquad x = \frac{\pi}{4} + k\pi$



$y = 1$

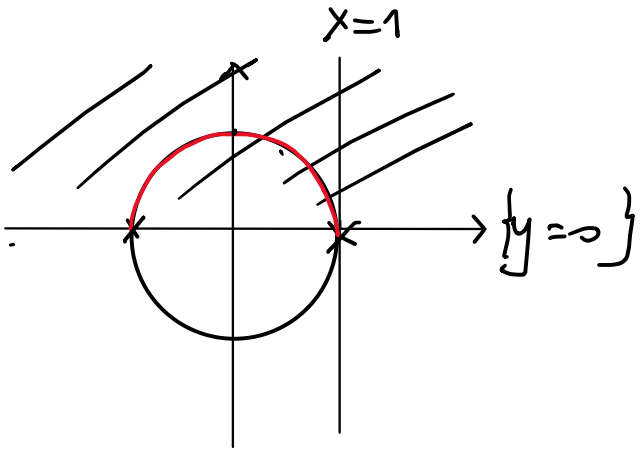
$\alpha = 45° (=) \quad \theta = \frac{\pi}{4}$

$\tan x \geq 0$

$x = 1$

$0 \leq x < \frac{\pi}{2} \vee \frac{\pi}{2} < x \leq \pi$

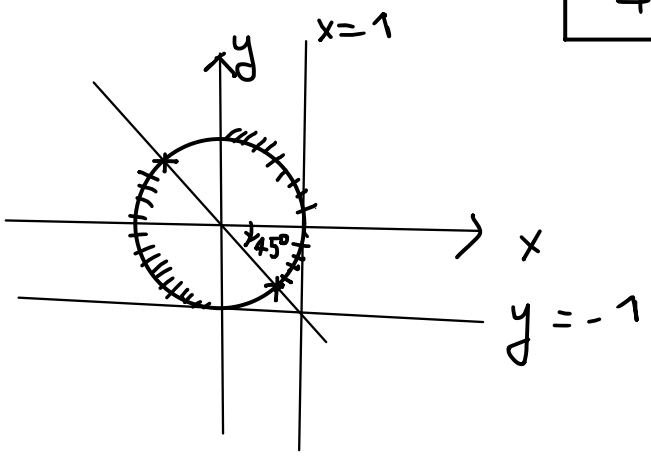
$0 \le x < \frac{\pi}{2} \quad \vee \quad \frac{\pi}{2} < x \le \pi$



$$k\pi \le x < \frac{\pi}{2} + k\pi \quad \vee \quad \frac{\pi}{2} + k\pi < x \le \pi + k\pi$$

$\tan x \ge -1 \iff$

$$-\frac{\pi}{4} + k\pi \le x < \frac{\pi}{2} + k\pi$$



Useful relations:

- $$1 + \tan^2 x = \frac{1}{\cos^2 x} \quad , \forall x \neq k\pi + \frac{\pi}{2}$$

$$1 + \tan^2 x = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

- $$1 + \cot^2 x = \frac{1}{\sin^2 x} \quad , \quad \forall x \neq k\pi$$

$$1 + \cot^2 x = 1 + \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$