

Exercises: sheet 3, ex 1, b)

$$\sqrt{x} - \sqrt{3x-3} = -\sqrt{x-3}$$

existence conditions: $\begin{cases} x \geq 0 \\ 3x-3 \geq 0 \\ x-3 \geq 0 \end{cases} \Leftrightarrow \begin{cases} x \geq 0 \\ x \geq 1 \\ x \geq 3 \end{cases} \Leftrightarrow x \geq 3$

$$\sqrt{x} + \sqrt{x-3} = \sqrt{3x-3} \Leftrightarrow x + x-3 + 2\sqrt{x(x-3)} = 3x-3 \Leftrightarrow 2\sqrt{x(x-3)} = 3x-2x \Leftrightarrow 2\sqrt{x(x-3)} = x$$

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2 \\ a &= \sqrt{x}, b = \sqrt{x-3} \\ 2ab &= 2\sqrt{x} \cdot \sqrt{x-3} = 2\sqrt{x(x-3)} \\ [\sqrt{c} \cdot \sqrt{d} &= \sqrt{cd}, \forall c, d \geq 0] \end{aligned} \quad \left| \quad \begin{aligned} 4x(x-3) &= x^2 \\ 4x^2 - 12x &= x^2 \\ 3x^2 - 12x &= 0 \\ 3x(x-4) &= 0 \\ x=0 \vee x=4 \\ \text{remember } x &\geq 3 \Rightarrow \boxed{x=4} \text{ is the unique solution.} \end{aligned} \right.$$

sheet 1, ex 3, f)

$$\sqrt{x^2-4x} \leq x+3 \Leftrightarrow \begin{cases} x^2-4x \leq (x+3)^2 \\ x^2-4x \geq 0 \end{cases} \Leftrightarrow \begin{cases} x^2-4x \leq x^2+9+6x \\ x^2-4x \geq 0 \end{cases} \Leftrightarrow \begin{cases} -10x \leq 9 \\ x^2-4x \geq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x \geq -9/10 \\ x(x-4) \geq 0 \end{cases} \begin{matrix} \text{I} \\ \text{II} \end{matrix} \Leftrightarrow \begin{cases} x \geq -9/10 \\ x \leq 0 \vee x \geq 4 \end{cases} \Leftrightarrow \boxed{-9/10 \leq x \leq 0 \vee x \geq 4}$$

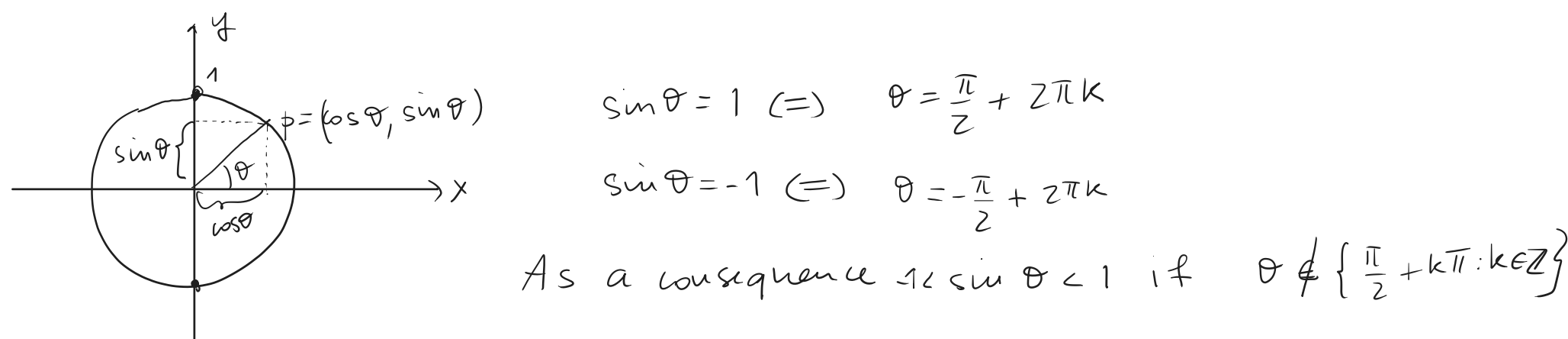
II $x(x-4) \geq 0$
 $x(x-4) = 0 \Leftrightarrow x=0 \vee x=4$

	0	4	
x	-	+	+
x-4	-	-	+
	+	-	+

Sheet 3, ex 2, f)

$$4\sin^2(5x)\cos^2(5x) < 1 \Leftrightarrow (2\sin(5x)\cos(5x))^2 < 1 \Leftrightarrow \sin^2(10x) < 1 \quad [\sin 2\theta = 2\sin\theta\cos\theta]$$

$$\Leftrightarrow -1 < \sin(10x) < 1 \quad [y^2 < 1 \Leftrightarrow -1 < y < 1] \Leftrightarrow 10x \notin \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\} \Leftrightarrow x \notin \left\{ \frac{\pi}{20} + \frac{k\pi}{10} : k \in \mathbb{Z} \right\}$$



$$4 + \tan^2(5x)\cos^2(5x) < 1 \Leftrightarrow \begin{cases} 4 \frac{\sin^2(5x)}{\cos^2(5x)} \cos^2(5x) < 1 \\ \cos(5x) \neq 0 \end{cases} \Leftrightarrow \begin{cases} \sin^2(5x) < 1/4 \\ \cos^2(5x) \neq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -\frac{1}{2} < \sin(5x) < \frac{1}{2} \\ \cos(5x) \neq 0 \end{cases} \begin{matrix} \text{I} \\ \text{II} \end{matrix} \Leftrightarrow \begin{cases} -\frac{\pi}{6} + 2\pi k < 5x < \frac{\pi}{6} + 2\pi k \vee \frac{5\pi}{6} + 2\pi k < 5x < \frac{7\pi}{6} + 2\pi k \\ 5x \neq \frac{\pi}{2} + k\pi \end{cases}$$

$$\begin{aligned} (\cos \theta = 0 \Leftrightarrow \theta = \frac{\pi}{2} + k\pi) \\ \Leftrightarrow \begin{cases} -\frac{\pi}{6} + k\pi < 5x < \frac{\pi}{6} + k\pi \\ 5x \neq \frac{\pi}{2} + k\pi \end{cases} \\ \Leftrightarrow \begin{cases} -\frac{\pi}{30} + \frac{k\pi}{5} < x < \frac{\pi}{30} + \frac{k\pi}{5} \\ x \neq \frac{\pi}{10} + \frac{k\pi}{5} \end{cases} \end{aligned}$$

$$x^2 - 5x + 6 < 0 \Leftrightarrow 2 < x < 3$$

$$x^2 - 5x + 6 = 0 \Leftrightarrow (x-3)(x-2) = 0 \Leftrightarrow x = 2 \vee x = 3$$

$$x_{\pm} = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2} = \frac{6}{2} / \frac{4}{2} = 3/2$$

$$ax^2 + bx + c = a(x-x_-)(x-x_+)$$

	2	3
x-2	-	+
x-3	-	+
	+	+

$$x-2 > 0 \Leftrightarrow x > 2$$

$$x-3 > 0 \Leftrightarrow x > 3$$

Limits from above and from below:

Def: let $f: X \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a function. let $y_0 \in \mathbb{R}$ and $x_0 \in \mathcal{A}(X)$.

We say that $\lim_{x \rightarrow x_0} f(x) = y_0^+$ if

right neighbourhood $J \subseteq \mathbb{R}$ of y_0 , $\exists$ a neighbourhood $I \subseteq \mathbb{R}$ of x_0 s.t.

$\forall x \in (X \cap I) \setminus \{x_0\}$, we have $f(x) \in J$.

We say that $\lim_{x \rightarrow x_0} f(x) = y_0^-$ if

left neighbourhood $J \subseteq \mathbb{R}$ of y_0 , $\exists$ a neighbourhood $I \subseteq \mathbb{R}$ of x_0 s.t.

$\forall x \in (I \cap X) \setminus \{x_0\}$, we have $f(x) \in J$.

Example: $x_0 = +\infty$, assume that $+\infty \in \mathcal{A}(X)$.

We say that $\lim_{x \rightarrow +\infty} f(x) = y_0^+$ if

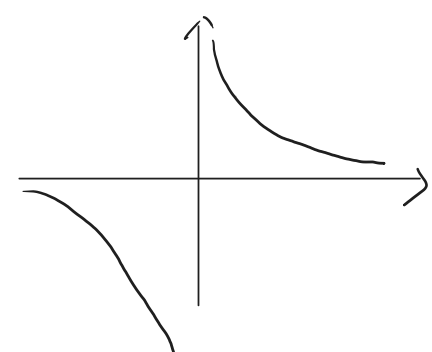
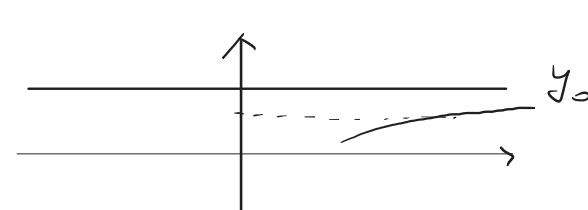
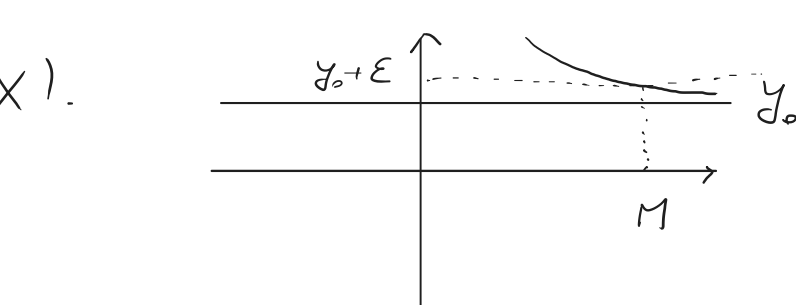
$\forall \varepsilon > 0 \exists M > 0 : \forall x \in X, x > M$, we have $y_0 < f(x) < y_0 + \varepsilon$.

and $\lim_{x \rightarrow +\infty} f(x) = y_0^-$ if

$\forall \varepsilon > 0 \exists M > 0 : \forall x \in X, x > M$, we have $y_0 - \varepsilon < f(x) < y_0$.

Examples: $f(x) = \frac{1}{x}$; $\lim_{x \rightarrow +\infty} \frac{1}{x} = 0^+$

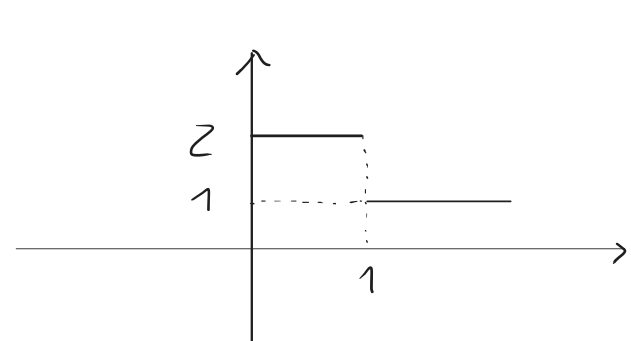
$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0^-$



if $y_0 \in \mathbb{R}, x_0 \in \mathbb{R}$

$\lim_{x \rightarrow x_0^+} f(x) = y_0^+$

$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in X, 0 < x - x_0 < \delta$, we have $y_0 < f(x) < y_0 + \varepsilon$.



$$f(x) = \begin{cases} 2 & x \in (0, 1) \\ 1 & x \in (1, \infty) \end{cases}$$

$\lim_{x \rightarrow 1^+} f(x) = 1$, $\lim_{x \rightarrow 1^-} f(x) = 2 \Rightarrow \nexists \lim_{x \rightarrow 1} f(x)$.

If $\lim_{x \rightarrow x_0} f(x) = y_0^+ \Rightarrow \lim_{x \rightarrow x_0} f(x) = y_0$.

Operations with limits:

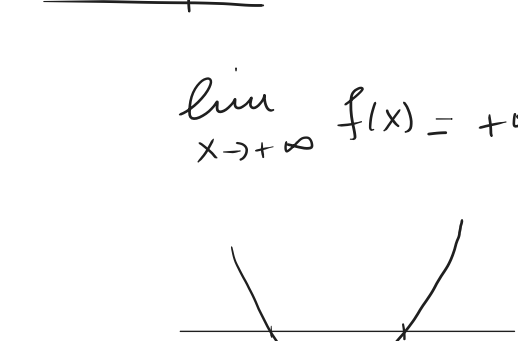
Th: let $f, g: X \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be functions. let $x_0 \in \mathcal{A}(X)$. Assume that

$\exists \lim_{x \rightarrow x_0} f(x) = y_0$ and $\exists \lim_{x \rightarrow x_0} g(x) = z_0$. If $(y_0, z_0) \notin \{(+\infty, -\infty), (-\infty, +\infty)\}$

$\Rightarrow \lim_{x \rightarrow x_0} (f(x) + g(x)) = y_0 + z_0$.

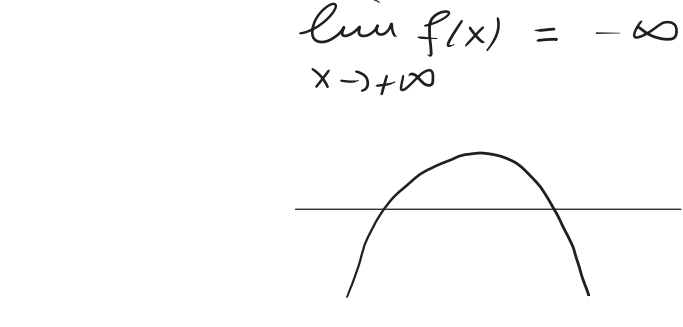
Example: $f(x) = x^2$, $g(x) = -2x$

$\lim_{x \rightarrow +\infty} f(x) = +\infty$, $\lim_{x \rightarrow +\infty} g(x) = -\infty$; $\lim_{x \rightarrow +\infty} (f(x) + g(x)) = \lim_{x \rightarrow +\infty} (x^2 - 2x) = +\infty$



$f(x) = -x^2$, $g(x) = 2x$

$\lim_{x \rightarrow +\infty} f(x) = -\infty$, $\lim_{x \rightarrow +\infty} g(x) = +\infty$, $\lim_{x \rightarrow +\infty} (f(x) + g(x)) = -\infty$



We say that $+\infty - \infty$ is an indefinite form.