

SOLUTIONS TO THE EXERCISES, SHEETS 1 – 5

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1. SHEET 1

(1) Exercise 1:

$$\mathbb{R} \setminus A = (-\infty, -1) \cup (1, \infty), \mathbb{R} \setminus B = [0, 1) \cup (1, \infty),$$

$$\mathbb{R} \setminus C = (-\infty, 1) \cup (1, 2) \cup (2, \infty), \mathbb{R} \setminus D = (-\infty, 1).$$

(2) Exercise 2:

(a) $x = \frac{5}{7}$

(b) $x = \frac{1}{2}$

(c) $x = -\frac{3}{2} \vee x = 1$

(d) $x = \frac{4}{3}$

(e) There is no solution

(3) Exercise 3:

(a) $-2 < x < 1$

(b) $x \neq \frac{1}{2}$

(c) $x < -4 \vee \frac{1}{3} < x < 1$

(d) The inequality is satisfied $\forall x \in \mathbb{R}$.

(e) $-2 \leq x \leq 1 \vee 1 \leq x \leq 2$

(f) $-\frac{9}{10} \leq x \leq 0 \vee x \geq 4$

(4) Exercise 4:

(a) f is both injective and surjective and $f^{-1}(y) = \sqrt[3]{y}$.

(b) f is both injective and surjective and $f^{-1}(y) = e^y$.

(c) f is neither injective nor surjective, since $f(0) = f(2) = 3$ and equation $f(x) = 0$ has no solution.

(d) f is injective since

$$\begin{cases} t^2 = s^2 \\ t^3 = s^3 \end{cases} \implies t = s$$

but not surjective since $(0, 1) \notin f(\mathbb{R})$.

2. SHEET 2

- (1) Exercise 1:
- (a) $A \cap B = \{3\}$
 - (b) $A \cap B = [-3, -1) \cup [0, 2)$
 - (c) $A \cap B = [-3, -1) \cup [0, 2)$
 - (d) $A \cap B = \emptyset$
- (2) Exercise 2:
- (a) $x = \frac{4}{3}$
 - (b) $x = \log_2(3 + \sqrt{10})$
 - (c) $x = \frac{1}{2} \vee x = -\frac{1}{2}$
 - (d) $x = -2 \vee x = 1$
- (3) Exercise 3:
- (a) $0 \leq x \leq 2$
 - (b) $x = -1 \vee 1 - \sqrt{2} \leq x \leq 1 + \sqrt{2}$
 - (c) $x < -\sqrt{3} \vee \sqrt{3} < x < 2 \vee x > 8$
 - (d) $-\sqrt{2 \log 2} \leq x \leq \sqrt{2 \log 2}$
 - (e) $x < -\frac{1}{\sqrt{2}} \vee \frac{1}{\sqrt{2}} < x \leq 3$
 - (f) $-\frac{5}{3} \leq x < 1 \vee 2 < x \leq 3$
 - (g) $\frac{1}{27} \leq x \leq 3$
- (4) Exercise 4:
- (a) $\log 2 < x \leq \sqrt{5} - 1$
 - (b) $-1 < x < 0 \vee 0 < x \leq \log 5$
- (5) Exercise 4:
- (a) f is both injective and surjective and $f^{-1}(y) = \frac{\sqrt[3]{y}-1}{2}$.
 - (b) f is injective but not surjective since, for any $y \geq 0$, equation $\sqrt[4]{x} = y$ has exactly 1 solution $x = y^4 \in [0, \infty)$, while, for $y < 0$, equation $\sqrt[4]{x} = y$ has no solution. Note that, for $y > 0$, equation $\sqrt[4]{x} = y$ has exactly 2 solutions $x = \pm y^4 \in \mathbb{R}$, but we are interested in the positive one only, since we consider the function on the domain $[0, \infty)$.

3. SHEET 3

- (1) Exercise 1:
- (a) $x = 0 \vee x = -2 \vee x = 4 \vee x = 6$
 - (b) $x = 4$
 - (c) $x = 2 \vee x = -\frac{3}{2}$
 - (d) $x \in (\cup_{k \in \mathbb{Z}} \{\frac{\pi}{3} + k\pi\}) \cup (\cup_{k \in \mathbb{Z}} \{-\frac{\pi}{3} + k\pi\}) \cup (\cup_{k \in \mathbb{Z}} \{\frac{\pi}{4} + 2k\pi\}) \cup (\cup_{k \in \mathbb{Z}} \{\frac{3\pi}{4} + 2k\pi\})$
- (2) Exercise 2:

- (a) $x = -2 \vee x \geq -1$
 - (b) $x \leq -1 \vee x \geq \frac{3}{4}$.
 - (c) $0 < x \leq 1$
 - (d) $x \in \cup_{k \in \mathbb{Z}} (\frac{5\pi}{4} + 2\pi k, \frac{7\pi}{4} + 2\pi k)$
 - (e) $x \in \cup_{k \in \mathbb{Z}} (\pi(1 + 2k), 2\pi(1 + k))$
 - (f) $x \in \mathbb{R} \setminus (\cup_{k \in \mathbb{Z}} \{\frac{\pi}{20} + \frac{\pi k}{10}\})$
- (3) Exercise 3: $D(f) = \mathbb{R} \setminus (\cup_{k \in \mathbb{Z}} \{\frac{\pi}{2} + k\pi\})$. Set set of zeros of f is

$$(\cup_{k \in \mathbb{Z}} \{k\pi\}) \cup (\cup_{k \in \mathbb{Z}} \{\frac{\pi}{4} + k\pi\}).$$

- (4) Exercise 4: $D(f) = \mathbb{R}$, the set of zeros of f is $\{-\frac{1}{2}, \frac{3}{2}\}$, $f^{-1}([0, \infty)) = (-\infty, -\frac{1}{2}] \cup [\frac{3}{2}, \infty)$. f is not injective since $f(-\frac{1}{2}) = f(\frac{3}{2}) = 0$.

$D(g) = \mathbb{R}$, the set of zeros of g is given by $\cup_{k \in \mathbb{N} \setminus \{0\}} \{\pm \sqrt{k\pi - 1}\}$ and g is not surjective, since $-1 \leq \sin(x^2 + 1) \leq 1$, for any $x \in \mathbb{R}$.

4. SHEET 4

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- (1) Exercise 1:
 - (a) $\mathbb{R} \setminus A = [0, \infty)$
 - (b) $(-2, -1) \cup \{4\}$.
- (2) Exercise 2:
 - (a) There exists no solution
 - (b) $x = -1 \vee x = 0 \vee x = 5$
 - (c) $0 \leq x \leq \log_3 4$
 - (d) $x = \frac{1}{4}$
 - (e) $x \in \cup_{k \in \mathbb{Z}} (\frac{\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k)$
- (3) Exercise 3:
 - (a) $D(f) = \mathbb{R} \setminus \{1\}$
 - (b) The zeros of f are $x = -1$ and $x = 0$.
 - (c) f is not injective since $f(0) = f(-1) = 0$.
 - (d) $f^{-1}((-\infty, 1)) = D(f) = \mathbb{R} \setminus \{1\}$.
- (4) Exercise 4:
 - (a) $D(f) = (-1, \infty)$
 - (b) $f : D(f) \rightarrow \mathbb{R}$ is invertible and $f^{-1}(y) = e^{2y} - 1$.

5. SHEET 5

- (1) Exercise 1:
 - (a) $\mathbb{R} \setminus A = [0, 1]$

- (b) $\emptyset$.
- (2) Exercise 2:
 - (a) $-4 < x < -2$
 - (b) $x < \sqrt[3]{-2} \vee x > -1$
 - (c) $0 < x < \frac{1}{9} \vee x > \frac{1}{3}$
 - (d) $x \in \{\frac{\pi k}{4} : k \in \mathbb{Z}\}$
 - (e) $x \in \cup_{k \in \mathbb{Z}}(\frac{\pi}{2} + k\pi, \pi(1 + k))$
- (3) Exercise 3:
 - (a) $D(f) = (-\infty, -1) \cup (1, \infty)$
 - (b) f never vanishes.
 - (c) f is even.
 - (d) f is not injective because it is even.
 - (e) $f^{-1}((1, \infty)) = (-\sqrt{2}, -1) \cup (1, \sqrt{2})$.
- (4) Exercise 4:
 - (a) $D(f) = (0, \infty)$
 - (b) $f(D(f)) = \mathbb{R}$, $f : D(f) \rightarrow \mathbb{R}$ is invertible and $f^{-1}(y) = e^{\sqrt[3]{y+1}}$.

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