THE GRASSMANN ALGEBRAS: DERIVATIONS AND IDENTITIES

VINCENZO C. NARDOZZA

Abstract. We survey some classical results on the polynomial identities satisfied by Grassmann algebras, with a list of recent new results on their derivations. Other new results concern the differential polynomial identities of the Grassmann algebra E of an infinite-dimensional vector space over a field of characteristic zero, arising under the derivation action of some Lie subalgebras of $Der(E)$.

1. Introduction

Grassmann algebras constitute a very important class of algebras, not only in Algebra: in Physics they provide a basic model for the notion of superalgebra, playing a fundamental role in the Bose-Fermi supersymmetry; they occur also in the so-called SuperAnalysis (or SuperMathematics) invented during the 70's by Berezin, a nontrivial analogue of Analysis in which anti-commuting variables appear on the same footing as the usual ones, providing a Mathematical setup associated with the ideas of supersymmetry [Be87].

In PI-Theory, the Grassmann algebras play a central role especially in characteristic zero, after Kemer's results on this subject; moreover the description of the polynomial identities of the Grassmann algebra associated to an infinite dimensional vector space over a field of characteristic zero was, historically, the first non trivial achievement among the (very few) known descriptions of the polynomial identities of concrete algebras.

Being so central, other kinds of identities have been investigated, through the years: the ones satisfied by finite dimensional ones, or those arising in case of positive characteristic, even in the extreme case of finite fields; also polynomial identities arising from superimposed structures, such as graded identities, or $*$-identities have been considered. Actually, all these variations on the theme of polynomial identities are embraced in a more general notion: identities taking into account a generalized Hopf algebra action on the considered algebra.

In the last few years, a further member of this family entered the game and received considerable attention: the so-called differential identities. Although introduced in the late 70's by Kharchenko, several papers investigating differential identities of concrete, relevant algebras appeared recently (in chronological order, [GR19], [Ri20], [DVN22], [MR23], [Na23], [Ri23], [RSV23], [Ar24]). The paper [Ri20] dealt with the Grassmann algebra under the action of a finite set of inner derivations. However, Grassmann algebras differ from the other studied algebras

2020 *Mathematics Subject Classification.* Primary 15A75, 17B60 Secondary 16E45, 16W25.
Key words and phrases. Grassmann algebra, derivation, differential polynomial identities.
Dedicated to Vesselin Drensky, for his 75-th birthday.

in two significant features: they may be infinite-dimensional, and possess non-inner derivations. Actually, very few is known about ordinary derivations of the Grassmann algebras, the only results dating back to [Dj78].

In the present paper, the focus will be mainly on the Grassmann algebra E of a vector space of denumerable dimension over a field of characteristic zero (in PI-Theory, one often refers to E as to THE Grassmann algebra), but sometimes more general assumptions will occur. We will survey some new results on the derivations of Grassmann algebras, and on the differential identities arising when involving certain non-inner derivations.

The sectioning of the paper is the following: after a quick recall on the Grassmann algebras and their general properties, the classical results about ordinary identities of Grassmann algebras are presented from the point of view of the proper ones: this makes it possible to provide really easy proofs of important known results, and this section may be defined of "didactical" nature; many of these proofs have been already provided in the classical book of Drensky [DrB], although in a maybe different form. Section 4 deals with the description of the derivations of the Grassmann algebras: starting from the one provided by Djoković in [Dj78], some new results are announced. Section 5 provides a list of new results concerning the differential polynomial identities of E under the action of some inner derivations on their own (in particular, rediscovering the results of [Ri20]) and together with some outer ones, generating a nonabelian solvable Lie subalgebra of $Der(E)$. Generators, cocharacters, codimensions and exponents according to the selected set of derivation are announced. These results are based on a joint work with Di Vincenzo [DVN25].

I was tempted, but decided of not including in this paper on the Grassmann algebras all the results concerning other types of identities (for instance, graded identities), because the paper would become pointlessly excessively long: there are at least two amenable surveys I would recommend about these topics, [Ce16], [dS12]. Instead, I felt compelled to include a short tribute to Hermann Grassmann, whose innovative and groundbreaking ideas were recognized only decades after he passed away.

2. The Grassmann algebras

Throughout the paper, let F denote a field; all vector spaces, algebras and tensor products are meant over F, unless explicitly indicated. If V is any vector space, its Grassmann (or Exterior) algebra is the factor algebra $T(V)/\mathscr{A}(V)$, where $T(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$ is the tensor (unital) algebra of V, and $\mathscr{A}(V)$ is the twosided ideal of $T(V)$ generated by the simple tensors $v \otimes v$ for $v \in V$. This is in fact a special instance of a more general construction, available for any module M of a commutative ring C, producing a wide range of algebras.

Example 2.1. Let C_n be the cyclic group of order n, seen as a $\mathbb{Z}$-module. Then its Grassmann algebra is isomorphic to the ring $\mathbb{Z}[x]/(nx)$.

We will denote $E(V)$ the Grassmann algebra of a vector space V, and simply by E the very important case when $\dim V = \aleph_0$. This definition of $E(V)$ is purely abstract, but in order of making computations one needs to have a basis, or at least a set of generators of $E(V)$; in fact, if $\mathscr{E}$ is any totally ordered basis of V, the set $\mathscr{B}$ of all cosets $e_1 \dots e_n := e_1 \otimes \cdots \otimes e_n + \mathscr{A}(V)$, for $n \in \mathbb{N}$ and $e_1, \dots, e_n \in \mathscr{E}$

such that $e_1 < e_2 < \cdots < e_n$, is a basis of $E(V)$. It follows that if $\dim V = n$ then $\dim E(V) = 2^n$, since its basis elements are in a bijective correspondence with the power set of $\mathscr{E}$. Moreover, if $char.F \neq 2$, then $E(V)$ can be presented as the (associative, unital) algebra generated by $\mathscr{E}$ with relations $e_1e_2 + e_2e_1 = 0$ for all $e_1, e_2 \in \mathscr{E}$.

Since both $T(V)$ and the ideal $\mathscr{A}(V)$ are $\mathbb{N}$-graded, $E(V)$ inherits a natural $\mathbb{N}$-grading; by the way, a natural and far more important $\mathbb{Z}_2$-grading is inherited, as well, by collecting together the $\mathbb{N}$-components of even degrees, and those of odd degrees. In terms of $\mathscr{B}$, let $\mathscr{B}_0$ be set of all elements of $\mathscr{B}$ of even length, and $\mathscr{B}_1$ be the set of the remaining ones; then the even component $E_0(V)$ of $E(V)$ is the span of $\mathscr{B}_0$, and $E_1(V)$ the span of elements of $\mathscr{B}_1$. This turns $E(V)$ into a superalgebra.

Although very close to being commutative (and actually so it is, when $char.F = 2$), $E(V)$ distinguishes between left and right ideals, and one has to distinguish between left and right divisibility among its elements. By the way, for homogenous elements, that is for elements $a \in E_0(V) \cup E_1(V)$, one has

Lemma 2.2. *If $a, b \in E(V)$ and a is $\mathbb{Z}_2$-homogeneous, then a left-divides b if and only if a right-divides b.*

So, if a is $\mathbb{Z}_2$-homogeneous, then it makes sense to write $a \mid b$: it means, equivalently, that there exist $q, \overline{q} \in E$ such that $aq = b = \overline{q}a$. A useful and easy relation between divisibility and annihilation properties is the following:

Lemma 2.3. *Let $e \in \mathscr{E}$ and let $a \in E(V)$. Then $e \mid a$ if and only if $ea = 0$.*

The proof is extremely simple, but one has to notice that this does not hold when a homogeneous element u replaces a generator e: for instance, $e_1e_2(e_1 + e_2) = 0$, but $e_1e_2 \nmid (e_1 + e_2)$.

In fact, in the original idea of Grassmann, the construction of $E(V)$ included a (well realized) tentative to translate into algebraic terms the geometric idea of linear dependence: $ab = 0$ in $E(V)$ is, in some sense, the algebraic form of linear dependence between geometric entities (see [Br12] for a clearer picture).

Also, $E(V)$ is in some sense resiliently associative: in general, if A is an associative algebra, one can turn it into a Lie algebra, usually denoted $A^{(-)}$, or into a Jordan algebra $A^{(+)}$, simply replacing the associative multiplication by the bracket product $[a, b] := ab - ba$ or by the (normalized) symmetric product $a \circ b := \frac{1}{2}(ab + ba)$. When applied to $E(V)$, it turns out that both $E(V)^{(-)}$ and $E(V)^{(+)}$ are still non-trivial associative structures, although at the same time Lie and Jordan respectively, as it has to be.

Structurally, $E(V)$ is a local superalgebra, whose center is exactly $E_0(V)$ apart when $\dim V = n \equiv 1 \pmod 2$, in which case the vector subspace $Fe_1 \dots e_n$ sums up to $E_0(V)$ to constitute the center. The $\mathbb{Z}_2$-grading on $E(V)$ induces a superalgebra structure on $End_F(E(V))$, with

$$End_0(E(V)) := \{a \in End_F(E(V)) \mid E_i(V)^a \subseteq E_i(V),\ i \in \mathbb{Z}_2\}$$
$$End_1(E(V)) := \{a \in End_F(E(V)) \mid E_i(V)^a \subseteq E_{i+1}(V),\ i \in \mathbb{Z}_2\};$$

notice we are adopting the exponential notation for the action of linear operators of $E(V)$, so that their composition $f \circ g$ means *first f, then g*.

3. The classical identities

The Grassmann algebra E plays a key role in PI-Theory, especially in characteristic zero, as outlined by Kemer's results [Ke91]; this is not only because E is involved in the classification of the T-prime T-ideals, but also because any PI-algebra is PI-equivalent to the Grassmann envelope of a suitable finite dimensional superalgebra ([Ke91], Theorem 2.3); thus E provides a sort of bridge between finite dimensional (super)algebras and infinite dimensional ones, not only in the ordinary case, but in the case of $*$-algebras as well [AGK17], [DVN24]. Even in characteristic $p > 2$, E turned out to be crucial too (see for instance [Ke93, Ke93i, Ke95, Ke95i]). Recently, a generalization of the Grassmann algebras which is well behaved, from PI-point of view, over any commutative ring (even in characteristic 2) has been provided in [DKBV20].

Here we recall briefly that an element $f(x_1, \ldots, x_n)$ of the free (associative, unital) algebra $F\langle X\rangle$ generated by a countable set X of indeterminates is a polynomial identity for an algebra A if $f(a_1, \ldots, a_n) = 0$ for all possible substitutions of the involved indeterminates by elements $a_1, \ldots, a_n \in A$. The set of such polynomials constitutes a two-sided ideal $Id(A)$ of $F\langle X\rangle$ which is invariant under all algebra endomorphisms (a Totally invariant ideal, or a T-ideal, for short). A special class of polynomial identities is constituted by the multilinear ones, that is by those polynomials of degree n spanned by monomials $x_{\sigma(1)} \ldots x_{\sigma(n)}$, for some $n \in \mathbb{N}$ and $\sigma \in S_n$, the symmetric group of order n. More precisely, if P_n denotes the $n!$-dimensional vector space of multilinear polynomials of degree n, the "slices" $P_n \cap Id(A)$ for n in $\mathbb{N}$ completely determine $Id(A)$ in case $char.F = 0$. Moreover, P_n is an isomorphic realization of the regular S_n-module, and since $P_n \cap Id(A)$ is an S_n-submodule, one can get also a description of the structure of the multilinear identities satisfied by A by the representation theory of S_n. In practice, however, it is far more convenient to study the smaller factor modules $P_n(A) := P_n/(P_n \cap Id(A))$, that is the multilinear non-identities of A, by complete reducibility; actually, these spaces can be considered for any T-ideal I instead of $Id(A)$ – in this case, one writes $P_n(I)$, with a slight abuse of notation.

In the beginning of the 70's Regev [Re72] had an illuminating insight, and considered as a valuable statistics the codimension sequence $c_n(A) := \dim P_n(A)$ (for all $n \in \mathbb{N}$). He was able to prove that A is PI (that is: $Id(A) \neq 0$) if and only if the codimension sequence $c_n(A)$ is exponentially bounded, and deduced that the tensor product of two PI-algebras is again PI. The properties of the codimension sequence were successively refined by Kemer, who proved that it has to grow either polynomially or exponentially [Ke78], and culminated in the papers of Giambruno and Zaicev [GiZa98, GiZa99] who confirmed a conjecture raised by Amitsur, stating that the limit of the sequence $\sqrt[n]{c_n(A)}$ does exist, and it is an integer, $\exp(A)$, called the PI-exponent of A. A thorough treatment of the whole story, together with many other results of combinatorial nature and in PI-Theory in general, is available in the book [GZb].

Getting back to E, soon after the introduction of codimension theory in the study of polynomial identities, Krakowski and Regev were able to prove that all elements of $Id(E)$ follow from a single identity, that is $Id(E)$ is the least T-ideal of $F\langle X\rangle$ containing that single identity [KrRe73]. The generating identity is often called the Grassmann identity, or more directly the triple commutator, since it is the polynomial $[x, y, z] := [x, y]z - z[x, y]$. The original proof employed the newborn codimension theory, and in an oversimplified picture the basic idea of the proof was the following: let I be the T-ideal generated by $[x, y, z]$; of course, $I \subseteq Id(E)$, and comparing the codimensions they proved that $2^{n-1} \leqslant c_n(E) \leqslant c_n(I) \leqslant 2^{n-1}$ for all $n \in \mathbb{N}$; in characteristic zero, this implies that $I = Id(E)$.

A bit later, Olsson and Regev [OlRe76] also provided the S_n-structure $\chi_n(E)$ of $P_n(E)$, that is the so-called cocharacter sequence of E: $\chi_n(E)$ is the sum of irreducible characters of S_n corresponding to partitions $[\![n-j, 1^j]\!]$, that is hooks of amplitude 1, for all $0 \leqslant j < n$, each with multiplicity 1.

It has to be mentioned that these were the first detailed results on the identities of a concrete, non trivial algebra, in the sense that so far just the cases of commutative or nilpotent algebras were known. On the other hand, although ingenious and innovative, the proofs of these results were rather complicated. In fact, several years later, Di Vincenzo [DV91] provided new proofs of them, together with the generators of $Id(E(V))$ and the cocharacter sequences when V is a finite dimensional vector space.

Actually, a really short and direct proof of them is available, and relies on a shift of perspective, mainly due to Drensky: indeed, while $char.F = 0$ allows the reduction to multilinear polynomials, the fact that E is a unital algebra allows a further reduction to proper multilinear polynomials. The notion of proper polynomials can be traced back to papers of Malcev [Ma50] and Specht [Sp50] in 1950, but their employment in PI-theory occurred much later, in a paper of Volichenko [Vo78]; starting from Volichenko's idea, Drensky turned proper polynomials into a powerful tool, using them extensively with outstanding results. In modern days, proper polynomials and their generalizations fitting several situations are among the must-know notions for PI-theorists. Technically, proper polynomials in $F\langle X\rangle$ are (a subalgebra isomorphic to) the universal enveloping algebra of the derived (or commutator) ideal of the free Lie algebra over X; in practical terms, they constitute the unital subalgebra of $F\langle X\rangle$ generated by the higher commutators $[x_1, \ldots, x_n]$, for $n \geqslant 2$ and $x_1, \ldots, x_n \in X$; so a proper polynomial is a linear combination of products of higher commutators, in a sort of melding of associative and Lie products of the generators of $F\langle X\rangle$. Of course, proper polynomials of degree n which are in P_n are called proper multilinear polynomials, and constitute an S_n-submodule of P_n, usually denoted Γ_n. An excellent reference about proper polynomials is Drensky's book [DrB], starting from the basic notions introduced in section 4.3.

The point in using proper multilinear polynomials is in that the slices $\Gamma_n \cap Id(A)$, for $n \in \mathbb{N}$, uniquely determine the whole $Id(A)$, just as for $P_n \cap Id(A)$; moreover, the factor modules $\Gamma_n(A) = \Gamma_n/(\Gamma_n \cap Id(A))$ are easier to study and Drensky [Dr84] proved they are Young-related to the modules $P_n(A)$ by an easy formula. Precisely, if $\xi_n(A)$ denotes the S_n-character of $\Gamma_n(A)$, then $\chi_n(A)$ is obtained by

$$\chi_n(A) = \sum_{k=0}^{n} [\![n-k]\!] \hat{\otimes} \xi_k(A),$$

where $[\![n-k]\!]$ is the trivial S_{n-k}-character and the summand $[\![n-k]\!] \hat{\otimes} \xi_k(A)$ is the induced S_n-character (Young-Pieri rule). The codimension sequence $c_n(A)$ and the proper codimension sequence $\gamma_n(A) := \dim \Gamma_n(A)$ are also related by the formula

$$c_n(A) = \sum_{k=0}^{n} \binom{n}{k} \xi_k(A).$$

These relations may result in describing the classical sequences $(\chi_n(A))_n$ and $(c_n(A))_n$ more easily, but sometimes also in accepting $(\xi_n(A))_n$ and $(\gamma_n(A))_n$ as the only reasonable description of the multilinear polynomial identities of A because the classical ones may be too complicated.

The effectiveness of proper polynomials is most evident in studying the classical identities of E: this was already present in Drensky's book (see [DrB], Sec. 5.1), here I propose it (in a slight different form) for convenience of the reader. So, let I be the T-ideal generated by $[x_1, x_2, x_3]$, hence $I \subseteq Id(E)$. A useful ancillary identity, that is an element of I (also called a consequence of $[x_1, x_2, x_3]$) is

Lemma 3.1. $[x_1, x_3][x_2, x_4] + [x_1, x_2][x_3, x_4] \equiv 0 \pmod{I}$.

Proof. The polynomial $[x_1x_4, x_2, x_3]$ belongs to I, and can be rewritten as

$$[x_1x_4, x_2, x_3] = [x_1, x_2, x_3]x_4 + [x_1, x_2][x_3, x_4] + [x_1, x_3][x_2, x_4] + x_1[x_2, x_3, x_4].$$

Then just notice that the first and the last summands are clearly in I. $\square$

Theorem 3.2. *$Id(E) = I$; in particular, $\Gamma_n(E) = 0$ if n is odd, and $\Gamma_n(E)$ is spanned by $[x_1, x_2] \dots [x_{n-1}, x_n]$ if n is even.*

Proof. If $f \in \Gamma_n$ then it is a linear combination of products of higher commutators; by the way all higher commutators of length $\geqslant 3$ are in I, so working modulo I the polynomial f is either 0 or a product of ordinary (of length 2) commutators. Then if n is odd only the former case is possible, and $\Gamma_n(I) = 0$. If instead n is even, then f is a linear combination of products of type $[x_{i_1}, x_{i_2}] \dots [x_{i_{n-1}}, x_{i_n}]$, with $\{i_1, \dots, i_n\} = \{1, \dots, n\}$. However, each such product can be straightened to $[x_1, x_2] \dots [x_{n-1}, x_n]$, modulo I and up to a sign, since $[x_j, x_i] = -[x_i, x_j]$ and $[x_u, x_j][x_i, x_v] \equiv -[x_u, x_i][x_j, x_v] \pmod{I}$ by the ancillary identity of Lemma 3.1.

So $\Gamma_n(I)$ is either 0 or it is spanned by $[x_1, x_2] \dots [x_{n-1}, x_n]$, and since $I \subseteq Id(E)$, the same holds for $\Gamma_n(E)$. Finally, in case $n \equiv 0 \pmod{2}$, consider the specialization $x_i \to e_i \in \mathscr{E}$ for all $i \in \mathbb{N}$: these polynomials evaluate to $2^n e_1 \dots e_n \neq 0$, so they are not in $Id(E)$, proving that $\Gamma_n(I) = \Gamma_n(E)$ for all even n (and for all odd n, too, of course).

This means that $Id(E)$ and I contain the same proper multilinear polynomials, so they coincide. $\square$

Corollary 3.3. *For all $n \in \mathbb{N}$, it holds*

$$\gamma_n(E) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod{2} \\ 1 & \text{if } n \equiv 0 \pmod{2} \end{cases}.$$

For the ordinary codimensions, it holds $c_n(E) = 2^{n-1}$ for all $n \in \mathbb{N}$; in particular, $\exp(E) = \lim_n \sqrt[n]{c_n(E)} = 2$.

Proof. Using the formula relating proper and ordinary codimensions, $c_n(E)$ is the binomial sum relative to even indexes, that is

$$c_n(E) = \sum_{k \text{ even}} \binom{n}{k} = 2^{n-1}.$$

$\square$

The multilinear structure of $Id(E)$ follows equally easily:

Theorem 3.4. *For all even $n \geqslant 2$, $\xi_n(E)$ is the sign character. Hence, for all $n \in \mathbb{N}$, the ordinary n-th character is*

$$\chi_n(E) = \sum_{k=0}^{n-1} [\![n-k, 1^k]\!].$$

Proof. For even $n \geqslant 2$ we already know that $\Gamma_n(E)$ is an S_n-module spanned by $[x_1, x_2] \dots [x_{n-1}, x_n]$. Being 1-dimensional, it has to be either the trivial module or the sign module. Since however the transposition $(1\ 2)$ changes the generating vector into its opposite, $\Gamma_n(E)$ is isomorphic to the sign representation of S_n.

Then, the ordinary characters $\chi_n(E)$ follow by applying the Young-Pieri rule to the summands $[\![n-2k]\!]\hat{\otimes}\xi_{2k}(E) = [\![n-2k]\!]\hat{\otimes}[\![1^{2k}]\!]$. □

Incidentally, these arguments also provide a description of what happens in case $\dim V = n$:

Corollary 3.5. *Let V be of dimension n; then $Id(E(V))$ is generated by the polynomials $[x, y, z]$ and $[x_1, x_2] \dots [x_{m-1}, x_m]$ for minimal even m such that $m > n$. In particular, the Grassmann algebras of a vector space of dimension $2k$ and the one of a vector space of dimension $2k + 1$ are PI-equivalent.*

Proof. All goes as in case of E, with the only difference that all polynomials $[x_1, x_2] \dots [x_{m-1}, x_m]$ for which $m > n$ are in $Id(E(V))$: in fact, multilinearity implies that it is enough to test them against a basis of $E(V)$, and although being true that $\dim E(V) = 2^n$ yet these polynomials vanish when evaluated on a basis of $E(V)$ because at least one element of $\mathscr{E}$ has to occur twice, turning the product to zero. Then, simply notice that chosen the minimal, even, m, all products involving more than m commutator factors are a consequence of $[x_1, x_2] \dots [x_{m-1}, x_m]$. □

It is a good point to remark that, even if the Grassmann algebras of finite dimensional vector spaces are not as important as E in Kemer's Theory, yet they are important algebras in the general PI-Theory. Indeed, Drensky and Carini (in a joint work, at the moment unpublished) proved that the Grassmann algebras of $2k$-dimensional vector spaces are among the so-called fundamental algebras, together with their finite tensor products; fundamental algebras are special finite dimensional algebras, constituting the building blocks of finitely generated PI-algebras up to PI-equivalence (see [AGPRb], Sec. 17.2.3). More precisely, any finitely generated PI-algebra is PI-equivalent (*i.e.* it has the same polynomial identities) to a finite direct sum of fundamental algebras.

To conclude this section, one may also prove that in fact the identities of E do not depend on the characteristic (> 2) of F, in case F is infinite:

Theorem 3.6. *(see* [GiKo01]*, Theorem 6)*
Let F be infinite, of characteristic $p > 2$. Then $Id(E)$ is generated by the Grassmann identity, as well.

Proof. The same reasoning as before yields that the multilinear proper polynomials in $Id(E)$ are the ones in the T-ideal I generated by $[x, y, z]$, but this is no longer enough to conclude, because of the characteristic of the field F. However, since F is infinite the proper polynomial identities in $Id(E)$ generate the whole T-ideal $Id(E)$, as well (see [DrB] Prop. 4.3.3.(ii)). So one has just to consider what happens for a proper polynomial which is not multilinear, and the question is easily settled noticing that any product $[x_1, x_2][x_1, x_3]$ is a consequence of the Grassmann identity: in fact, under the substitution $x_3 \to x_1$ and $x_4 \to x_3$ in the ancillary identity of Lemma 3.1, one gets

$$[x_1, x_2][x_1, x_3] \equiv -[x_1, x_1][x_2, x_3] = 0 \pmod{I}.$$

This means that any proper polynomial which is not multilinear is in I, and we may conclude. $\square$

4. Derivations of Grassmann algebras

Any (possibly non-associative) algebra A comes with an associative unital algebra, $End_F(A)$; inside $End_F(A)$ lies the Lie subalgebra of (ordinary) derivations of A, $Der(A)$, whose elements are the linear operators d such that the so-called Leibniz rule is fulfilled, that is

$$(ab)^d = a^d b + ab^d \qquad \text{for all } a, b \in A.$$

Although in student's life this notion is first met when dealing with real analysis, it is of a purely algebraic nature, and in fact it occurs quite often in Algebra: a well rooted branch of Algebra, fathered by Ritt in the 40's, is called *Differential Algebra* after his book [Rt50]; a few years later, Kaplansky [Kap57] wrote a book on that subject, and years after Ritt's death his former student Kolchin provided a unified version [Kol73]. Easy instances on common use of derivations in an Algebra setup, are the following

Example 4.1.

- If $A = F[x]$, the algebra of univariate polynomials, the linear map defined by $(x^k)^\partial := kx^{k-1}$ for all $k \geqslant 0$ is usually named the formal derivation on $F[x]$, and occurs, for instance, in deciding if a given polynomial has multiple roots or not.
- If L is a Lie-algebra, the adjoint maps $ad_a : L \to L$ defined by $x^{ad_a} := [xa]$ (Lie product in L) are derivations of L, for all $a \in L$. The adjoint representation of a Lie algebra L, that is the homomorphism $ad : L \to \mathfrak{gl}(L)$, constitutes a central theme in Lie Theory.
- If A is any associative algebra, any element $a \in A$ defines a derivation $\cdot^a$, defined by $x^a := [x, a]$ (the commutator $xa - ax$); such derivations are called inner.

Notice that, despite being very similar, ad_a and $\cdot^a$ are different notions: $[xa]$ is the Lie product of a Lie algebra, while $[x, a]$ is related to an associative product. More explicitly, the Leibniz rule for $\cdot^a$ writes $[xy, a] = [x, a]y + x[y, a]$, while ad_a is the (right) multiplication by a in the Lie algebra L, and the Leibniz rule is essentially the Jacobi rule. This often causes misunderstandings. Fortunately, when turning A into a Lie algebra $A^{(-)}$ by considering the bracket product $[x, y]$, then $\cdot^a$ is still a derivation of $A^{(-)}$ and, in fact, it coincides with $ad_a \in Der(A^{(-)})$.

Inner derivations (the maps $\cdot^a$, or ad_a in case of a Lie algebra) form a Lie ideal of $Der(A)$, denoted $IDer(A)$; sometimes, they constitute the whole $Der(A)$: this happens, for instance, for the full matrix algebras $M_n(F)$ or the upper triangular matrix algebras $UT_n(F)$, but also for all finite dimensional semisimple associative and Lie algebras over a field of characteristic zero [Ho42].

About the Grassmann algebras, their ordinary derivations have been considered in a paper of Djoković [Dj78]. He did not discuss further the case *char.*$F = 2$, because in this case every linear map $V \to E(V)$ uniquely extends to a derivation of $E(V)$; he also underlined he was concerned with *ordinary* derivations of $E(V)$, not with other types of derivations taking into special account the superalgebra structure of $E(V)$. In fact, one could consider also *anti-derivations* or *skew-derivations*:

- an anti-derivation ∂ on $E(V)$ is an element of $End_i(E(V))$ such that $(ab)^\partial = a^\partial b + (-1)^{i|a|} ab^\partial$, for any homogeneous element a (of $\mathbb{Z}_2$-degree $|a|$) and any $b \in E(V)$; these were known to Djoković, and studied in [Ka77];
- a skew-derivation ∂ on $E(V)$ is an element of $End_i(E(V))$ such that $(ab)^\partial = a^\partial b + (-1)^{|a|} ab^\partial$ for homogeneous element a and any $b \in E(V)$; hence ∂ is a graded operator in case $i = 0$, as well, but it is not an ordinary derivation. They have been studied in [Ba09].

Sticking to ordinary derivations, the results of Djoković are the only ones I am aware of, and if *char.*$F \neq 2$ one can summarize those results in a single statement

Theorem 4.2. *Let $E(V)$ be the Grassmann algebra of the vector space V. Then $Der(E(V)) = Der_0(E(V)) \oplus Der_1(E(V))$, where $Der_i(E(V))$ is a vector subspace of $End_i(E(V))$. Moreover*

1. *$[Der_0(E(V)), Der_1(E(V))] \subseteq Der_1(E(V))$;*
2. *$IDer(E(V))$ is an abelian ideal of $Der(E(V))$, contained in $Der_1(E(V))$;*
3. *$IDer(E(V)) = Der_1(E(V))$ if and only if* $\dim V \neq \aleph_0$.

Notice that all these statements but the last one hold for any superalgebra $A = A_0 \oplus A_1$ satisfying the anticommutative law $yx = (-1)^{ij}xy$ ($x \in A_i$ and $y \in A_j$), as remarked by Djoković himself. Unfortunately, in case of E, the most relevant one for PI-Theory, there are non-inner derivations within $Der_1(E)$, besides those in $Der_0(E)$; also, a more detailed description of elements in $Der_i(E(V))$ would be useful.

The results of Djoković can be refined, and a bit more can be said. First of all, any set-theoretic map $f : \mathscr{E} \to E_1(V)$ can be extended into a derivation δ_f by simply imposing F-linearity and the Leibniz rule on f. Let Δ be the set of such derivations. Then, in case f satisfies the relation

$$e_1 e_2 (e_2^f - e_1^f) = 0 \qquad \text{for all } e_1, e_2 \in \mathscr{E}, \tag{1}$$

it gives rise to another derivation, d_f, uniquely defined by assigning $e^{d_f} := ee^f$ for all $e \in \mathscr{E}$ and imposing linearity and the Leibniz rule on d_f. Let D denote this second set of derivations.

Example 4.3. Let e be any fixed element in $\mathscr{E}$, and let us consider the constant map $\varepsilon := \varepsilon_e : \mathscr{E} \to \{e\}$. It gives rise to a derivation $\delta_e \in \Delta$ by defining on an arbitrary element $w = e_1 \dots e_m \in \mathscr{B}$

$$w^{\delta_e} := ee_2 \dots e_m + e_1 e e_3 \dots e_m + \cdots + e_1 \dots e_{m-1} e$$

and then by linear extension to a derivation of $E(V)$.

By the way, since for any $e_1, e_2 \in \mathscr{E}$ it is true that $e_1 e_2 (e_2^\varepsilon - e_1^\varepsilon) = 0$, ε gives rise to another derivation $d_e \in D$, as well, by defining

$$w^{d_e} = (e_1 e) e_2 \dots e_m + e_1 (e_2 e) e_3 \dots e_m + \cdots + e_1 e_2 \dots e_{m-1} (e_m e).$$

Notice that d_e is the inner derivation induced by $e/2$, that is $d_e = [\cdot\,, e/2]$.

These two basic ways of producing derivation on $E(V)$ are, in fact, the only possible ones:

Proposition 4.4. $\Delta = Der_0(E(V))$, *and* $D = Der_1(E(V))$.

In particular, when V is finite dimensional, one has

Corollary 4.5. *Let V be a finite dimensional vector space of dimension n.*

(1) *If n is even, then*

$$\dim D = 2^{n-1}, \quad \dim \Delta = n2^{n-1} \quad \textit{and} \quad \dim Der(E) = (n+1)2^{n-1};$$

(2) *if n is odd, then*

$$\dim D = 2^{n-1} - 1, \quad \dim \Delta = n2^{n-1} \quad \textit{and} \quad \dim Der(E) = (n+1)2^{n-1} - 1.$$

Qualitatively, elements of D and Δ can be characterized by their action on the $\mathbb{Z}_2$-graded components of $E(V)$:

Lemma 4.6. *Let ∂ be any derivation of $E(V)$. Then*

- *$\partial \in D$ if and only if $E(V)^{\partial} \subseteq E_0(V)$;*
- *$\partial \in \Delta$ if and only if $E_i(V)^{\partial} \subseteq E_i(V)$, for all $i \in \mathbb{Z}_2$.*

While there is great freedom in building elements of Δ, it is quite difficult to keep the condition (1) under control: it is clearly verified just when f is a constant function. Di Vincenzo produced a concrete, non trivial example of an element of D which is a non-inner derivation of E:

Example 4.7. (Di Vincenzo)
Let us fix a total order on $\mathscr{E}$, so that $\mathscr{E} = \{e_1, e_2, \dots\}$, and denote $w_n := e_1 \dots e_n$ for all $n \in \mathbb{N}$ (the product of the first n elements of $\mathscr{E}$). Then set $e_1^a := 0$ and define recursively $e_{2i}^a = e_{2i+1}^a := e_{2i-1}^a + w_{2i-1}$ for $i \geqslant 1$. That is, the sequence $(e_i^a \mid i \geqslant 1)$ is the following:

$$(0, w_1, w_1, w_1 + w_3, w_1 + w_3, w_1 + w_3 + w_5, \dots).$$

Then a satisfies the condition (1), so the map sending $e_i \to e_i e_i^a$ uniquely defines an outer derivation $d \in D$.

In fact, inner derivations can be easily characterized:

Corollary 4.8. *A derivation d of E is an inner derivation if and only if it is an element of D induced by a constant function $f : \mathscr{E} \to E_1$.*

Both in case V is finite dimensional or V has uncountable dimension, Djoković results guarantee that $IDer(E(V)) = D$ is an abelian Lie ideal of $Der(E(V))$, but in case $\dim V = \aleph_0$ it just says that $IDer(E)$ is an abelian ideal of $Der(E)$, contained in D. Actually, more can be said, here as well:

Theorem 4.9. *D, Δ are Lie subalgebras of $Der(E)$, and D is a maximal abelian Lie subalgebra of $Der(E)$.*

The first part of this statement, that D and Δ are not mere subspaces of $Der(E)$, is easily checked out, and holds whatever $\dim V$ is; the significant point is that the whole D, and not just $IDer(E)$, is an abelian Lie ideal, but it is something more: a maximal abelian Lie subalgebra of $Der(E)$. In proving this result, the inner derivations d_i, induced by the elements $e_i/2$ (that is the maps $d_{e_i} = [\cdot, e_i/2]$ seen in Example 4.3) were the tools doing the trick.

Remark 4.10. What if $\dim V = n$? The arguments used in proving the previous Theorem work in this case, too, and show that if n is an even number, nothing changes; in case n is odd, instead, D is no longer maximal. More precisely, the element $\delta \in \Delta$ induced by the constant map $e_i \to e_1 \dots e_n$ commutes with all elements of D, so that $D \oplus F\delta$ replaces D as a maximal abelian subalgebra.

All the new results of this section, together with the proofs and other details, have been submitted for publication and will appear elsewhere.

5. Differential identities of E

The notion of *differential identities of an algebra A* was brought up at the end of the 70's by Kharchenko, with two renowned papers [Kh78, Kh79], and concerns identical relations involving ordinary elements of A together with "differential" elements, that is images of elements of A under some selected set of derivations of A. For instance, Schwarz's Theorem about the inversion of partial derivations may be considered a well known example of differential identity.

To express the intervention of derivations in more polite and precise form, let L be a Lie algebra: one says L acts by derivation on A (for short: A is turned into an L-algebra) if a Lie-homomorphism $\varphi : L \to Der(A)$ has been assigned. This situation can be expressed in an associative setup: denoting $U(L)$ the universal enveloping algebra of L, A turns into a right $U(L)$-module. In fact, φ defines uniquely an algebra homomorphism φ^* from $U(L)$ to the subalgebra generated by $\varphi(L)$ inside $End_F(A)$ (actually, inside the associative subalgebra generated by $Der(A)$).

Algebra homomorphisms between L-algebras preserving the L-actions (that is the $U(L)$-module structure) are called L-homomorphisms.

In order to talk about differential polynomial identities, one has to define a free object in the class of L-algebras. This construction is similar to the construction of the free algebra $F\langle X\rangle$, but a bit more involved. So, let X be a denumerable set of indeterminates, let W be the vector space $FX \otimes U(L)$ and consider the tensor algebra $T(W)$, which will be denoted $F\langle X|L\rangle$. This algebra inherits a natural L-algebra structure, and it is free on X in the class of L-algebras, that is for any L-algebra A, any set-theoretic function $X \to A$ uniquely extends to an L-homomorphism $F\langle X|L\rangle \to A$.

From the computational point of view, if a basis $\mathscr{U}$ of $U(L)$ has been fixed (for instance, by fixing a totally ordered basis in L and then using the Poincaré-Birkhoff-Witt Theorem) and $X = \{x_1, x_2, \dots\}$, one can consider the new set $X^L := \{x_i^u \mid i \geqslant 1,\, u \in \mathscr{U}\}$; then the free associative algebra $F\langle X^L\rangle$ is a natural L-algebra, as well, and in fact it is L-isomorphic to $F\langle X|L\rangle$. In these settings, an element $x_i^u \in X^L$ (actually a rewriting of the simple tensor $x_i \otimes u$ of $F\langle X|L\rangle$) is called a *letter*, the number i is called the *name* and u is called the *differential exponent* of the letter x_i^u. As for the terminology, letters whose differential exponent is $1 \in U(L)$ are called *ordinary letters*, and we identify them with the indeterminates $x_i \in X$; the remaining letters are called *differential letters*. This also legitimates the term *differential polynomials* for elements of $F\langle X^L\rangle \cong F\langle X|L\rangle$.

If A is an assigned L-algebra, the set of differential polynomials $f(x_1, \dots, x_n) \in F\langle X^L\rangle$ vanishing under all substitutions of its indeterminates by elements $a_1, \dots, a_n \in A$ is an ideal, denoted $Id^L(A)$, stable under all L-endomorphisms of $F\langle X^L\rangle$. The elements of $Id^L(A)$ are called the *L-differential identities of* A, and $Id^L(A)$ is called the T_L-ideal of differential identities of A.

Assuming, as we will in this section, $char.F = 0$, the description of $Id^L(A)$ can be carried on through the multilinear parts of it, that is by employing differential multilinear polynomials forming the sets denoted P_n^L; better yet, when A is unital (as in our case $A = E$) by employing differential proper multilinear polynomials, denoted Γ_n^L. Each of them is a direct generalizations of the ordinary spaces P_n and Γ_n, respectively, and precisely P_n^L is the linear span of the monomials $x_{\sigma(1)}^{u_1} \dots x_{\sigma(n)}^{u_n}$ for some $\sigma \in S_n$ and $u_1, \dots, u_n \in \mathscr{U}$, while Γ_n^L is the subspace of all differential multilinear polynomials in the unital subalgebra of $F\langle X^L \rangle$ generated by higher commutators together with the differential letters (which we consider as commutators of length 1).

Example 5.1. If d is any derivation of an algebra A, the Grassmann identity $[x_1, x_2, x_3]$ is a (differential) multilinear proper polynomial; the differential polynomial $f = x_1^d x_2 x_3 - x_1^d x_3 x_2$ is multilinear and proper, because $f = x_1^d [x_2, x_3]$; the polynomials $g = x_1 x_2^d x_3 - x_3 x_2^d x_1$, $h = x_1 x_2^d x_3 - x_1 x_3 x_2^d$ are multilinear, but not proper ones.

Informally speaking, a differential multilinear polynomial is proper if and only if its ordinary letters occur in commutators only.

While addressing the reader to [DVN22, Na23] for the details, here we recall that $Id^L(A)$ is generated by the differential proper multilinear polynomials it contains, just as in the ordinary case. Once they are known, several pleasant information may become available: the natural renaming action of S_n turns both P_n^L and Γ_n^L into left S_n-module, having intersections with $Id^L(A)$ as submodules, and giving rise to factor S_n-modules customarily denoted $P_n^L(A)$ and $\Gamma_n^L(A)$. They can be decomposed into a direct sum of irreducible S_n-modules, indexed by the partitions $\lambda = [\![\lambda_1, \dots, \lambda_r]\!] \vdash n$; by abusing the notation to ease of reading, we denote both the irreducible S_n-module indexed by λ and its irreducible character by λ. The S_n-cocharacters $\chi_n^L(A)$ and $\xi_n^L(A)$, of $P_n^L(A)$ and $\Gamma_n^L(A)$ respectively, are still Young-related; more precisely, it holds

$$\chi_n^L(A) = \sum_{k=0}^{n} [\![n-k]\!] \hat{\otimes} \xi_k^L(A)$$

for all $n \in \mathbb{N}$. The codimensions $c_n^L(A)$ and the proper codimensions $\gamma_n^L(A)$ of $P_n^L(A)$, $\Gamma_n^L(A)$ are thus related by

$$c_n^L(A) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k^L(A).$$

As for the ordinary polynomial identities, the L-codimension sequence $(c_n^L(A))_{n\in\mathbb{N}}$ provides in some sense a measure on how big $Id^L(A)$ is, and is the upright generalization of the classical one. However, here a few words of caution are needed: it is entirely possible that $c_n^L(A)$ (equivalently, $\gamma_n^L(A)$), is infinite, a kind of phenomenon not possible for other enriched algebra structures.

Example 5.2. Let $\delta \in \Delta$ be induced by the identity map on $\mathscr{E}$, so that for all $w \in \mathscr{B}$ it holds $w^\delta = len(w)w$. Setting $L := F\delta$, the 1-dimensional Lie algebra generated by δ, it follows that $\Gamma_n^L(E)$ is infinite-dimensional for all $n \geqslant 1$. Indeed, since $U(L)$

is (isomorphic to) the free univariate polynomial algebra, one can check that $\Gamma_n^L(E)$ contains all the vectors $x_1^{\delta^k}[x_2,x_3]\dots[x_{n-1},x_n]$ (if n is odd) or $[x_1^{\delta^k},x_2]\dots[x_{n-1},x_n]$ (if n is even) for all $k \geqslant 1$, linearly independent modulo the Grassmann identity. It also follows that $Id^L(E)$ is generated by the Grassmann identity only (as a T_L-ideal!).

The safe situation is when A is a finite dimensional algebra; in this case, it has been proved that not only $c_n^L(A)$ is exponentially bounded, but the limit $\exp^L(A) := \lim_n \sqrt[n]{c_n^L(A)}$ does exist [Go13] and equals the ordinary exponent of A, as proved at first for L finite dimensional semisimple [GoKc14], then for all Lie algebras ([Ri23], Theorem 3.7). This, of course, does not fit the case of E.

In a joint paper with Di Vincenzo ([DVN25], to appear), we considered some suitable Lie subalgebras of $Der(E)$, and described the corresponding differential identities of E.

The first subalgebra we studied was quite general: let $d_1,\dots,d_m$ be arbitrary non zero elements in D, and let D_m be the (abelian) Lie subalgebra they generate in $Der(E)$. Although the concrete ideal $Id^{D_m}(E)$ depends on which concrete derivations d_i have been selected, for all of them it holds

Lemma 5.3. *The polynomials*

$$x^{d_id_j},\ [x^{d_i},y],\ [x,y,z] \qquad (\text{ for all } i,j\in\{1,\dots,m\})$$

are in $Id^{D_m}(E)$.

Useful consequences of these polynomials are then available:

Corollary 5.4. *The polynomials*

(1) $x^{d_i}y^{d_i}$
(2) $x^{d_i}y^{d_j}+y^{d_i}x^{d_j}$
(3) $x^{d_i}[y,z]+y^{d_i}[x,z]$
(4) $[x,y][z,t]+[x,z][y,t]$

(for all $i,j\in\{1,\dots,m\}$) are consequences of the polynomials in the preceding Lemma.

Then, let us pick elements of D which are inner derivations; namely, pick m elements $q_1,\dots,q_m$ in $\mathscr{B}_1$, and denote by $d_1,\dots,d_m$ the inner derivations they define; we distinguished just between the two basic situations when the q_i's have pairwise disjoint supports (thus recovering the investigations in [Ri20]), and when the q_i's supports constitute an ascending chain.

Theorem 5.5. *Let $q_1,\dots,q_m\in\mathscr{B}_1$ be chosen such that $Supp(q_i)\cap Supp(q_j)=\varnothing$ for all distinct $i,j\in\{1,\dots,m\}$. Then $Id^{D_m}(E)$ is generated, as a T_L-ideal, exactly by the polynomials of Lemma 5.3. Moreover, the vectors*

$$[x_1,x_2]\dots[x_{2h-1},x_{2h}]x_{2h+1}^{d_{j_1}}\dots x_n^{d_{j_s}},\quad d_{j_1}<\dots<d_{j_s} \text{ and } 2h+s=n$$

form a basis for $\Gamma_n^{D_m}(E)$.

As a byproduct of the previous result, it holds

Corollary 5.6. *The proper differential codimensions are*

$$\gamma_n^{D_m}(E)=\sum_{n\leqslant s\equiv n}\binom{m}{s},$$

where $s \equiv n$ means $s \equiv n \pmod 2$.

Of course, when $n \geqslant m$, it holds $\gamma_n^{D_m}(E) = 2^{m-1}$, that is $\gamma_n^{D_m}(E)$ is a constant, while if $n < m$ then $\gamma_n^{D_m}(E) < 2^{m-1}$. Passing to differential codimensions, one gets

Corollary 5.7. *For all $n \in \mathbb{N}$ it holds $2^n \leqslant c_n^{D_m}(E) \leqslant 2^{m-1}2^n$.*

Also the S_n-action on $\Gamma_n^{D_m}(E)$, and so on $P_n^{D_m}(E)$, is now clear:

Proposition 5.8. $\xi_n^{D_m}(E) = \gamma_n^{D_m}(E)[\![1^n]\!]$.
As a consequence, just the irreducible characters $[\![n]\!]$, each with multiplicity 1, and $[\![1+n-k, 1^{k-1}]\!]$ with multiplicity $\gamma_k^{D_m}(E) + \gamma_{k-1}^{D_m}(E)$ for all $k \geqslant 2$, occur in the decomposition of $\chi_n^{D_m}(E)$.

Now let us consider the, in some sense, opposite case:

Proposition 5.9. *Let $q_1, \dots, q_m \in \mathscr{B}_1$ such that $Supp(q_1) \subseteq \cdots \subseteq Supp(q_m)$. Then $Id^{D_m}(E)$ is generated by the polynomials of Lemma 5.3 together with the polynomials $x^{d_i}y^{d_j}$ (for all $i, j \in \{1, \dots, m\}$).*
Moreover, the vectors $[x_1, x_2] \dots [x_{n-2}, x_{n-1}]x_n^{d_i}$, with $i \in \{1, \dots, m\}$ (if n is odd) or $[x_1, x_2] \dots [x_{n-1}, x_n]$ (if n is even) constitute a basis of $\Gamma_n^{D_m}(E)$. In particular, for all $n \in \mathbb{N}$, it holds

$$\gamma_n^{D_m}(E) = \begin{cases} 1 & \text{if } n \text{ is even} \\ m & \text{if } n \text{ is odd} \end{cases}.$$

Also, $\xi_n^{D_m}(E) = \gamma_n^{D_m}(E)[\![1^n]\!]$ for all $n \in \mathbb{N}$.

Hence, for the codimension and cocharacter sequences, one has

Corollary 5.10. *For any $n \in \mathbb{N}$ it holds $c_n^{D_m}(E) = (m+1)2^{n-1}$ and the irreducible characters occurring in $\chi_n^{D_m}(E)$ are the hooks $[\![1+a, 1^{n-a-1}]\!]$, with multiplicity $m+1$.*

Remarkably, both in the disjoint-support case and in the chain-support case it holds

Theorem 5.11. *The differential exponent $exp^{D_m}(E) := \lim_n \sqrt[n]{c_n^{D_m}(E)}$ does exist and equals the ordinary exponent of E.*

A more challenging framework is the following:
for any $m \geqslant 1$, pick m elements $e_1, \dots, e_m$ in $\mathscr{E}$, and let f_i be the constant function $\mathscr{E} \to \{e_i\}$. We know that any f_i gives rise to a derivation $\delta_i \in \Delta$, but also to an inner derivation $d_i \in D$, the one induced by $e_i/2$ (these were precisely the derivation doing the trick in the proof of Theorem 4.9). Let D_m and Δ_m be the vector spaces spanned by the d_i's and δ_i's, respectively. It turns out that, although both D_m and Δ_m are associative (non unital) subalgebras of $End_F(E)$, their direct sum $L_m := D_m \oplus \Delta_m$, with respect to the bracket product, is a $2m$-dimensional Lie subalgebra sitting in $Der(E)$.

Writing from now on L instead of L_m, its multiplication table is defined by the products

$$[d_i, d_j] = 0 \qquad [\delta_i, \delta_j] = \delta_j - \delta_i, \qquad [d_i, \delta_j] = d_j \qquad (i, j \leqslant m);$$

also, L has a nice structure: it is a solvable, non nilpotent, Lie algebra of index 3.

The algebra L is isomorphic to the abstract Lie algebra $\hat{L}$ generated by $2m$-elements s_i, t_j with relations $[s_i, s_j] = s_j - s_i$, $[t_i, t_j] = 0$ and $[t_i, s_j] = t_j$, whose universal enveloping algebra is an infinite dimensional noncommutative algebra; fixing the order $s_1 < \cdots < s_m < t_1 < \cdots < t_m$ among the basis elements of $\hat{L}$, the Poincaré-Birkhoff-Witt Theorem provides the canonical basis of $U(\hat{L})$ consituted by all the semistandard words on the alphabet $\{s_i, t_j \mid i, j \in \{1, \ldots, m\}\}$. Actually, the selected elements δ_i, d_j generate a subalgebra of $End_F(E)$ which is a homomorphic image of $U(\hat{L})$ with a kernel K. One can prove that K is the two-sided ideal of $U(\hat{L})$ generated by

$$\mathscr{K} := \{s_i s_j - s_j,\, t_i t_j,\, s_i t_j + s_j t_i \mid i, j \in \{1, \ldots, m\}\}.$$

Hence the associative algebra $Alg(L)$ generated by L inside $End_F(E)$ has the set

$$\{1 = Id_E,\, \delta_i,\, d_j,\, \delta_h d_k \mid i, j, h, k \in \{1, \ldots, m\},\, h < k\}$$

as an F-basis.

By the previous discussion, each $x \in X$ gives rise just to $m(m+3)/2$ differential letters x^{δ_i}, x^{d_j} and $x^{\delta_i d_j}$ (with $i < j$) to be considered, because all other x^k (for $k \in K$) are differential identities induced by K; with obvious meaning, we will call them δ-letters, d-letters and δd-letters, respectively. Also, the generators of K provide us with the identities

$$x^{\delta_i \delta_j} - x^{\delta_j}, \qquad x^{d_i d_j}, \qquad x^{\delta_i d_j} + x^{\delta_j d_i}$$

of E under the L-action (which we call the *structural differential identities* of E under L, in the terminology of [Na23]). Further, non structural, identities are the Grassmann identity $[x, y, z]$, and the commutators $[x^{d_i}, y]$ for all $i \leqslant m$, inherited from the fact that $d_i \in D$. In fact, it holds

Theorem 5.12. *$Id^L(E)$ is generated by the differential polynomials*

$$x^{\delta_i \delta_j} - x^{\delta_j}, \qquad x^{d_i d_j}, \qquad x^{\delta_i d_j} + x^{\delta_j d_i}, \qquad [x^{d_i}, y], \qquad [x, y, z] \qquad (i, j \leqslant m).$$

The detailed proof will appear in [DVN25], but here I will give a schematic idea of it: at first several useful ancillary identities, involving the δ_i's, have to be added to those listed in Corollary 5.4. The complete list of the new ones is the following:

(1) (involving δ-letters only)

$$x^{\delta_i} y^{\delta_j} + x^{\delta_j} y^{\delta_i} \text{ (in particular } x^{\delta_i} y^{\delta_i}),$$
$$x^{\delta_i}[y^{\delta_j}, z] + x^{\delta_j}[y^{\delta_i}, z],$$
$$[x^{\delta_i}, y^{\delta_j}][w^{\delta_h}, z^{\delta_k}] + [x^{\delta_i}, w^{\delta_h}][y^{\delta_j}, z^{\delta_k}];$$

(2) (involving δ-letters and d-letters only)

$$x^{\delta_i} y^{d_j} + x^{\delta_j} y^{d_i} \text{ (in particular: } x^{\delta_i} y^{d_i} \text{ and } x^{d_i} y^{\delta_i})$$

(3) (involving δd-letters only)

$$[x^{\delta_i d_j}, y], \quad x^{\delta_{\sigma(i)} d_{\sigma(j)}} y^{\delta_{\sigma(h)} d_{\sigma(k)}} + (-1)^\sigma x^{\delta_i d_j} y^{\delta_h d_k}$$

(4) (miscellaneous)

$$y^{\delta_i d_j}[x, z] + x^{d_j}[y^{\delta_i}, z],$$
$$x^{d_{\sigma(i)}} y^{\delta_{\sigma(j)} d_{\sigma(h)}} + (-1)^\sigma x^{d_i} y^{\delta_j d_h}, \quad x^{\delta_{\sigma(i)}} y^{\delta_{\sigma(j)} d_{\sigma(h)}} + (-1)^\sigma x^{\delta_i} y^{\delta_j d_h}$$

where σ is any permutation of the involved indexes.

In the next step, working modulo the T_L-ideal generated by the candidate generators in Theorem 5.12, we selected a (huge) set of elements of Γ_n^L, for all $1 \leqslant n \in \mathbb{N}$, spanning $\Gamma_n^L(E)$. Finally, we proved that the selected vectors are, in fact, linearly independent modulo $Id^L(E)$, thus proving the Theorem.

In selecting this basis of $\Gamma_n^L(E)$, we have to split the cases $n > m$ and $n \leqslant m$.

Definition 5.13. For all $n \geqslant 1$ let $\mathscr{S}_n^{(1)} \subseteq \Gamma_n^L$ be constituted by the following vectors:

$$[x_{a_1}, x_{a_2}] \dots [x_{a_{h-1}}, x_{a_h}][x_{b_1}^{\delta_{i_1}}, x_{b_2}^{\delta_{i_2}}] \dots [x_{b_{k-1}}^{\delta_{i_{k-1}}}, x_{b_k}^{\delta_{i_k}}] x_{c_1}^{\delta_{j_1}} \dots x_{c_r}^{\delta_{j_r}} x_{a_{h+1}}^{d_{\ell_1}} \dots x_{a_{h+s}}^{d_{\ell_s}}$$

(when $h \equiv k \equiv 0 \pmod 2$) or

$$[x_{a_1}, x_{a_2}] \dots [x_{a_h}, x_{b_1}^{\delta_{i_1}}] \dots [x_{b_{k-1}}^{\delta_{i_{k-1}}}, x_{b_k}^{\delta_{i_k}}] x_{c_1}^{\delta_{j_1}} \dots x_{c_r}^{\delta_{j_r}} x_{a_{h+1}}^{d_{\ell_1}} \dots x_{a_{h+s}}^{d_{\ell_s}}$$

(when $h \equiv k \equiv 1 \pmod 2$),
for all h, k, r, s non-negative integers such that

- $h + k + r + s = n$, $1 \leqslant h \equiv k \pmod 2$;
- $a_1 < \dots a_h < a_{h+1} < \dots < a_{h+s}$;
- $b_1 < \dots < b_k$ and $c_1 < \dots < c_r$;
- $1 \leqslant i_1 < \dots < i_k < j_1 < \dots < j_r < \ell_1 < \dots < \ell_s \leqslant m$.

Since elements of $\mathscr{S}_n^{(1)}$ are quite awkward, a more comfortable way of referring to them is to look at them as a product $cc_\delta w_\delta w_d$, where c is the product of the 2-commutators in ordinary letters only, c_δ is the product of 2-commutators involving at least a δ-letter, w_δ is the standard word on the selected δ-letter and w_d is the standard word on the involved d-letters. Notice that no δd-letter occurs in the selected vectors. Also, notice at least an ordinary letter has to occur, and that the differential exponents form a standard (possibly empty) sequence.

In case $m \geqslant n$ a complementary set of vectors is needed because it may happen that no ordinary letter occurs:

Definition 5.14. Let $\mathscr{S}_n^{(2)} \subseteq \Gamma_n^L$ be the constituted by the following vectors:

$$x_{a_1}^{\delta_{p_1} d_{q_1}} \dots x_{a_h}^{\delta_{p_h} d_{q_h}} [x_{a_{h+1}}^{\delta_{i_1}}, x_{a_{h+2}}^{\delta_{i_2}}] \dots [x_{a_{h+k-1}}^{\delta_{i_{k-1}}}, x_{a_{h+k}}^{\delta_{i_k}}] x_{b_1}^{\delta_{j_1}} \dots x_{b_r}^{\delta_{j_r}} x_{c_1}^{d_{\ell_1}} \dots x_{c_s}^{d_{\ell_s}}$$

for all h, k, r, s non-negative integers such that

- $h + k + r + s = n$, $k \equiv 0 \pmod 2$;
- $a_1 < \dots a_h < a_{h+1} < \dots < a_{h+k}$;
- $b_1 < \dots < b_r$ and $c_1 < \dots < c_s$;
- $p_1 < q_1 < \dots < p_h < q_h < i_1 < \dots < i_k < j_1 < \dots < j_r < \ell_1 < \dots < \ell_s$.

In the same style than before, an element of $\mathscr{S}_n^{(2)}$ can be read off as a product $w_{\delta d} c_\delta w_\delta w_d$; notice that here no ordinary letter appears, so $\mathscr{S}_n^{(2)}$ is actually the empty set when $n > m$. Finally,

Definition 5.15. For all $n \geqslant 1$ let $\mathscr{S}_n := \mathscr{S}_n^{(1)} \cup \mathscr{S}_n^{(2)}$. The elements in $\mathscr{S}_n$ are called the *normal forms* of the elements of $\Gamma_n^L(E)$.

Having at hand a complete F-basis for each $\Gamma_n^L(E)$, we turned our attention in describing its differential proper S_n-character; this also needs a separate treatment in case $n > m$ or $n \leqslant m$: if $n > m$, $\Gamma_n^L(E)$ is spanned by vectors of $\mathscr{S}_n^{(1)}$ only, while if $n \leqslant m$ vectors of $\mathscr{S}_n^{(2)}$ play their role too. By the way, the resulting structure of $\Gamma_n^L(E)$ can be synthetically expressed in the following

Theorem 5.16. *The n-th differential proper S_n-cocharacter of $Id^L(E)$ is*

$$\xi_n^L(E) = \sum_{(h,k,l,r,s)} \binom{m}{n-h+l} [\![1^{h+s}]\!] \hat{\otimes} [\![r]\!] \hat{\otimes} [\![1^{k+l}]\!],$$

where (h,k,l,r,s) ranges among the 5-multisets of class n satisfying

(1) $h \equiv k \pmod 2$*;*
(2) *if $h > 0$ then $l = 0$.*

In principle, the differential cocharacters $\chi_n^L(E)$ could be determined via their relation with the proper ones and the Littlewood-Richardson rule. In practice, however, the explicit decomposition would result too complicate to be considered a better description.

About the differential proper codimension sequence, it holds

Corollary 5.17. *For all $n \in \mathbb{N}$ the differential proper n-th codimension of E under the derivation action of L is*

$$\gamma_n^L(E) = \sum_{(h,k,l,r,s)} \binom{m}{n-h+l} \binom{n}{h+s \quad r \quad k+l}.$$

where the (h,k,l,r,s) runs among all the 5-multisets of class n satisfying the relations

(1) $h \equiv k \pmod 2$
(2) *if $h > 0$ then $l = 0$.*

Unfortunately, there is no closed formula expressing the general term of this sequence, mainly because no closed formula may exist for a generic, partial sum of entries in a row of the Pascal triangle. By the way, exploiting the relation between $c_n^L(E)$ and $\gamma_n^L(E)$, and after some careful manipulations, we got

Theorem 5.18. *The limit $\exp^L(E) := \lim_n \sqrt[n]{c_n^L(E)}$ does exist, and precisely $\exp^L(E) = 2$.*

So, even if E is not a finite dimensional algebras, nor L semisimple, yet the differential exponent of E equals the ordinary one.

6. Tribute to Grassmann

I think anyone learned the name Grassmann because of the linear algebra Theorem bearing his name; this is right, of course. By the way, the feeling is that often the name Grassmann seems limited to this contribution, and to the invention of the algebras we dealt with so far in this paper. Sometimes, it even happens to see his name misspelled in Grassman (actually, Vandermonde and Peirce share this kind of fate, too).

This is not the best place for any tentative of summarizing Grassmann's biography, nor I have the intention to; by the way, I was astonished in discovering how underestimated were he and his work by the mathematical community.

Hermann G. Grassmann never had academic courses in Mathematics, nor as a student nor as a teacher; his discoveries provided him with a single prize (for a work about a problem raised by Leibniz) after solicited by Möbius, a won hollowed by being the sole participant to the contest; it was almost despised by Kummer, when Grassmann applied for an academic position, by the statement *"commendably good material expressed in a deficient form"*.

Actually, Grassmann was simply too far ahead his contemporaries:

- he created what is now known as linear (and multilinear) algebra in 1844, when he was thirty-five; it became of common knowledge only after another Hermann, Weyl, who published formal definitions in 1920, almost 80 years after Grassmann. Actually, Weyl credited Peano, not Grassmann, for it: he was Peano, in 1890, who credited Grassmann for his ideas, but this passed unnoticed;
- in a time when euclidean geometry was the one and only possible Geometry, Grassmann tried to put Geometry into a new algebraic form and, doing this, he spoiled the number *three* of his magic, since the "new geometry" is not confined to three dimensions, but develops into arbitrary dimensions, even infinite;
- in the same work, he invented the Grassmann algebra, in a time when the abstract notion of an *algebra*, and even the language to express it, has yet to be invented; in 1878, Clifford recognized the Grassmann algebra and the Hamilton quaternions as examples of more general structures now known as Clifford algebras.

In 1862, Grassmann re-published his work from a different point of view, but the first recognition came only by Clifford an Gibbs in 1878, when Grassmann was dead (he died in 1877). However, already in 1870 Grassmann himself was not interested in Mathematics anymore, due to the hostile opinions against his ideas. He turned to Linguistics (with outstanding results: he presented a phonological rule which is still known as Grassmann's rule), and in the meantime he also published a theory on how colors mix, still taught and named Grassmann's laws.

As Fearnley-Sander put it ([FS79], just before Sec. 9, page 816)

> *"All mathematicians stand, as Newton said he did, on the shoulders of giants, but few have come closer than Hermann Grassmann to creating, single-handedly, a new subject."*

Let me address the interested reader to this short paper, or to the more extensive volumes [Br12, Br21] by John Browne, who dedicated himself to the Grassmann algebra project until, sadly, his death in 2021. A quick access to a wealth of information is also available at the website `https://www.grassmannalgebra.com`, formerly created and managed by Browne himself, and now run by the John Browne's Estate in his memory.

These few lines I wrote are my humble *Huldigung* to Grassmann and his work. History is giving him back all the credit he deserves. Unfortunately, a bit too late.

References

[AGK17] E. Aljadeff, A. Giambruno, Y. Karasik, *Polynomial identities with involution, superinvolutions and the Grassmann envelope*, Proc. Amer. Math. Soc. **145** (2017), 1843–1857

[AGPRb] E. Aljadeff, A. Giambruno, C. Procesi, A. Regev, *Rings with Polynomial Identities and Finite Dimensional Representations of Algebras*, AMS Colloquium Publications Vol. **66** (2000)

[Ar24] S. Argenti, *Lie semisimple algebras of derivations and varieties of PI-algebras with almost polynomial growth*, Proc. Amer. Math. Soc. **152 (10)** (2024), 4217–4229

[Ba09] V. V. Bavula, *Derivations and skew derivations of the Grassmann algebras*, J. Algebra Appl., **8 (6)** (2009), 805–827

[Be87] F. A. Berezin, *Introduction to SuperAnalysis*, Kluver Academic Publishers (1987)

[Br12] J. Browne, *Grassmann algebra Volume 1: Foundations: Exploring extended vector algebra with Mathematica* (2012)

[Br21] J. Browne, *Grassmann algebra Volume 2: Extensions* (2021) – Abridged draft edition

[Ce16] L. Centrone, *On the graded identities of the Grassmann algebra*, J. Algebra Comb. Discrete Appl. **4 (2)** (2016), 165–180

[dS12] V. R. T. da Silva, *On ordinary and $\mathbb{Z}_2$-graded polynomial identities of the Grassmann algebra*, Serdica Math. J. **38** (2012), 417–432

[Dj78] D. Ž. Djoković, *Derivations and automorphisms of exterior algebras*, Can. J. Math. **XXX (6)** (1978), 1336–1344

[DKBV20] G. Dor, A. Kanel–Belov, U. Vishne, *The Grassmann algebra over arbitrary rings and minus sign in arbitrary characteristic*, Trans. Amer. Math. Soc., Series B **7** (2020), 227–253

[DrB] V. Drensky, *Free Algebras and PI-Algebras: Graduate Course in Algebra* Springer-Verlag Singapore (2000)

[Dr84] V. Drensky, *Codimensions of T-ideals and Hilbert series of relatively free algebras*, J. Algebra **91** (1984), 1–17

[DV91] O. M. Di Vincenzo, *A note on the identities of the Grassmann algebras*, Boll. Unione Mat. Ital., **A 7 (5) n. 3** (1991), 307–315

[DVN22] O. M. Di Vincenzo, V. Nardozza, *Differential Polynomial Identities of Upper Triangular Matrices Under the Action of the Two-Dimensional Metabelian Lie Algebra*, Algebr. Represent. Theory**(25)** (2022), 187–209

[DVN24] O. M. Di Vincenzo, V. Nardozza, *Minimal $*$-varieties and superinvolutions*, Israel J. Math **265** (2025), 311–346 https://doi.org/10.1007/s11856-024-2657-2

[DVN25] O. M. Di Vincenzo, V. Nardozza, *Differential Identities of the Grassmann Algebra*, J. Pure App. Algebra (2025) (to appear)

[FS79] D. Fearnley-Sander, *Hermann Grassmann and the Creation of Linear Algebra*, The American Mathematical Monthly, **86 (10)** (1979), 809–817, https://doi.org/10.1080/00029890.1979.11994921

[GiKo01] A. Giambruno, P. Koshlukov, *On the identities of the Grassmann algebras in characteristic $p > 0$*, Israel J. Math. **122** (2001), 305–316

[GR19] A. Giambruno, C. Rizzo, *Differential identities, 2×2 upper triangular matrices and varieties of almost polynomial growth*, J. Pure Appl. Algebra **223** (2019), 1710–1727

[GZb] A. Giambruno, M.V. Zaicev, *Polynomial identities and asymptotic methods*, AMS Mathematical Surveys and Monographs, vol. 122. Providence, R.I. (2005)

[GiZa98] A. Giambruno, M. V. Zaicev, *On codimension growth of finitely generated associative algebras*, Adv. Math. **140** (1998), 145–155

[GiZa99] A. Giambruno, M. V. Zaicev, *Exponential codimension growth of PI algebras: an exact estimate*, Adv. Math. **142** (1999), 221–243

[Go13] A. S. Gordienko, *Asymptotics of H-identities for associative algebras with a generalized Hopf action*, J. Pure Appl. Algebra **393** (2013), 92–101

[GoKc14] A. S. Gordienko, M. V. Kochetov, *Derivations, Gradings, Actions of Algebraic Groups, and Codimensions Growth of Polynomial Identities* Algebra Represent. Theory **17** (2014), 539–563

[Ho42] G. Hochschild, *Semi-simple algebras and generalized derivations*, J. Math. **64** (1942), 677–694

[Ka77] V. G. Kac, *Lie superalgebras*, Advances in Mathematics **26** (1977), 8–96

[Kap57] I. Kaplanski, *An Introduction to Differential Algebra*, Hermann, Paris (1957)

[Ke78] A. R. Kemer, *T-ideals with power growth of the codimensions are Specht*, Sibirsk. Math. Zh. **19** (Russian), translation in Siberian Math. J. **19** (1978), 37–48

[Ke91] A. R. Kemer, *Ideals of identities of associative algebras*, Transl. Math. Monogr., **vol 87** (1991) Am. Math. Soc., Providence RI

[Ke93] A. R. Kemer, *Identities of algebras over a field of characteristic p*, Rendiconti del Circolo Matematico di Palermo, Supplemento **31** (1993), 133–148

[Ke93i] A. R. Kemer, *The standard identity in characteristic p. A conjecture of I. B. Volichenko*, Israel J. Math **81** (1993), 343–355

[Ke95] A. R. Kemer, *Remarks on the prime varieties*, Israel J. Math. **96** (1995), 341–356

[Ke95i] A. R. Kemer, *Multilinear identities of the algebras over a field of characteristic p*, International Journal of Algebra and Computations **5 (2)** (1995), 189–197

[Kh78] V.K. Kharchenko, *Differential identities of prime rings*, Algebra and Logic **17 (2)** (1978), 155–168

[Kh79] V.K. Kharchenko, *Differential identities of semiprime rings*, Algebra and Logic **18 (1)** (1979), 58–80

[KrRe73] D. Krakowski, A. Regev, *The Polynomial Identities of the Grassmann Algebra*, Trans. Amer. Math. Soc. **181** (1973), 429–438

[Kol73] E.R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, New York, London (1973)

[Ma50] A. I. Malcev, *On algebras defined by identities* (Russian), Mat. Sb. **26** (1950), 19–33

[MR23] F. Martino, C. Rizzo, *Differential Identities and Varieties of Almost Polynomial Growth*, Israel J. Math. **254** (2023), 243–274

[Na23] V. Nardozza, *Differential Polynomial Identities of Upper Triangular Matrices of Size Three*, Linear Algebra Appl. **663** (2023), 142–177

[OlRe76] J. B. Olsson, A. Regev, *Colength sequence of some T-ideals*, J. Algebra **38** (1976), 100–111

[Re72] A. Regev, *Existence of Identities in $A \otimes B$*, Israel J. Math **11** (1972), 131–152

[Re94] A. Regev, *Young-Derived Sequences of S_n-Cocharacters*, Adv. Math. **106** (1994), 169–197

[Rt50] J. F. Ritt, *Differential Algebra*, Amer. Math. Soc. Colloq. Publ., Vol. 33, Amer. Math. Soc., New York (1950)

[Ri20] C. Rizzo, *The Grassmann algebra and its differential identities*, Algebr. Represent. Theory **23** (2020), 125–134

[Ri23] C. Rizzo, *Differential codimensions and exponential growth*, Linear Algebra Appl. **675** (2023), 294–311

[RSV23] C. Rizzo, R. B. dos Santos, A. C. Vieira, *Differential identities and polynomial growth of the codimensions*, Algebr. Represent. Theory **26** (2023), 2001–2014

[Sp50] W. Specht, *Gesetze in Ringen I*, Math. Z. **52** (1950), 557–589

[Vo78] I. B. Volichenko, *The T-ideal generated by the element $[X_1, X_2, X_3, X_4]$* (Russian), Preprint no. 22, Institut Matemat. Akad. Nauk BSSR, Minsk (1978)

Dipartimento di Matematica, Università degli Studi di Bari, via Orabona 4, 70125 Bari, Italia

Email address: vincenzo.nardozza@uniba.it