

Qualitative properties of functions:

Boundedness:

Def: Let $f: X \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a function. Set

$$\inf_{x \in X} f(x) = \inf f(X)$$

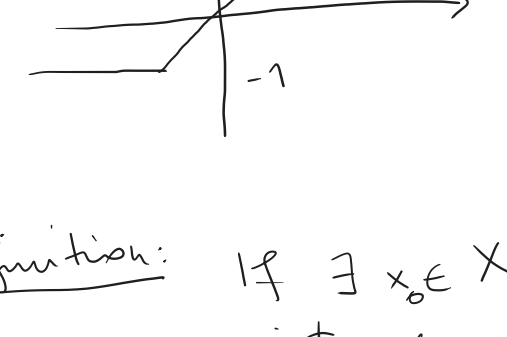
$$\sup_{x \in X} f(x) = \sup f(X)$$

We say that a function $f: X \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is bounded below if $\inf_{x \in X} f(x) \in \mathbb{R}$, bounded above if $\sup_{x \in X} f(x) \in \mathbb{R}$ and bounded if it is both bounded below and bounded above.

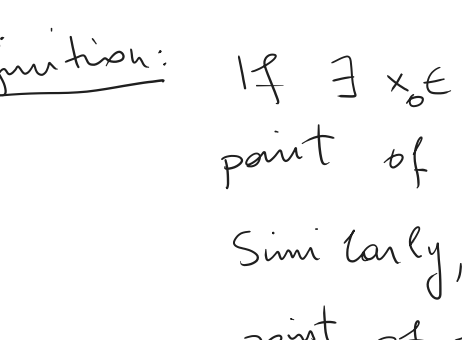


$f(x) = e^x$ is bounded below, but not bounded above.

In fact, $\inf_{x \in \mathbb{R}} f(x) = \inf f(\mathbb{R}) = \inf (0, \infty) = 0$



$f(x) = \log x$ is neither bounded above, nor bounded below



$f(x) = \begin{cases} x & \text{for } -1 \leq x \leq 1 \\ 1 & \text{for } x > 1 \\ -1 & \text{for } x < -1 \end{cases}$ is bounded

$$\sup_{x \in \mathbb{R}} f(x) = 1, \inf_{x \in \mathbb{R}} f(x) = -1$$

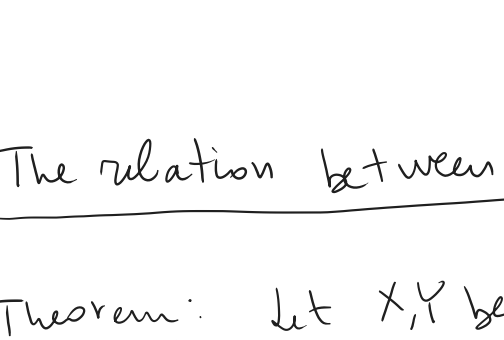
Definition: If $\exists x_0 \in X: f(x_0) = \inf_{x \in X} f(x)$, we say that x_0 is a minimum point of f and $f(x_0)$ is the minimum of f in X .

Similarly, if $\exists x_0 \in X: f(x_0) = \sup_{x \in X} f(x)$, we say that x_0 is a maximum point of f and $f(x_0)$ is the maximum of f in X .

Example: The function achieves its minimum, that is -1, and its maximum, that is 1.

The function $f(x) = e^x$ has neither a minimum nor a max.

The function $f(x) = x^2$



$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$f(\mathbb{R}) = [0, \infty)$$

$$\min_{x \in \mathbb{R}} f(x) = 0$$

$x=0$ is the unique minimum point.

f has no maximum

$$\sup_{x \in \mathbb{R}} f(x) = +\infty (= \sup f(\mathbb{R}) = \sup ([0, \infty)))$$

The relation between monotonicity and invertibility of a function:

Theorem: Let X, Y be set, let $f: X \rightarrow Y$ be a function. Then

f is invertible $\Leftrightarrow f$ is both injective and surjective $\Leftrightarrow f$ is bijective

Reminder: $f: X \rightarrow Y$ is injective if $\forall x_1, x_2 \in X, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

f is surjective if $\forall y \in Y, \exists x \in X: y = f(x)$.

Proof: (i) f is invertible $\Rightarrow \exists f^{-1}: Y \rightarrow X: f \circ f^{-1} = Id_Y$ and $f^{-1} \circ f = Id_X$.

$\forall y \in Y, y = f(f^{-1}(y)) \Rightarrow$ setting $x := f^{-1}(y)$ we have $y = f(x) \Rightarrow f$ is surjective.

$f(x_1) = f(x_2), x_1, x_2 \in X \Rightarrow f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \Rightarrow x_1 = x_2 \Rightarrow f$ is injective.

(ii) $\forall y \in Y$, we need to define $f^{-1}(y) \in X$.

Let $y \in Y$. f is surjective $\Rightarrow \exists x \in X: y = f(x)$. f is injective $\Rightarrow$ such an element $x \in X$ is unique. For this reason, we can define $x := f^{-1}(y)$.

$$f \circ f^{-1}(y) = f(x) = y \text{ and } f^{-1} \circ f(x) = f^{-1}(y) = x.$$

Roughly speaking: a function is invertible if you can change the role of the input with the one of the output.

Proposition: Let $f: X \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a strictly monotone function. Then $f: X \rightarrow f(X)$ is invertible.

[Strictly monotone means either strictly increasing or strictly decreasing]

Proof: $f: X \rightarrow f(X)$ is surjective (because we restricted the target to $f(X)$).

$f: X \rightarrow f(X)$ is injective. Assume by contradiction that it is not.

$$\Rightarrow \exists x_1 \neq x_2, x_1, x_2 \in X: f(x_1) = f(x_2).$$

$\Rightarrow$ Either $x_1 < x_2$, or $x_1 > x_2$. Assume, for example that $x_1 < x_2$ and f is increasing $\Rightarrow f(x_1) < f(x_2)$. This is a contradiction. $\square$

Example: $f(x) = e^x, f: \mathbb{R} \rightarrow \mathbb{R}$ is not invertible.

$f: \mathbb{R} \rightarrow (0, \infty)$ is invertible, with inverse $f^{-1}: (0, \infty) \rightarrow \mathbb{R}$

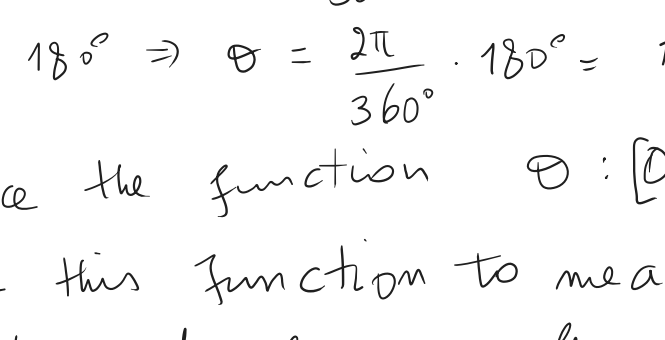
$$x \mapsto e^x$$

$$x \mapsto \log x$$

$$[e^{\log x} = x, \log e^x = x] \quad \forall x > 0 \quad \forall x \in \mathbb{R}$$

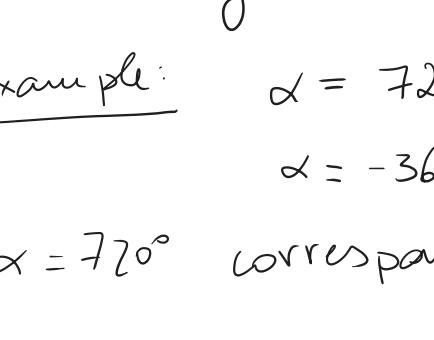
The graph of the inverse function:

The graph of the inverse function can be obtained by reflecting the original graph w.r. to the straight line $y=x$.



Trigonometric functions

$$S^1 := \{(x, y) \in \mathbb{R}^2: x^2 + y^2 = 1\}.$$



To any angle $\alpha \in (0, 360^\circ)$, we can associate an arc of length $\theta = \theta(\alpha)$ on S^1 .

How long is $\theta(\alpha)$? The length of the whole circle is 2π , and it corresponds to the angle $\alpha = 360^\circ$.

$$\frac{\alpha}{\theta} = \frac{360^\circ}{2\pi} \Rightarrow \theta = \frac{2\pi}{360^\circ} \alpha$$

$$\alpha = 90^\circ \Rightarrow \theta = \frac{2\pi}{360^\circ} \cdot 90^\circ = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$$

$$\alpha = 180^\circ \Rightarrow \theta = \frac{2\pi}{360^\circ} \cdot 180^\circ = \pi$$

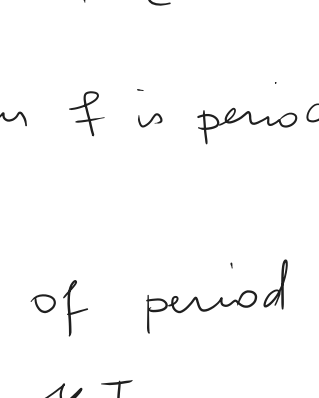
Since the function $\theta: [0, 360^\circ) \rightarrow [0, 2\pi)$ is invertible, we can use this function to measure angles.

In other words, if an angle measures α° , we can say that it measures θ radians.

α actually varies in the whole $\mathbb{R}$ (and θ too)

Example: $\alpha = 720^\circ$ identifies the point $(1, 0)$

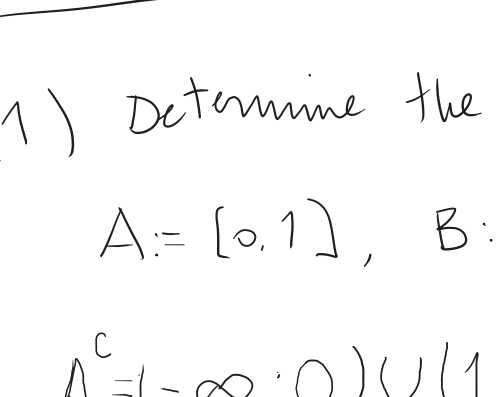
$\alpha = -360^\circ$ identifies the point $(1, 0)$



$\alpha = 720^\circ$ corresponds to $\theta = 4\pi \left[\frac{2\pi}{360^\circ} \cdot 720^\circ = 2\pi \cdot 2 \right]$

$\alpha = -360^\circ$ corresponds to $\theta = -2\pi$

Definition (sin and cosine)



$$\alpha \mapsto \theta(\alpha) \in \mathbb{R}$$

The half line associated to the angle α crosses S^1 exactly at 1 point P .

By definition, we call the coordinates of P $\cos \theta$ and $\sin \theta$ respectively. In other words

$$P = (\cos \theta, \sin \theta)$$

We have defined two functions $\theta \in \mathbb{R} \mapsto \cos \theta \in \mathbb{R}$ which are known

$$\theta \in \mathbb{R} \mapsto \sin \theta \in \mathbb{R}$$

as sine and cosine.

Properties: $\sin$ and $\cos$ are bounded, because $|\sin \theta| \leq 1, \forall \theta \in \mathbb{R}$

and $|\cos \theta| \leq 1, \forall \theta \in \mathbb{R}$.

$$\sin^2 \theta + \cos^2 \theta = 1, \forall \theta \in \mathbb{R}. \text{ (because the point } P = (\cos \theta, \sin \theta) \in S^1 \text{).}$$

Periodicity: Let $f: X \rightarrow \mathbb{R}$ be a function, $X \subseteq \mathbb{R}$. Then f is periodic of period $T > 0$ if $f(x+T) = f(x), \forall x \in \mathbb{R}$.

Remark: Let $f: X \rightarrow \mathbb{R}, X \subseteq \mathbb{R}$, be a periodic function of period $T > 0$.

Then $\forall k \in \mathbb{Z}, f$ is also periodic of period kT .

$$(k>0) \quad f(x+kT) = f(x+\underbrace{T+\dots+T}_{k \text{ times}}) = f(x_1+T) = f(x_1) \quad x_1 = x + \underbrace{T+\dots+T}_{k-1 \text{ times}}$$

Examples: $\sin$ and $\cos$ are periodic of period 2π .

Sheet 1

(1) Determine the complement of the following sets in $\mathbb{R}$.

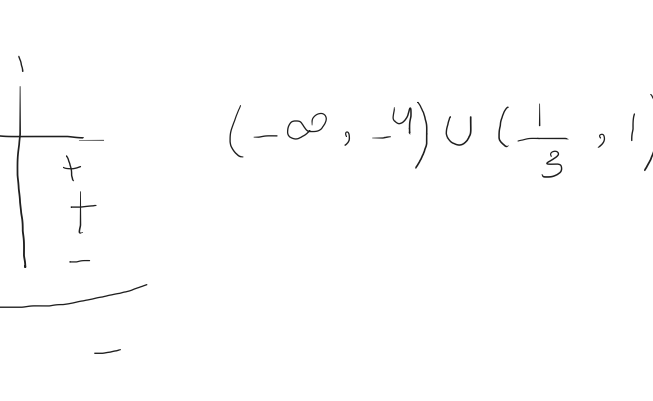
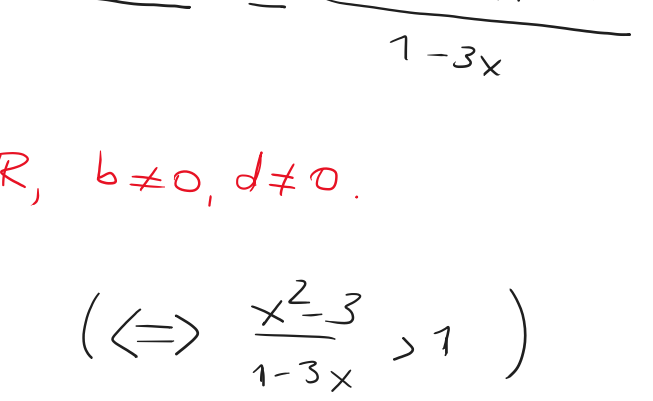
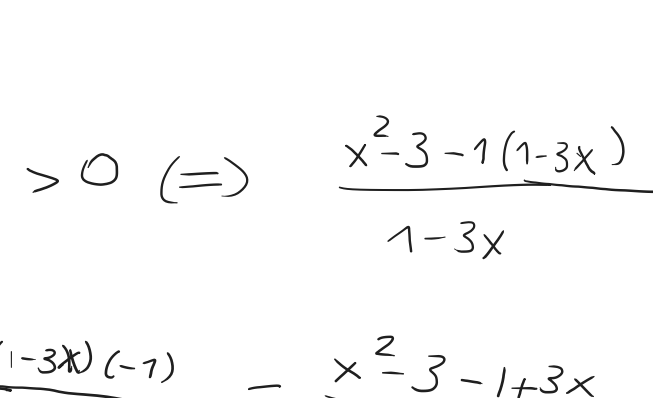
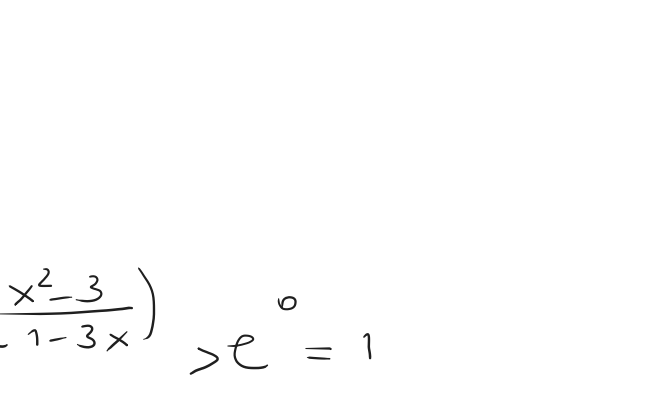
$$A := [0, 1], B := (-\infty, 0) \cup \{1\}, C = \{1\} \cup \{2\}, D = [1, \infty)$$

$$A^c = (-\infty, 0) \cup (1, \infty)$$

$$B^c = [0, 1) \cup (1, \infty)$$

$$C^c = (-\infty, 1) \cup (2, +\infty)$$

$$D^c = (-\infty, 1)$$



(2) Find the real solutions to the following equations:

$$2 - 6x = x - 3$$

$$2 + 3 = 7x$$

$$x = \frac{5}{7}$$

$$\frac{4x-1}{2} - 2(x-1) = x+1$$

$$4x-1-4(x-1) = 2x+2$$

$$4x-1-4x+4 = 2x+2$$

$$3 = 2x+2$$

$$1 = 2x \Rightarrow x = \frac{1}{2}$$

$$2x^2 + 10x + 6 = 0 \Rightarrow x^2 + 5x + 3 = 0 \Rightarrow x = \frac{-5 \pm \sqrt{25-12}}{2} = \frac{-5 \pm \sqrt{13}}{2}$$

$$|x-1| + 2x - 3 = 2$$

$$|x-1| = 2-2x+3$$

$$-x+1 = 5-2x \Rightarrow x = 4$$

$$+x-1 = 5-2x \Rightarrow 3x = 6 \Rightarrow x = 2$$

$$e^x, e \approx 2.7 \dots$$

$$e^x > 0 \quad \forall x \in \mathbb{R}$$

(#) $15x-4 + e^x = 0$: the equation has no solution.

$$|5x-4| - e^x = 0 \Leftrightarrow |5x-4| = e^x$$

$$f(x) = g(x)$$

$$f(x) = 0 \Leftrightarrow x = \frac{4}{5}$$

$$f(0) = 4 > g(0) = 1$$

(3)

$$\log \left(\frac{x^2-3}{1-3x} \right) > 0 \Leftrightarrow e^{\log \left(\frac{x^2-3}{1-3x} \right)} > e^0 = 1$$

$$\Leftrightarrow \frac{x^2-3}{1-3x} > 1 \Leftrightarrow \frac{x^2-3}{1-3x} - 1 > 0 \Leftrightarrow \frac{x^2-3-1(1-3x)}{1-3x} > 0$$

$$\frac{x^2-3}{1-3x} - 1 = \frac{x^2-3}{1-3x} - \frac{1}{1} = \frac{1(x^2-3) + (1-3x)(-1)}{1-3x} = \frac{x^2-3-1+3x}{1-3x} = \frac{x^2+3x-4}{1-3x}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd} \quad \forall a, b, c, d \in \mathbb{R}, b \neq 0, d \neq 0.$$

we need to solve $\frac{x^2+3x-4}{1-3x} > 0 \Leftrightarrow \frac{x^2-3}{1-3x} > 1$

$$\Leftrightarrow \frac{(x+4)(x-1)}{1-3x} > 0$$

$$(-\infty, -4) \cup (1, \frac{1}{3}) \cup (1, \infty)$$