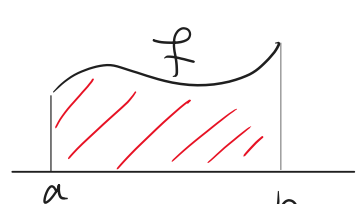


Indefinite integrals = inverse operation to differentiation.

We write $F(x) = \int f(x) dx$ if $F'(x) = f(x)$.

$(x)' = 1 \Rightarrow \int 1 dx = x + c$

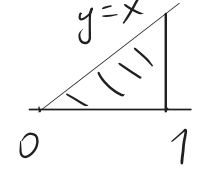
Riemann integration: computation of the area below a graph.



$$f: [a, b] \rightarrow \mathbb{R}, \quad \int_a^b f(x) dx = \text{red Area.}$$

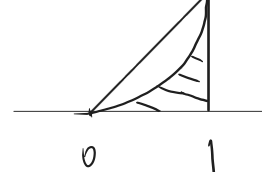
$$\int_a^b f(x) dx = F(b) - F(a), \text{ where } F \text{ is a primitive of } f \text{ (continuous)}$$

$$[F' = f]$$



$$\text{Area} = \frac{1 \cdot 1}{2} = \frac{1}{2}$$

$$\int_0^1 x dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$



$$\text{Area} = \int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

$$\int x dx = \frac{x^{n+1}}{n+1} + c$$

Exercises: Sheet 7

1.a) $f(x) = \tanh^{-1}(x)$, where $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$, $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$

$$f = g^{-1}, \quad (g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))}$$

$$\tanh'(x) = \frac{(\sinh x)'}{\cosh x} + \sinh x \left(\frac{1}{\cosh x} \right)' = \frac{(\sinh x)'}{\cosh x} - \frac{\sinh x (\cosh x)'}{(\cosh x)^2} = 1 - \frac{\sinh x}{\cosh^2 x} = 1 - (\tanh x)^2$$

$$(\sinh)' = \left(\frac{e^x - e^{-x}}{2} \right)' = \frac{1}{2} (e^x)' - \frac{1}{2} (e^{-x})' = \frac{e^x}{2} - \frac{1}{2} (-e^{-x}) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$(\alpha f + \beta g)' = \alpha f' + \beta g'$$

$$(a^x)' = (e^{\log a^x})' = (e^{x \log a})' = e^{x \log a} \log a = e^{\log a^x} \log a = (\log a) a^x$$

$$(2x)' = 2(x)' = 2 \cdot 1 = 2 \quad [(e^x)' = e^x]$$

Chain rule: $(f \circ g)' = (f' \circ g) \cdot g' \Rightarrow (e^{-x})' = e^{-x} (-1) = -e^{-x}$

$$f(y) = e^y, g(x) = -x; \quad g'(x) = (-1)(x)' = -1$$

$$(\cosh x)' = \left(\frac{e^x + e^{-x}}{2} \right)' = \frac{1}{2} (e^x)' + \frac{1}{2} (e^{-x})' = \frac{e^x}{2} + \frac{1}{2} (-e^{-x}) = \frac{e^x - e^{-x}}{2} = \sinh x$$

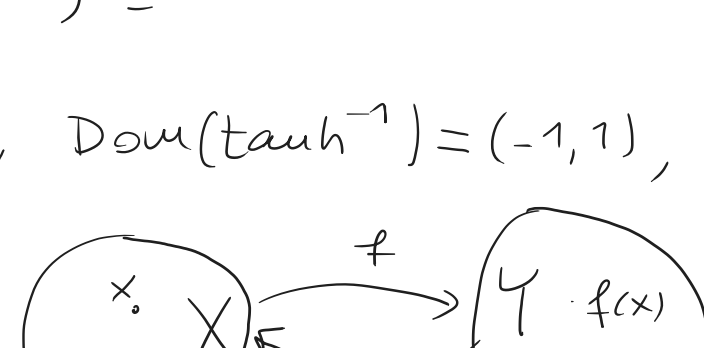
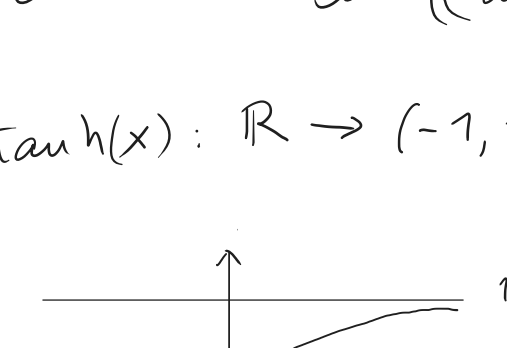
$$(\tanh^{-1})'(x) = \frac{1}{(\tanh)'(\tanh^{-1}(x))} = \frac{1}{1 - \tanh^2(\tanh^{-1}(x))} = \frac{1}{1 - (\tanh(\tanh^{-1}(x)))^2} = \frac{1}{1 - x^2}$$

$$g(g^{-1}(x)) = x$$

Determine $\text{Dom}((\tanh)^{-1})$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$$

$$\tanh(x): \mathbb{R} \rightarrow (-1, 1), \quad \text{Dom}(\tanh^{-1}) = (-1, 1), \quad \tanh^{-1}(-1, 1) = \text{Im}(\tanh^{-1}) = \mathbb{R}$$



$$1.b) f(x) = \tan\left(\frac{1}{2(x^2+1)}\right)$$



$$\text{Dom}(f) = \left\{ x \in \mathbb{R} : \frac{1}{2(x^2+1)} \neq \frac{\pi}{2} + k\pi \right\} = \left\{ x \in \mathbb{R} : \frac{1}{x^2+1} \neq \pi + 2\pi k : k \in \mathbb{Z} \right\}$$

$$= \left\{ x \in \mathbb{R} : x^2 + 1 \neq \frac{1}{\pi + 2\pi k} : k \in \mathbb{Z} \right\} = \left\{ x \in \mathbb{R}^2 : x^2 \neq -1 + \frac{1}{\pi + 2\pi k} : k \in \mathbb{Z} \right\} = \mathbb{R}$$

$$-1 + \frac{1}{\pi(1+2k)} \geq 0 \Leftrightarrow \frac{1}{\pi(1+2k)} \geq 1 \Leftrightarrow \frac{1 - \pi(1+2k)}{\pi(1+2k)} \geq 0$$

$$\begin{cases} 1 - \pi(1+2k) \geq 0 \\ 1+2k > 0 \end{cases} \quad \begin{cases} 1 \geq \pi(1+2k) \\ k \geq 0 \end{cases} \Leftrightarrow \begin{cases} 1+2k \leq \frac{1}{\pi} \\ k \geq 0 \end{cases} \Leftrightarrow \begin{cases} 2k \leq \frac{1}{\pi} - 1 < 0 \\ k \geq 0 \end{cases} \text{ impossible}$$

$$\begin{cases} 1 - \pi(1+2k) \leq 0 \\ 1+2k < 0 \end{cases} \quad \begin{cases} 1 \leq \pi(1+2k) \\ k < 0 \end{cases} \quad \begin{cases} \frac{1}{\pi} \leq 1+2k \\ k < 0 \end{cases} \quad \begin{cases} \frac{1}{\pi} - 1 \leq 2k \\ k < 0 \end{cases} \quad \frac{1-\pi}{\pi} \leq 2k < 0$$

$$\frac{1-\pi}{2\pi} \leq k < 0, \quad -1 < \frac{1-\pi}{2\pi} < 0 \text{ impossible}$$

$$0 < \frac{1}{2(x^2+1)} < \frac{1}{2} < \frac{\pi}{2}, \forall x \in \mathbb{R} \Rightarrow \text{Dom}(f) = \mathbb{R}$$

$$\left(\tan \frac{1}{2(x^2+1)} \right)' = \left(1 - \tan^2 \left(\frac{1}{2(x^2+1)} \right) \right) \left(\frac{1}{2(x^2+1)} \right)' = \left(1 - \tan^2 \left(\frac{1}{2(x^2+1)} \right) \right) \frac{1}{2} \left(\frac{-1}{1+x^2} \right)' =$$

$$\frac{d}{dx} \left(\tan y \right) = 1 - \tan^2 y = \left(1 - \tan^2 \left(\frac{1}{2(x^2+1)} \right) \right) \frac{1}{2} \frac{-1}{(x^2+1)^2} 2x$$

$$= \left(1 - \tan^2 \left(\frac{1}{2(x^2+1)} \right) \right) \frac{-x}{(1+x^2)^2}$$

$$1.3) f(x) = x \sin\left(\frac{1}{x}\right)$$

$$f = f_1 \cdot f_2, \quad (f_1 \cdot f_2)' = f_1' f_2 + f_2' f_1$$

$$\text{D}(f) = \mathbb{R} \setminus \{0\}$$

$$f'(x) = (x)' \sin\left(\frac{1}{x}\right) + \left(\sin\left(\frac{1}{x}\right)\right)' x = 1 \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right) \cdot x \left(-\frac{1}{x^2}\right) = \sin\left(\frac{1}{x}\right) - \frac{1}{x} \cos\left(\frac{1}{x}\right)$$

$$\left(\sin\left(\frac{1}{x}\right)\right)' = \frac{1}{x^2} \cos\left(\frac{1}{x}\right) \text{ chain rule}$$

$$1.4) f(x) = (x^2-1)^{\frac{1}{x-1}}, \quad f(x) = f_1(x) f_2(x), \quad \text{D}(f) = \{x \in \mathbb{R} : f_1(x) > 0\}$$

$$\text{D}(f) = (-\infty, -1) \cup (1, \infty)$$

$$f(x) = e^{\log(x^2-1)^{\frac{1}{x-1}}} = e^{\frac{\log(x^2-1)}{x-1}} = e^{g(x)}$$

$$f'(x) = (e^{g(x)})' = e^{g(x)} g'(x) = (x^2-1)^{\frac{1}{x-1}} \left(\frac{\log(x^2-1)}{x-1} \right)' = (x^2-1)^{\frac{1}{x-1}} \left(\frac{(\log(x^2-1))' (x-1) - \log(x^2-1) (x-1)'}{(x-1)^2} \right)$$

$$(a \cdot b)'(x) = a'(b(x)) \cdot b'(x) = \frac{(x^2-1)^{\frac{1}{x-1}}}{(x-1)^2} \left(\frac{1}{x^2-1} (2x)(x-1) - \log(x^2-1) \cdot 1 \right)$$

$$a(y) = e^y, \quad b(x) = g(x) \Rightarrow a'(y) = e^y$$

$$(a \cdot b)'(x) = (e^{g(x)})' = a'(g(x)) \cdot g'(x) = e^{g(x)} \cdot g'(x) = \frac{(x^2-1)^{\frac{1}{x-1}}}{(x-1)^2} \left(\frac{2x}{x-1} - \log(x^2-1) \right)$$

$$\left(\frac{f_1}{f_2} \right)' = \frac{f_1'}{f_2} + f_1 \left(\frac{1}{f_2} \right)' = \frac{f_1'}{f_2} - f_1 \frac{f_2'}{f_2^2} = \frac{f_1' f_2 - f_1 f_2'}{f_2^2}$$

$$\lim_{h \rightarrow 0} \frac{\sinh h}{h} = 1, \quad \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

$$2.a) \text{ Draw the graph of } f(x) = \frac{1}{2}x^2 + \frac{1}{4}|x| - \frac{1}{3}|x|^3$$

$$\text{I) } \text{D}(f) = \mathbb{R}$$

$$\text{II) Symmetry: } f(-x) = f(x) \Rightarrow f \text{ is even.}$$

$$\text{III) Zeros of } f: x=0 \text{ and?}$$

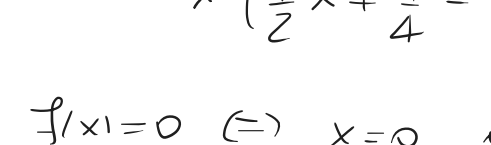
$$f(x) = x \left(\frac{1}{2}x + \frac{1}{4} - \frac{1}{3}x^2 \right), \quad \forall x > 0$$

$$f(x) = 0 \Leftrightarrow x=0 \text{ or } \frac{1}{2}x + \frac{1}{4} - \frac{1}{3}x^2 = 0 \quad x_{\pm} = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4 \left(\frac{1}{4} \right) \left(-\frac{1}{3} \right)}}{-\frac{2}{3}} = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{3}}}{-\frac{2}{3}}$$

$$x_{\pm} = -\frac{3}{2} \left(-\frac{1}{2} \pm \sqrt{\frac{7}{3}} \right) = \frac{3}{4} \left(1 \pm \sqrt{\frac{7}{3}} \right), \quad x_- < 0 < x_+, \quad f(x) = 0 \Leftrightarrow \boxed{x=0 \vee x=x_+}$$

$$\text{IV) } f(x) > 0, \quad x > 0$$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \left(\frac{1}{2}x^2 + \frac{1}{4}x - \frac{1}{3}x^3 \right) = \lim_{x \rightarrow +\infty} x^3 \left(\frac{1}{2x} + \frac{1}{4x^2} - \frac{1}{3} \right) = -\infty$$



$$f(x) = x \left(\frac{1}{2}x + \frac{1}{4} - \frac{1}{3}x^2 \right), \quad \forall x > 0 \quad [f(x) = \frac{1}{2}x^2 + \frac{1}{4}x - \frac{1}{3}x^3]$$

$$f'(x) = x + \frac{1}{4} - x^2$$

$$\text{Which are the critical points? } f'(x) = 0 \quad \tilde{x}_{\pm} = \frac{-1 \pm \sqrt{1 - 4 \left(\frac{1}{4} \right) \left(-\frac{1}{3} \right)}}{-2} = \frac{-1 \pm \sqrt{\frac{7}{3}}}{-2}$$

$$\tilde{x}_{\pm} = \frac{1}{2} \pm \frac{\sqrt{7}}{2} = \frac{1}{2} (1 \pm \sqrt{7}); \quad \text{which of the two solutions is acceptable? } \tilde{x}_+ \text{ only.}$$

$$\tilde{x}_+ \geq x_+? \quad \tilde{x}_+ < x_+$$

$$\text{What kind of critical point is } \tilde{x}_+? \quad \tilde{x}_+ \text{ is a global max.}$$

$$\text{Where is } f \text{ convex-concave?}$$

$$f \text{ is convex if } f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall x, y \in I, \forall \lambda \in (0, 1)$$

$$f: I \rightarrow \mathbb{R}$$



$$z = \lambda x + (1-\lambda)y$$

$$\text{If } f \text{ is twice differentiable in } I, \quad f \text{ is convex} \Leftrightarrow f'' \geq 0 \text{ in } I$$

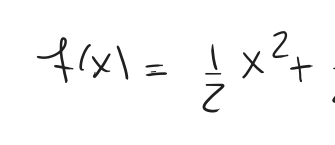
$$f(x) = \frac{1}{2}x^2 + \frac{1}{4}x - \frac{1}{3}|x|^3, \quad \text{for } x > 0, \quad f(x) = \frac{1}{2}x^2 + \frac{1}{4}x - \frac{1}{3}x^3$$

$$f'(x) = x + \frac{1}{4} - x^2; \quad f''(x) = 1 - 2x \geq 0 \Leftrightarrow x \leq \frac{1}{2}$$

$$f \text{ is convex in } [0, \frac{1}{2}]; \quad f \text{ is concave in } [\frac{1}{2}, +\infty)$$

$$f(x) = \frac{1}{2}x^2 + \frac{1}{4}x - \frac{1}{p}|x|^p \quad p > 2$$

$$x > 0, \quad f(x) = \frac{1}{2}x^2 + \frac{1}{4}x - \frac{1}{p}x^p, \quad \text{D}(f) = \mathbb{R}, \quad f(0), \quad \lim_{x \rightarrow +\infty} f(x) = -\infty$$



$$f' = x + \frac{1}{4} - x^{p-1}, \quad f'' = 1 - (p-1)x^{p-2}$$

$$f''(0) = 1 > 0, \quad f'(x) = 0 \Leftrightarrow x = \left(\frac{1}{p-1} \right)^{\frac{1}{p-2}} > 0$$