

Limits of powers:

Thm: let $f, g: X \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be functions. let $x_0 \in \mathcal{A}(X)$. Assume that $\lim_{x \rightarrow x_0} f(x) = \alpha \in \overline{\mathbb{R}}$ and $\lim_{x \rightarrow x_0} g(x) = \beta \in \overline{\mathbb{R}}$. Assume that $(\alpha, \beta) \notin \{(0, 0), (1, \pm\infty), (\pm\infty, 0)\}$. Then $\lim_{x \rightarrow x_0} f(x)^{g(x)} = \alpha^\beta$.

Remark: $0^0, 1^{\pm\infty}, (\pm\infty)^0$ are indefinite forms.

Examples: $1^\infty, f(x) = 1+x, g(x) = \frac{1}{x}; \lim_{x \rightarrow 0^+} f(x) = 1, \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$

$$f(x)^{g(x)} = (1+x)^{\frac{1}{x}} = e^{\frac{\log(1+x)}{x}} = e^{\frac{1}{x} \log(1+x)} \rightarrow e \text{ as } x \rightarrow 0^+ \quad \log(a^b) = b \log a$$

we will see (when we treat derivatives) that $\lim_{x \rightarrow 0^+} \frac{\log(1+x)}{x} = 1 \quad \forall a > 0, \forall b \in \mathbb{R}$.

$$f(x) = 1+x, g(x) = \frac{1}{x^2}; f(x)^{g(x)} = (1+x)^{\frac{1}{x^2}} = e^{\frac{\log(1+x)}{x^2}} = e^{\frac{1}{x^2} \log(1+x)} = e^{\left(\frac{1}{x}\right) \left(\frac{\log(1+x)}{x}\right)} \xrightarrow{x \rightarrow 0^+} e^{1 \cdot 1} = e$$

$$g(x) = \frac{1}{\sqrt{x}}; f(x)^{g(x)} = (1+x)^{\frac{1}{\sqrt{x}}} = e^{\frac{\log(1+x)}{\sqrt{x}}} = e^{\frac{1}{\sqrt{x}} \log(1+x)} = e^{\left(\frac{1}{\sqrt{x}}\right) \left(\frac{\log(1+x)}{x}\right)} \xrightarrow{x \rightarrow 0^+} e^{1 \cdot 1} = e$$

$$0^0: f(x) = e^{-x}, g(x) = \frac{1}{\sqrt{x}}; f(x)^{g(x)} = (e^{-x})^{\frac{1}{\sqrt{x}}} = e^{-\frac{x}{\sqrt{x}}} = e^{-\sqrt{x}} \rightarrow e^0 = 1 \quad x \rightarrow \infty$$

$$[e^0 = e^{0^+} = e^0 = 1]$$

$$g(x) = \frac{1}{x}; f(x)^{g(x)} = (e^{-x})^{\frac{1}{x}} = e^{-1} = \frac{1}{e} \rightarrow \frac{1}{e} \quad x \rightarrow \infty$$

$$g(x) = -\frac{1}{\sqrt{x}}; f(x)^{g(x)} = (e^{-x})^{-\frac{1}{\sqrt{x}}} = e^{\frac{x}{\sqrt{x}}} = e^{\sqrt{x}} \rightarrow +\infty \quad x \rightarrow \infty$$

Limits of polynomials:

let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial. Then $\exists n \in \mathbb{N} \setminus \{0\}$ and coefficients $a_0, \dots, a_n \in \mathbb{R}, a_n \neq 0$

$$\text{s.t. } f(x) = a_0 + a_1 x + \dots + a_n x^n$$

We are interested in computing $\lim_{x \rightarrow \pm\infty} f(x)$.

Thm: if $a_n > 0 \Rightarrow \lim_{x \rightarrow \pm\infty} f(x) = \begin{cases} +\infty & \text{if } n \text{ is odd} \\ +\infty & \text{if } n \text{ is even} \end{cases}$

$$\text{if } a_n < 0 \Rightarrow \lim_{x \rightarrow \pm\infty} f(x) = \begin{cases} -\infty & \text{if } n \text{ is odd} \\ -\infty & \text{if } n \text{ is even} \end{cases}$$

proof: $a_n > 0$ $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = a_n x^n \left(1 + a_{n-1} \frac{x^{n-1}}{x^n} + \dots + \frac{a_0}{x^n}\right)$

$$= a_n x^n \left(1 + \frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n}\right) \xrightarrow{x \rightarrow \pm\infty} \begin{cases} \pm\infty & n \text{ is odd} \\ +\infty & n \text{ even} \end{cases}$$

Graphs:

let's start with polynomials:

$$f(x) = x^2$$

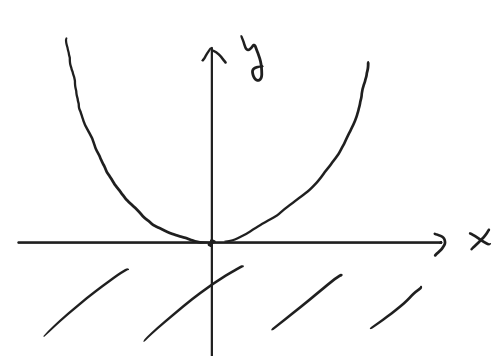
Step I Determine the domain of f . For which values of x are we allowed to compute $f(x)$?

If f is a polynomial $\Rightarrow D(f) = \mathbb{R}$ (the domain of a poly. is $\mathbb{R}$).

Step II Intersections with the x -axis (or the zeros of f). In other words, we need to solve equation $\boxed{f(x) = 0}$.

$$f(x) = x^2, \text{ which are the zeros of } f? \quad x = 0. \quad [x^2 = 0 \Leftrightarrow x = 0]$$

In other words $f^{-1}(\{0\}) = \{x \in D(f) : f(x) = 0\} = \{0\}$.



Step III: study the sign of f . In other words, study inequalities $f(x) > 0$ [and $f(x) < 0$]. (This is equivalent to find $f^{-1}((0, +\infty))$).

$$f(x) > 0 \Leftrightarrow x^2 > 0 \Leftrightarrow x \neq 0 \Leftrightarrow x \in \mathbb{R} \setminus \{0\}$$

Step VI: limit. What is $\mathcal{A}(D(f)) \setminus D(f)$?

Which are the accumulation points of the domain which do not belong to the domain? $+\infty$ and $-\infty$.

We need to compute $\lim_{x \rightarrow +\infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$.

$$\lim_{x \rightarrow +\infty} x^2 = +\infty, \lim_{x \rightarrow -\infty} x^2 = +\infty$$

Remark: $f(x) = x^2$ is even. So it is enough to study it in $[0, +\infty)$ and then reflect the graph w.r.to. the y -axis.

Example: $f(x) = x(1-x^2)$

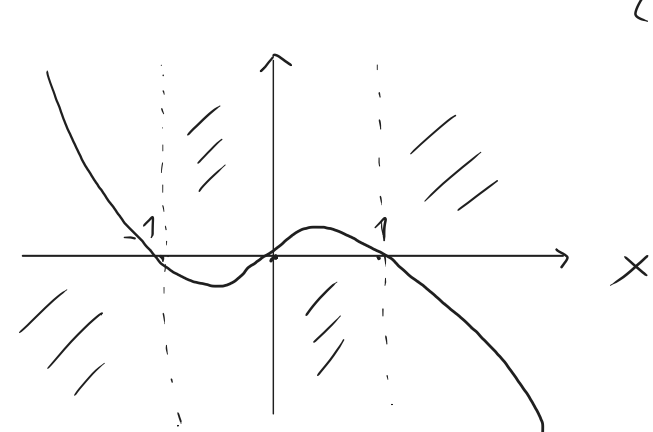
Step I the domain. $D(f) = \mathbb{R}$.

Step II the zeros of f . We need to solve $f(x) = 0 \Leftrightarrow x(1-x^2) = 0 \Leftrightarrow x = 0 \vee x = 1 \vee x = -1$

$$x(1-x^2) = 0 \Leftrightarrow x(1-x)(1+x) = 0 \quad [a^2 - b^2 = (a-b)(a+b), \forall a, b \in \mathbb{R}]$$

Step III: the sign of f . $f(x) > 0 \Leftrightarrow x(1-x^2) > 0 \Leftrightarrow x(1-x)(1+x) > 0$

$$\Leftrightarrow x \in (-\infty, -1) \cup (0, 1)$$



Step IV limits: $\lim_{x \rightarrow -\infty} f(x) = -\infty$

$$[f(x) = x \cdot x^3] \quad \lim_{x \rightarrow +\infty} f(x) = +\infty$$

Remark: f is odd.

	-1	0	1
$1-x$	+	+	-
$1+x$	-	+	+
x	-	-	+
	$\oplus$	$\ominus$	$\oplus$

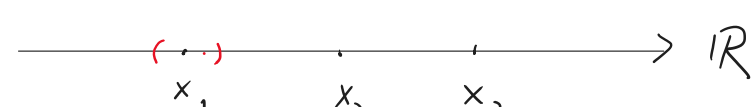
Rational functions

let $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be polynomials, g non-constant. let $h(x) = \frac{f(x)}{g(x)}$.

We are interested in the limits of h as $x \rightarrow x_0$, for $x_0 \in \mathcal{A}(D(h)) \setminus D(h)$.

let $Z(g) = \{x \in \mathbb{R} : g(x) = 0\}$. Then $D(h) = \{x \in \mathbb{R} : g(x) \neq 0\} = \mathbb{R} \setminus Z(g)$.

$$\Rightarrow \mathcal{A}(D(h)) = \overline{\mathbb{R}}, \quad \mathcal{A}(D(h)) \setminus D(h) = \{\pm\infty\} \cup Z(g)$$



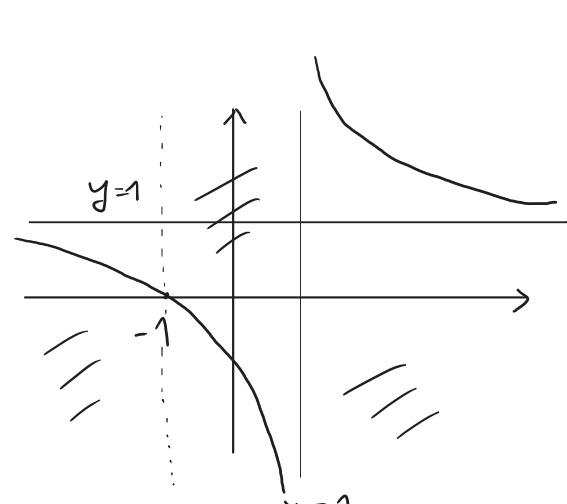
so we need to compute $\lim_{x \rightarrow \pm\infty} h(x)$, $\lim_{x \rightarrow x_0} h(x), \forall x_0 \in Z(g)$.

$$h(x) = \frac{x+1}{x-1}$$

$$\textcircled{i} D(h) = \mathbb{R} \setminus \{1\}$$

$$\textcircled{ii} \text{ The intersections with the } x\text{-axis. } h(x) = 0 \Leftrightarrow \frac{x+1}{x-1} = 0 \Leftrightarrow x \in D(h) \wedge x+1 = 0 \Leftrightarrow x = -1$$

$$\textcircled{iii} \text{ The sign of } h: h(x) > 0 \Leftrightarrow \frac{x+1}{x-1} > 0$$



	-1	1
$x+1$	-	+
$x-1$	-	-
	$\oplus$	$\oplus$

IV the limits:

$$\lim_{x \rightarrow +\infty} \frac{x+1}{x-1} = \lim_{x \rightarrow +\infty} \frac{x(1+\frac{1}{x})}{x(1-\frac{1}{x})} = 1$$

$$\lim_{x \rightarrow -\infty} \frac{x+1}{x-1} = \lim_{x \rightarrow -\infty} \frac{x(1+\frac{1}{x})}{x(1-\frac{1}{x})} = 1$$

$y=1$ is an asymptote (horizontal)

$$\lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = \frac{2}{x-1} = \frac{2}{0^-} = -\infty$$

$$\lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = +\infty$$

$x=1$ is a vertical asymptote.