MATEMATICA - LEZIONE 20

mercoledì 4 novembre 2026 09:09

ESEMPI DI RIEPILOGO SUI LIMITI

1) $\lim_{x \to +\infty} x^4 - \frac{1}{2} x^3 =$

$= \lim_{x \to +\infty} x^4 \left(1 - \frac{1}{2} \left(\frac{1}{x} \right) \right) = +\infty \cdot 1 = +\infty$

$\frac{1}{+\infty} = 0$

2) $\lim_{x \to +\infty} \left(\sqrt{x} - x^{\frac{5}{2}} \right) = \lim_{x \to +\infty} \left(x^{\frac{1}{2}} - x^{\frac{5}{2}} \right)$

$= \lim_{x \to +\infty} x^{\frac{5}{2}} \left(\frac{1}{x^2} - 1 \right) = +\infty \cdot (-1) = -\infty.$

$\frac{1}{x^2} \to 0$

3) $\lim_{x \to -\infty} \left(x^5 - x^4 \right) = \lim_{x \to -\infty} x^5 \left(1 - \frac{1}{x} \right) = -\infty \cdot 1 = -\infty$

$x^5 \to -\infty$, $\frac{1}{x} \to 0$, $1 - \frac{1}{x} \to 1$

4) $\lim_{x \to -\infty} \left(x^6 - x^4 \right) = +\infty$

5) $\lim_{x \to +\infty} \frac{x^3 + 2x^4}{7x + 1 + 5x^4} = \lim_{x \to +\infty} \frac{x^4 \left(\frac{1}{x} + 2 \right)}{x^4 \left(\frac{7}{x^3} + \frac{1}{x^4} + 5 \right)}$

$= \frac{2}{5}$

(o $\lim_{x \to +\infty} \frac{x^3 + 2x^4}{7x + 1 + 5x^4} = \lim_{x \to +\infty} \frac{2x^4}{5x^4} = \frac{2}{5}$)

6) $\lim_{x \to -\infty} \frac{x^5 + 2x^4}{1 + 4x^3 + 2x} = \lim_{x \to -\infty} \frac{x^5}{4x^3} = \lim_{x \to -\infty} \frac{1}{4} x^2$

$= \frac{1}{4} (-\infty)^2 = +\infty.$

Gerarchia degli infiniti per $x \to +\infty$

- $\forall\, a > 1$, $\log_a x \to +\infty$ e $\log_a$ cresce meno di tutte le potenze x^α con $\alpha > 0$. $\left(\lim_{x\to+\infty} \frac{\log_a x}{x^\alpha} = 0 \;/\; \lim_{x\to+\infty} \frac{x^\alpha}{\log_a x} = +\infty \right)$
- A sua volta $x^\alpha \to +\infty$ $\forall \alpha > 0$ ma x^α cresce più lentamente degli esponenziali a^x con base $a > 1$.

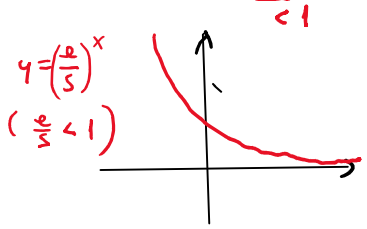
ESEMPI

- $\lim_{x\to+\infty} \left(x^{\frac{4}{3}} - 2^x \right) = \lim_{x\to+\infty} 2^x \left(\frac{x^{\frac{4}{3}}}{2^x} - 1 \right) = +\infty \cdot (-1) = -\infty$

  ($\frac{x^{\frac{4}{3}}}{2^x} \to 0$)

- $\lim_{x\to+\infty} \frac{e^x + x^4}{5^x + x^{\pi}} = \lim_{x\to+\infty} \frac{e^x}{5^x} = \lim_{x\to+\infty} \left(\frac{e}{5} \right)^x = 0.$

  ($\frac{e}{5} < 1$)



- Consideriamo la funzione

$$f(x) = \frac{e^{2x}}{x^2 - 1}$$

$x^2 \neq 1$
$x \neq \pm 1$

$$\text{Dom}(f) =]-\infty, -1[\cup]-1, 1[\cup]1, +\infty[$$

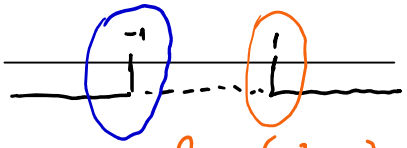
Facciamo i limiti agli estremi del dominio

- $\lim_{x\to+\infty} \frac{e^{2x}}{x^2 - 1} = \lim_{x\to+\infty} \frac{e^{2x}}{x^2}$ $\left(\frac{+\infty}{+\infty}\right)$

$$= \lim_{x\to+\infty} \frac{(e^2)^x}{x^2} = +\infty$$ (per la gerarchia degli infiniti)

- $\lim_{x \to 1} \frac{e^{2x}}{x^2-1} = \text{“}\frac{e^2}{0}\text{”}$ Bisogna studiare il segno del denominatore.

Segno di x^2-1



$\lim_{x \to 1^+} (x^2-1) = 0^+$

$\lim_{x \to 1^-} (x^2-1) = 0^-$

$\lim_{x \to -1^+} x^2-1 = 0^-$

$\lim_{x \to -1^-} (x^2-1) = 0^+$

$$\lim_{x \to 1^+} \frac{e^{2x}}{x^2-1} = \frac{e^2}{0^+} = +\infty$$

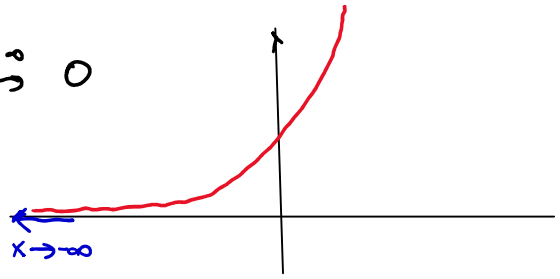
$$\lim_{x \to 1^-} \frac{e^{2x}}{x^2-1} = \frac{e^2}{0^-} = -\infty$$

$$\lim_{x \to -1^+} \frac{e^{2x}}{x^2-1} = \frac{e^{-2}}{0^-} = -\infty$$

$$\lim_{x \to -1^-} \frac{e^{2x}}{x^2-1} = \frac{e^{-2}}{0^+} = +\infty$$

$$\lim_{x \to -\infty} \frac{e^{2x}}{x^2-1} = \text{“}\frac{e^{-\infty}}{+\infty}\text{”} = \frac{0}{+\infty} = 0$$

Nota: $e^{2x} = (e^2)^x \xrightarrow{x \to -\infty} 0$



Teoremi sui limiti

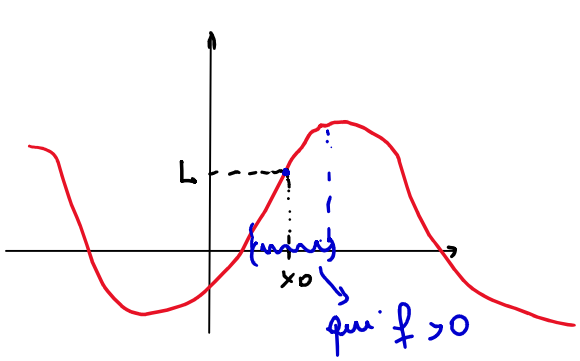
1) TEOREMA DI UNICITÀ DEL LIMITE.

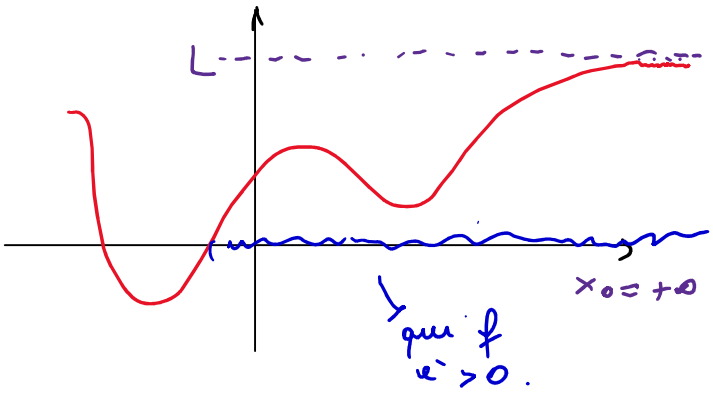
Sia $X \subseteq \mathbb{R}$, sia $f: X \to \mathbb{R}$. Sia $x_0 \in Dr(X)$.

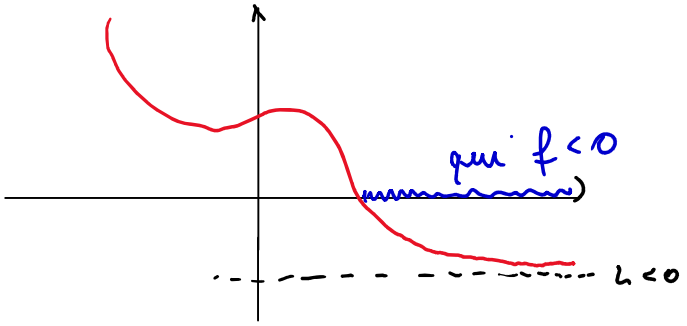
Se $\lim_{x \to x_0} f(x) = L_1$ e $\lim_{x \to x_0} f(x) = L_2$ con $L_1, L_2 \in \overline{\mathbb{R}}$, allora $L_1 = L_2$.

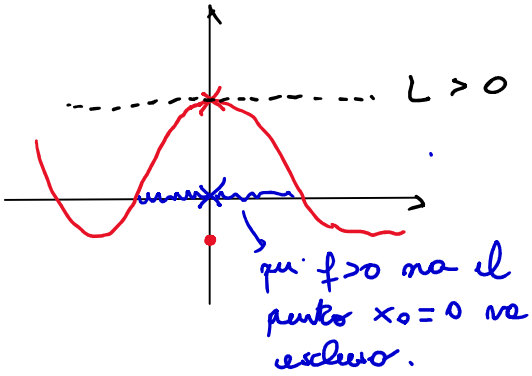
2) Permanenza del segno:

Se $\lim_{x \to x_0} f(x)$ ha un segno (+ o −) allora nei punti vicini a x_0, f deve avere lo stesso segno del limite









TEOREMA (DELLA PERMANENZA DEL SEGNO)

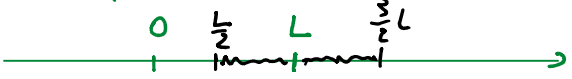
Sia $X \subseteq \mathbb{R}$, sia $f: X \to \mathbb{R}$. Sieno $x_0 \in D(X)$ e $L \in \overline{\mathbb{R}}$. Supponiamo che $\lim_{x \to x_0} f(x) = L$.

1) Se $L > 0$, allora $\exists\, U \in \mathcal{I}_{x_0}$ tale che $\forall x \in U \cap X \setminus \{x_0\}$: $f(x) > 0$.

2) Se $L < 0$, allora $\exists\, U \in \mathcal{I}_{x_0}$ t.c. $\forall x \in U \cap X \setminus \{x_0\}$: $f(x) < 0$.

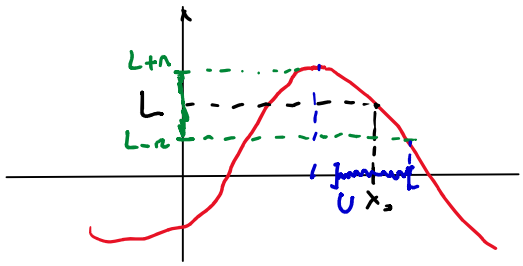
DIM Per semplicità supponiamo $L \in \mathbb{R}$.

1) Se $L \in \mathbb{R}$, $L > 0$.



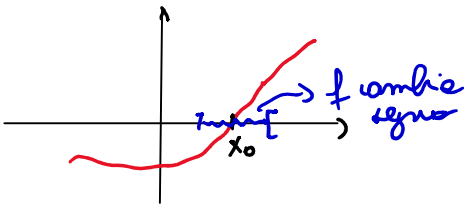
Se $r = \frac{L}{2}$, allora, $]L - r, L + r[\in \mathcal{I}_L$ e $]L - r, r + L[\subseteq]0, +\infty[$.

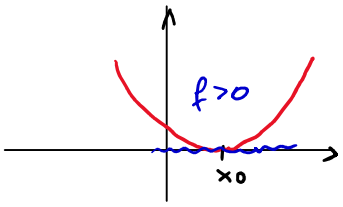
Siccome $]L-r, L+r[\in \mathcal{I}_L$ $\exists U \in \mathcal{I}_{x_0}$ t.c. $\forall x \in U \cap X \setminus \{x_0\}$. $f(x) \in]L-r, L+r[\subseteq]0, +\infty[$ $(f(x) > 0)$.

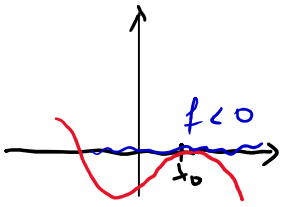


2) Uguale con $r = \frac{|L|}{2}$

E se $L = 0$? Non è possibile prendere il segno di f vicino a x_0.









f cambia segno infinite volte.

COROLLARIO

Siano X, f, x_0, L come nel teorema.

1) Se $\exists U \in \mathcal{I}_{x_0}$ t.c. $\forall x \in U \cap X \setminus \{x_0\}$: $f(x) \geq 0$, allora $L \geq 0$.

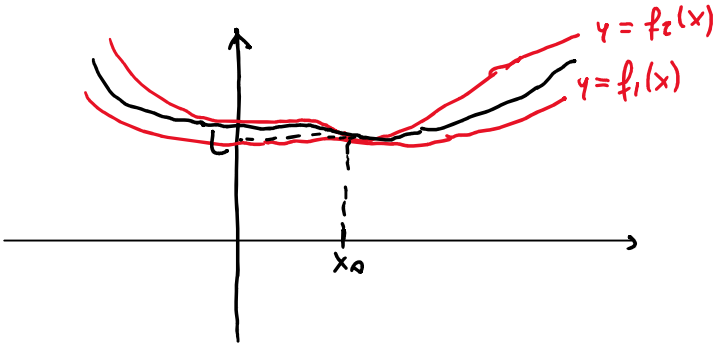
2) Se $\exists U \in \mathcal{I}_{x_0}$ t.c. $\forall x \in U \cap X \setminus \{x_0\}$: $f(x) \leq 0$, allora $L \leq 0$.

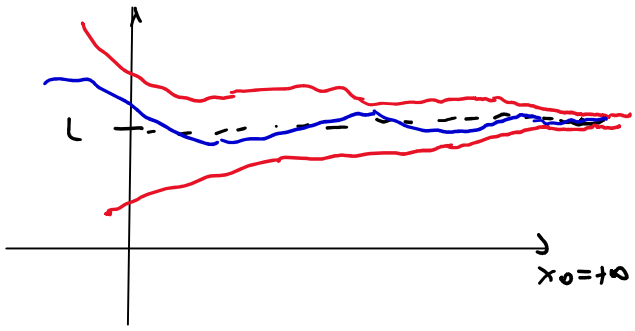
3) Se $\exists U \in \mathcal{I}_{x_0}$ t.c. $\forall x \in U \cap X \setminus \{x_0\}$: $f(x) \geq \alpha \in \mathbb{R}$ allora $L \geq \alpha$

4) Se $\exists U \in \mathcal{D}_{x_0}$ t.c. $\forall x \in U \cap X \setminus \{x_0\}$: $f(x) \leq \alpha \in \mathbb{R}$

allora $L \leq \alpha$

Teorema dei carabinieri





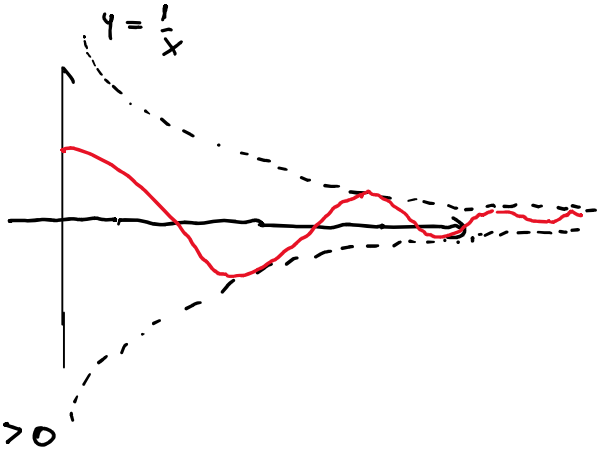
TEOREMA Sia $X \subseteq \mathbb{R}$, siano $f_1, f_2, g : X \longrightarrow \mathbb{R}$ e $x_0 \in D_r(X)$.

Assumiamo che:

1) $\exists\, U \in \mathcal{D}_{x_0}$ t.c. $\forall x \in U \cap X \setminus \{x_0\}$: $f_1(x) \leq g(x) \leq f_2(x)$

2) $\exists \lim_{x \to x_0} f_1(x) = \lim_{x \to x_0} f_2(x) = L \in \overline{\mathbb{R}}$

Allora $\exists \lim_{x \to x_0} g(x) = L$.



ESEMPIO

$$f(x) = \frac{\sin x}{x}, \quad x_0 = +\infty$$

$$-1 \leq \sin x \leq 1$$

$$\Rightarrow -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \quad \forall x > 0$$

$$\lim_{x \to +\infty} \frac{1}{x} = \frac{1}{+\infty} = 0 \quad , \quad \lim_{x \to +\infty} -\frac{1}{x} = -\frac{1}{+\infty} = 0$$

Per il teorema dei carabinieri, anche:

$$\lim_{x \to +\infty} \frac{\sin x}{x} = 0$$

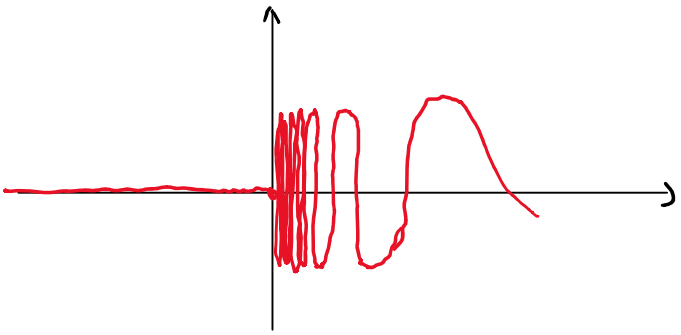
oss

Se $f(x)$ è una funzione limitata e $g(x)$ è infinitesima ($\lim_{x \to x_0} g(x) = 0$) allora

$\lim_{x \to x_0} f(x) g(x) = 0$.

Teorema di esistenza del limite

Abbiamo detto che ci sono casi in cui $\nexists \lim_{x \to x_0} f(x)$. Abbiamo visto che è anche possibile che $\nexists \lim_{x \to x_0^+} f(x)$ o $\nexists \lim_{x \to x_0^-} f(x)$



$$f(x) = \begin{cases} \sin \frac{1}{x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$\nexists \lim_{x \to 0^+} f(x)$.

Se escludiamo che i valori di f possano oscillare, richiedendo la monotonia, possiamo garantire l'esistenza del limite.

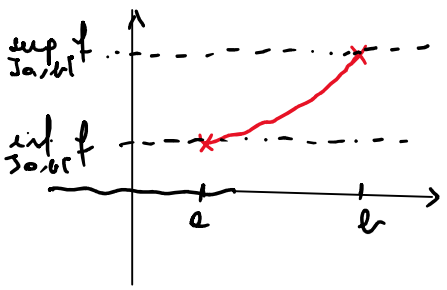
TEOREMA (DI ESISTENZA DEL LIMITE PER FUNZIONI MONOTONE)

Siano $a, b \in \mathbb{R}$, $a < b$. Sia $f :]a, b[\to \mathbb{R}$ una funzione.

1) Se f è monotona crescente in $]a, b[$ allora

$\exists \lim_{x \to b^-} f(x) = \sup_{]a,b[} f$

$\exists \lim_{x \to a^+} f(x) = \inf_{]a,b[} f$

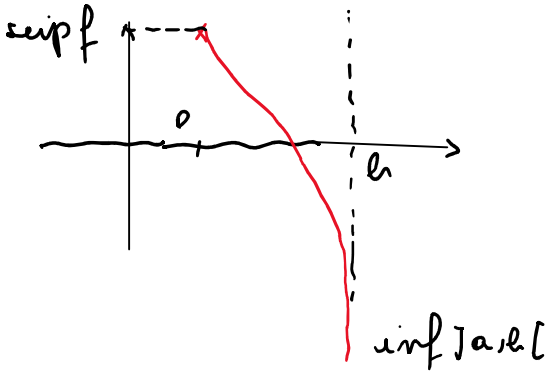


2) Se f è mon. decrescente in $]a,b[$

allora

$\exists \lim_{x \to b^-} f(x) = \inf_{]a,b[} f$

$\exists \lim_{x \to a^+} f(x) = \sup_{]a,b[} f$.



Teorema sul limite della composta

Siano $X, Y \subseteq \mathbb{R}$. Siano $f: X \to \mathbb{R}$ e $g: Y \to \mathbb{R}$ con $f(X) \subseteq Y$. Siano $x_0 \in Dr(X)$ $y_0 \in Dr(Y)$ e $L \in \overline{\mathbb{R}}$. Assumiamo

1) $\exists \lim_{x \to x_0} f(x) = y_0$

2) $\exists \lim_{y \to y_0} g(y) = L$

3) $\exists U \in \mathcal{I}_{x_0}$ t.c. $\forall x \in U \cap X \setminus \{x_0\}: f(x) \neq y_0$

$(\text{o } y_0 \in Y \text{ e } L = g(y_0))$

Allora $\lim_{x \to x_0} g(f(x)) = L$

Idea:

$$\lim_{x \to x_0} g(f(x)) \overset{y=f(x)}{=} \lim_{y \to y_0} g(y) = L$$

ESEMPI

- $\lim_{x \to +\infty} e^{\frac{x-1}{2x+1}}$

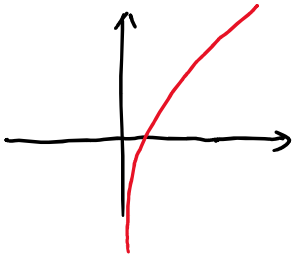
$$f(x) = \frac{x-1}{2x+1} \quad , \quad \lim_{x\to+\infty} f(x) = \frac{1}{2}$$

Il teorema ci dice che

$$\lim_{x\to+\infty} e^{\frac{x-1}{2x+1}} \overset{y=f(x)}{=} \lim_{y\to\frac{1}{2}} e^{y} = e^{\frac{1}{2}}$$

- $\lim_{x\to+\infty} \ln\left(\frac{1}{x} + e^{-x}\right) = \lim_{y\to 0} \ln y = -\infty$

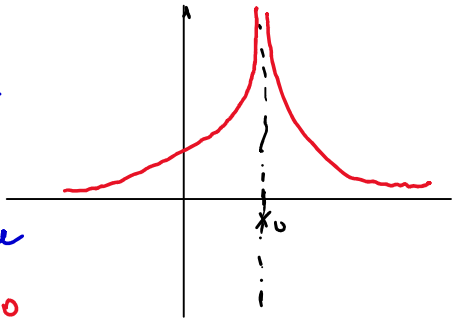
($\frac{1}{x} \to 0$, $e^{-x} \to 0$)



Asintoti di funzioni

Sono delle rette a cui il grafico di f si avvicina

Asintoti verticali per $x \to x_0 \in \mathbb{R}$.



Def: Siano $a, b \in \mathbb{R}$ con $a < b$.
Sia $f:]a, b[\longrightarrow \mathbb{R}$.
Sia $x_0 \in [a, b[$. Si dice che la retta $x = x_0$ è un ASINTOTO VERTICALE DESTRO (o PER DESTRA) di $f(x)$ se

$$\exists \lim_{x\to x_0^+} f(x) \in \{-\infty, +\infty\}$$

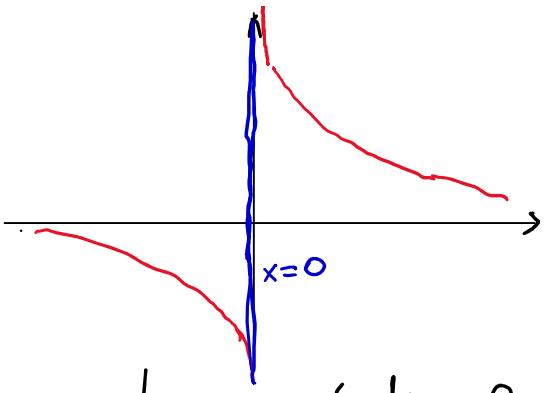
Sia $x_0 \in]a, b]$. Si dice che la retta $x = x_0$ è un ASINTOTO VERTICALE SINISTRO se $\exists \lim_{x\to x_0^-} f(x) \in \{-\infty, +\infty\}$.

ESEMPI

1) $f(x) = \frac{1}{x}$

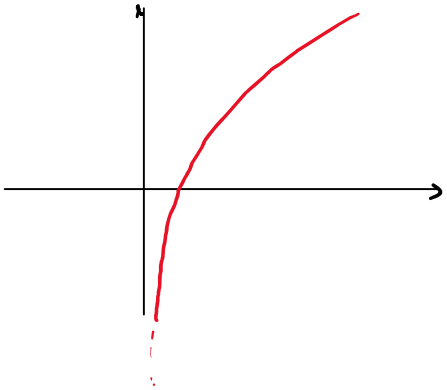
$\lim_{x \to 0^+} \frac{1}{x} = \frac{1}{0^+} = +\infty$

$\lim_{x \to 0^-} \frac{1}{x} = \frac{1}{0^-} = -\infty$



La retta $x = 0$ è un as. vert sia destro che sinistro

2) $f(x) = \ln x$



$\lim_{x \to 0^+} f(x) = -\infty$

$x = 0$ è un as. vert. destro.

Asintoti orizzontali per $x \to +\infty$ e $x \to -\infty$

Def: Sia $a \in \mathbb{R}$ e $f:]a, +\infty[\longrightarrow \mathbb{R}$ ($f:]-\infty, a[\to \mathbb{R}$)
si dice che la retta $y = L$ con $L \in \mathbb{R}$ è un
ASINTOTO ORIZZONTALE di f per $x \to +\infty$ ($x \to -\infty$)
se $\exists \lim_{x \to +\infty} f(x) = L \in \mathbb{R}$ ($\lim_{x \to -\infty} f(x) = L$)

ESEMPI

- $f(x) = \frac{1}{x}$

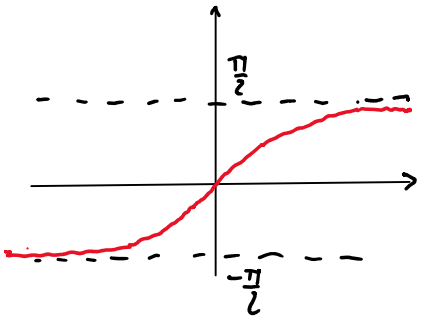
La retta $y = 0$ è un as. orizzontale sia per $x \to +\infty$ sia per $x \to -\infty$ perché

$\lim_{x \to +\infty} \frac{1}{x} = 0$, $\lim_{x \to -\infty} \frac{1}{x} = 0$

- $f(x) = \arctan x$

$\lim_{x \to +\infty} \arctan x = \frac{\pi}{2}$

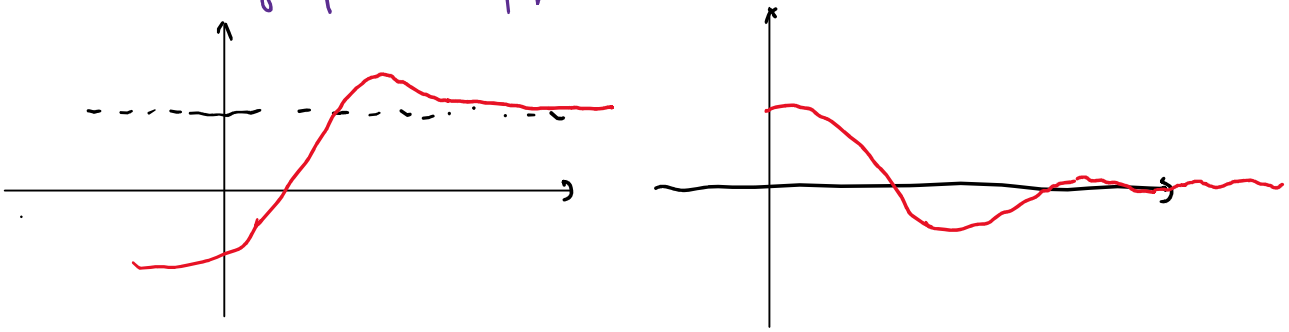
$\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$



La retta $y = \frac{\pi}{2}$ è un asintoto orizzontale per $x \to +\infty$

La retta $y = -\frac{\pi}{2}$ è un asintoto orizzontale per $x \to -\infty$

- Nota: Il grafico di f può intersecare i suoi asintoti



$f(x) = \frac{\sin x}{x}$ per $x \to +\infty$ la retta $y = 0$ è un asintoto orizzontale che viene attraversato infinite volte dal grafico di $f(x)$.

Asintoti obliqui

Def: Sia $a \in \mathbb{R}$, $f:]a, +\infty[\to \mathbb{R}$ (o $f:]-\infty, a[\to \mathbb{R}$)

si dice che la retta $y = mx + q$ con $m \neq 0$ è un ASINTOTO OBLIQUO di f per $x \to +\infty$ ($x \to -\infty$) se

$\lim_{x \to +\infty} (f(x) - mx - q) = 0$

$\left(\lim_{x \to -\infty} f(x) - mx - q \right)$

In tal caso:

$m = \lim_{x \to +\infty} \frac{f(x)}{x}$ $\qquad q = \lim_{x \to +\infty} (f(x) - mx)$

$\left(m = \lim_{x \to -\infty} \frac{f(x)}{x} \text{ e } q = \lim_{x \to -\infty} (f(x) - mx) \right)$

ESEMPIO

$f(x) = \sqrt{x^2+1}$

$\lim_{x \to +\infty} f(x) = +\infty$

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x \to +\infty} \sqrt{\frac{x^2+1}{x^2}}$$

$$= \lim_{x \to +\infty} \sqrt{1 + \frac{1}{x^2}} = \sqrt{1} = 1 \qquad (m = 1).$$

$$\lim_{x \to +\infty} \sqrt{x^2+1} - x = \lim_{x \to +\infty} \left(\sqrt{x^2+1} - x\right) \cdot \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}$$

$$= \lim_{x \to +\infty} \frac{x^2+1 - x^2}{\sqrt{x^2+1} + x} = \frac{1}{+\infty} = 0 \qquad (q = 0).$$

La retta $y = x$ è un asintoto obliquo.

