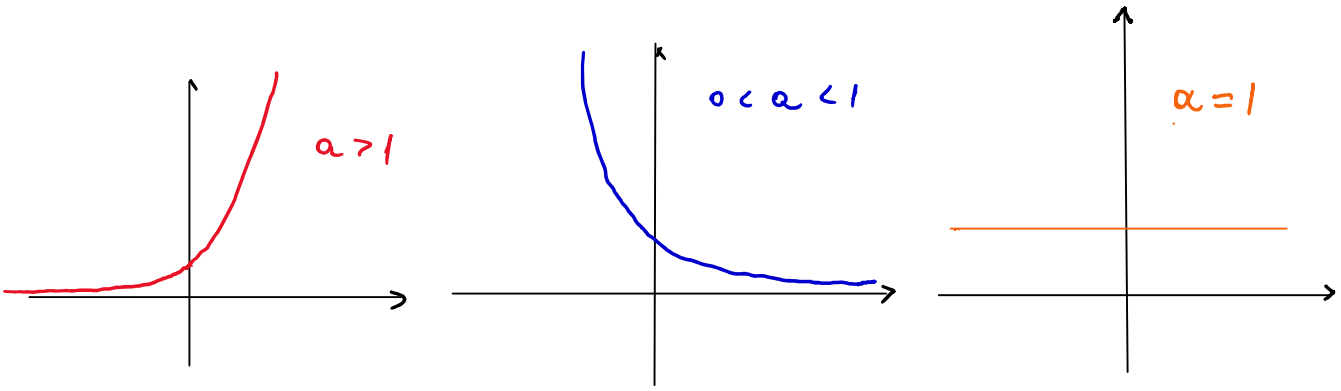
MATEMATICA - LEZIONE 19

lunedì 2 novembre 2026 09:07

Limiti di funzioni elementari

- Esponenziali $f(x) = a^x$ con $a > 0$

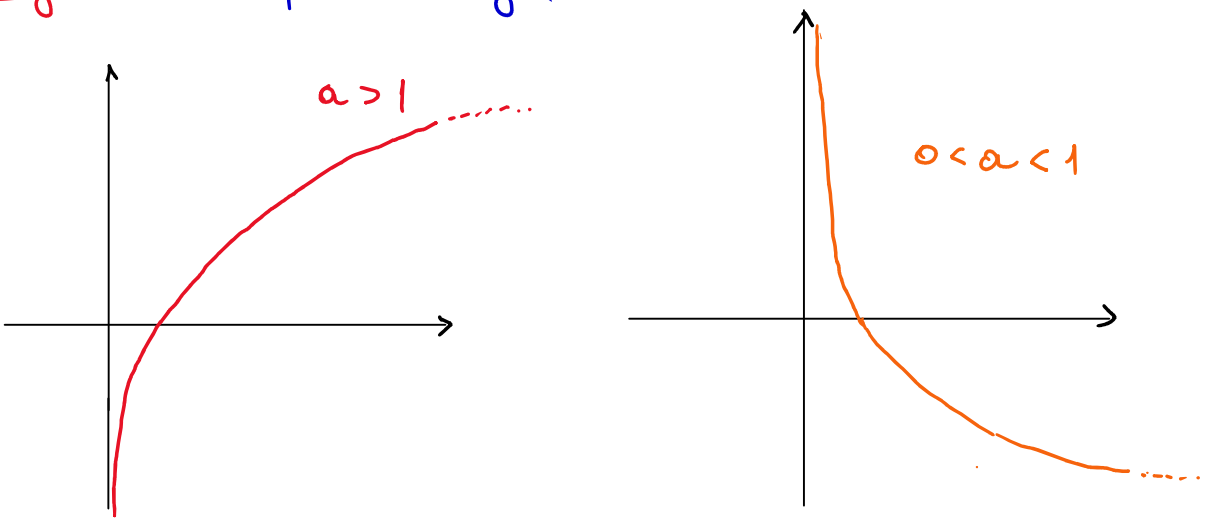


- $\forall\, x_0 \in \mathbb{R}: \lim_{x \to x_0} a^x = a^{x_0}$ (in tutti e tre i casi)

- $\lim_{x \to +\infty} a^x = \begin{cases} +\infty & \text{se } a > 1 \\ 0 & \text{se } 0 < a < 1 \\ 1 & \text{se } a = 1 \end{cases}$

- $\lim_{x \to -\infty} a^x = \begin{cases} 0 & \text{se } a > 1 \\ +\infty & \text{se } 0 < a < 1 \\ 1 & \text{se } a = 1 \end{cases}$

Logaritmi $f(x) = \log_a x$ con $a > 0$, $a \neq 1$

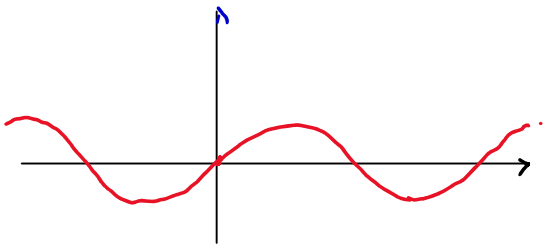


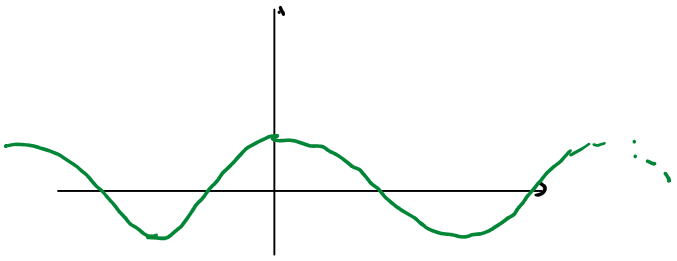
- $\lim_{x \to x_0} \log_a x = \log_a x_0$ se $x_0 \in \mathbb{R}$, $x_0 > 0$
- $\lim_{x \to +\infty} \log_a x = \begin{cases} +\infty & \text{se } a > 1 \\ -\infty & \text{se } 0 < a < 1 \end{cases}$
- $\lim_{x \to 0} \log_a x = \lim_{x \to 0^+} \log_a x = \begin{cases} -\infty & \text{se } a > 1 \\ +\infty & \text{se } 0 < a < 1 \end{cases}$

Funzioni trigonometriche

Seno e coseno

$f(x) = \sin x$ e $f(x) = \cos x$





- $\lim_{x \to x_0} \sin x = \sin x_0$
- $\nexists \lim_{x \to +\infty} \sin x$
- $\nexists \lim_{x \to -\infty} \sin x$

$\lim_{x \to x_0} \cos x = \cos x_0$

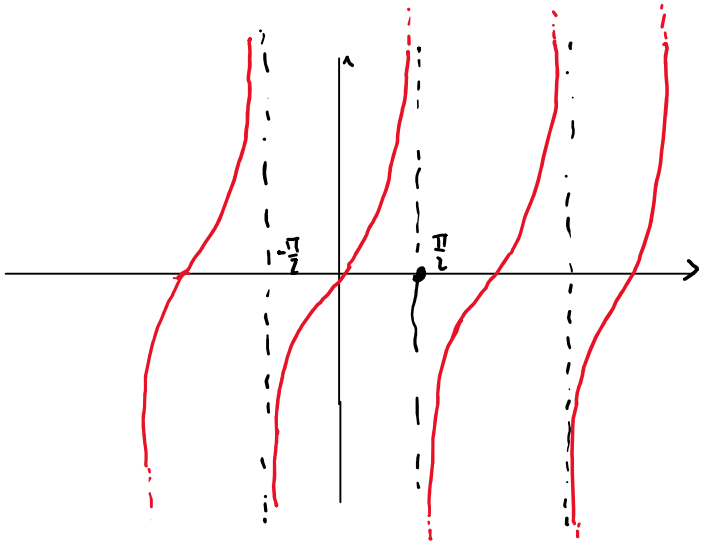
$\nexists \lim_{x \to +\infty} \cos x$

$\nexists \lim_{x \to -\infty} \cos x$

Tangente

$f(x) = \tan x$

- $\nexists \lim_{x \to +\infty} \tan x$
- $\nexists \lim_{x \to -\infty} \tan x$



- $\lim_{x \to x_0} \tan x = \tan x_0$

  se $x_0 \in Dom(f)$.

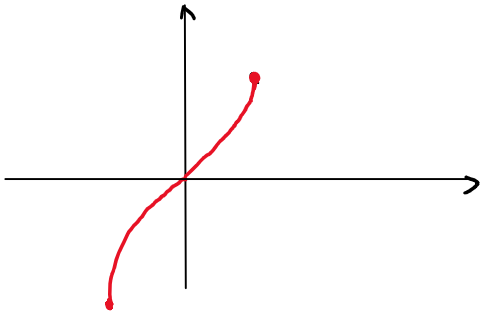
  $= \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \mid k \in \mathbb{Z} \right\}$

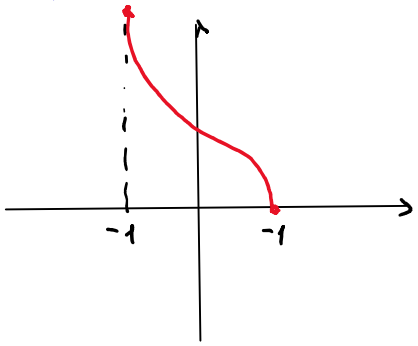
- $\forall\, k \in \mathbb{Z}$: $\lim_{x \to \left(\frac{\pi}{2} + k\pi\right)^-} \tan x = +\infty$

  $\lim_{x \to \left(\frac{\pi}{2} + k\pi\right)^+} \tan x = -\infty$

- Inverse delle funzioni trigonometriche

$f(x) = \arcsin x$ e $f(x) = \arccos x$





$\forall\, x_0 \in [-1, 1]$: $\lim_{x \to x_0} \arcsin x = \arcsin x_0$

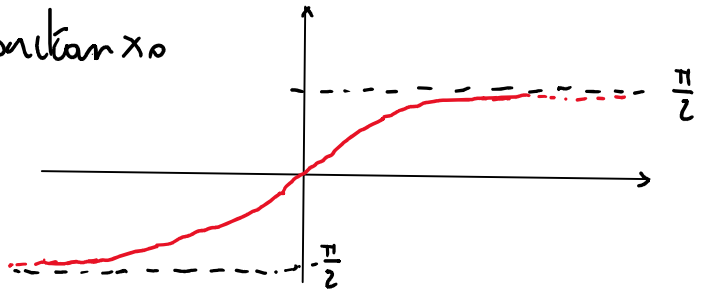
e $\lim_{x \to x_0} \arccos x = \arccos x_0$

- $f(x) = \arctan x$

$\forall\, x_0 \in \mathbb{R}$: $\lim_{x \to x_0} \arctan x = \arctan x_0$

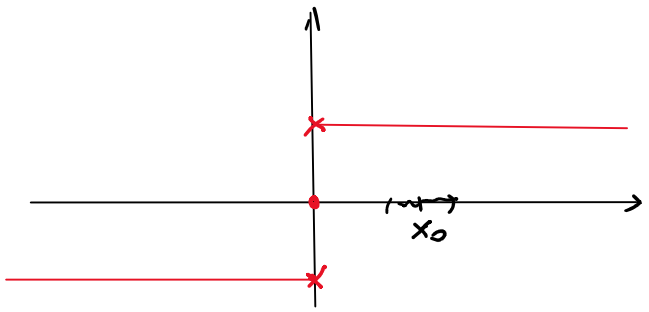
$\lim_{x \to +\infty} \arctan x = \frac{\pi}{2}$

$\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$



- Funzione segno

$$f(x) = \begin{cases} 1 & \text{se } x > 0 \\ -1 & \text{se } x < 0 \\ 0 & \text{se } x = 0 \end{cases}$$



Se $x_0 \in \mathbb{R}$, $x_0 > 0$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} 1 = 1$$

Se $x_0 \in \mathbb{R}$, $x_0 < 0$: $\lim_{x \to x_0} f(x) = -1$.

Se $x_0 = 0$. $\lim_{x \to 0^+} f(x) = 1$ e $\lim_{x \to 0^-} f(x) = -1$

(e $f(0) = 0$, quindi non è vero che $\lim_{x \to 0} f(x) = f(0)$)

Si noti anche che: $\lim_{x \to +\infty} f(x) = 1$ e $\lim_{x \to -\infty} f(x) = -1$

Operazioni tra limiti

TEOREMA

Siano $f, g : X \longrightarrow \mathbb{R}$, $X \subseteq \mathbb{R}$. Siano $x_0 \in Dr(X)$ e $L_1, L_2 \in \mathbb{R}$. Supponiamo $\lim_{x \to x_0} f(x) = L_1$ e $\lim_{x \to x_0} g(x) = L_2$. Allora

- $\lim_{x \to x_0} \left(f(x) \pm g(x) \right) = L_1 \pm L_2$
- $\lim_{x \to +\infty} f(x)\, g(x) = L_1 \cdot L_2$

- Se $L_2 \neq 0$, allora $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$

- Se $L_1 > 0$ e $L_2 \in \mathbb{R}$: $\lim_{x \to x_0} (f(x))^{g(x)} = L_1^{L_2}$

In molti casi il teorema vale anche quando L_1 e L_2 sono $\pm\infty$.

Operazioni tra infinità:

- $L_1 = +\infty$ e $L_2 \in \mathbb{R}$: $L_1 + L_2 = +\infty$
- $L_1 = -\infty$ e $L_2 \in \mathbb{R}$: $L_1 + L_2 = -\infty$
- $L_1 = L_2 = +\infty$: $L_1 + L_2 = +\infty$
- $L_1 = L_2 = -\infty$: $L_1 + L_2 = -\infty$
- $L_1 = +\infty$ e $L_2 > 0$: $L_1 L_2 = +\infty$ e $\frac{L_1}{L_2} = +\infty$
- $L_1 = +\infty$ e $L_2 < 0$: $L_1 \cdot L_2 = -\infty$ e $\frac{L_1}{L_2} = -\infty$
- $L_1 \in \mathbb{R}$ e $L_2 = \pm\infty$: $\frac{L_1}{L_2} = 0$

E se $L_2 = 0$? È necessario conoscere il segno del denominatore:

Def: Se $f: X \to \mathbb{R}$, $X \subseteq \mathbb{R}$. Se $x_0 \in D_r(X)$ e $\lim_{x \to x_0} f(x) = 0$, si dice che $f(x)$ è INFINITESIMA per $x \to x_0$. Se inoltre $\exists\, U \in \mathcal{I}_{x_0}$ tale che $f(x) > 0 \ \forall x \in U \cap X \setminus \{x_0\}$, si dice che f è INFINITESIMA POSITIVA e si scrive che $\lim_{x \to x_0} f(x) = 0^+$.

Se invece $\exists\, U \in \mathcal{I}_{x_0}$ t.c. $\forall x \in U \cap X \setminus \{x_0\}$: $f(x) < 0$, si dice che $f(x)$ è INFINITESIMA NEGATIVA per $x \to x_0$ e

si scrive $\lim_{x \to x_0} f(x) = 0^-$

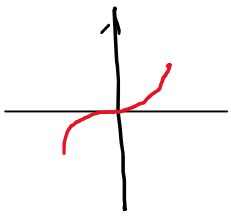
ESEMPI

1) $\lim_{x \to 0} x^2 = 0^+$

2) $\lim_{x \to -\infty} e^x = 0^+$

3) $\lim_{x \to 0} \left(\underbrace{\sqrt{1-x^2}}_{<1} - 1 \right) = 0^-$

4) $\lim_{x \to 0} x^3 = 0$ $\quad (\text{ne } 0^+ \text{ ne } 0^-)$



$\lim_{x \to 0^+} x^3 = 0^+$

$\lim_{x \to 0^-} x^3 = 0^-$

5) $\lim_{x \to 1^+} 1 - x = 0^-$ e $\lim_{x \to 1^-} 1 - x = 0^+$

<u>oss</u> Siano $f, g : X \to \mathbb{R}$ con $X \subseteq \mathbb{R}$ e sia $x_0 \in D_R(X)$.

Se $\lim_{x \to x_0} f(x) = L_1 \neq 0$ (anche $+\infty$ o $-\infty$) e $\lim_{x \to x_0} g(x) = 0^+$, allora:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{0^+} = \begin{cases} +\infty & \text{se } L_1 > 0 \\ -\infty & \text{se } L_1 < 0 \end{cases}$$

Se $\lim_{x \to x_0} g(x) = 0^-$, allora:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{0^-} = \begin{cases} -\infty & \text{se } L_1 > 0 \\ +\infty & \text{se } L_1 < 0. \end{cases}$$

- $\lim_{x \to 0^+} \dfrac{x-1}{x} = \dfrac{-1}{0^+} = -\infty$

Forme indeterminate

- $+\infty - \infty \quad (-\infty + \infty)$
- $\pm\infty \cdot 0$
- $\dfrac{0}{0}$ e $\dfrac{\pm\infty}{\pm\infty}$
- 0^0, $(+\infty)^0$, $1^{\pm\infty}$

Limiti con polinomi

ESEMPI

- $\lim_{x \to +\infty} (x^2 - x^3) \qquad (+\infty - (+\infty) \quad \text{f.i.})$

$= \lim_{x \to +\infty} x^3 \left(\underbrace{\frac{1}{x}}_{\to \frac{1}{+\infty} = 0} - 1 \right) = +\infty \cdot (-1) = -\infty.$

- $\lim_{x \to -\infty} x^2 - x^3 = \lim_{x \to -\infty} \underbrace{x^3}_{\to -\infty} \left(\underbrace{\frac{1}{x}}_{\to \frac{1}{-\infty} = 0} - 1 \right)$

$= -\infty \cdot (-1) = +\infty.$

In generale Se $p(x) = a_n x^n + \dots + a_1 x + a_0$ con $a_n \neq 0$, allora $\lim_{x \to \pm\infty} p(x) = \lim_{x \to \pm\infty} a_n x^n$

(**Infatti**: $a_n x^n + \dots + a_1 x + a_0$

$= a_n x^n \left(1 + \underbrace{\frac{a_{n-1}}{a_n} \frac{1}{x}}_{\to 0} + \dots + \underbrace{\frac{a_1}{x^{n-1}}}_{\to 0} + \underbrace{\frac{a_0}{a_n x^n}}_{\to 0} \right)$

Rapporti tra polinomi

- $\lim_{x\to+\infty} \dfrac{x^4+2x}{5x^4+3x^2+1} = \lim_{x\to+\infty} \dfrac{x^4\left(1+\frac{2}{x^3}\right)}{x^4\left(5+\frac{3}{x^2}+\frac{1}{x^4}\right)} = \dfrac{1}{5}$

  ($\frac{2}{x^3}\to 0$, $\frac{3}{x^2}\to 0$, $\frac{1}{x^4}\to 0$)

$$\left(\lim_{x\to+\infty} \frac{x^4+2x}{5x^4+3x^2+1} = \lim_{x\to+\infty} \frac{x^4}{5x^4} = \frac{1}{5}\right)$$

- $\lim_{x\to+\infty} \dfrac{x^4+2x}{x^4+3x^5-2} = \lim_{x\to+\infty} \dfrac{x^4}{3x^5} = \lim_{x\to+\infty} \dfrac{1}{3x} = \dfrac{1}{+\infty} = 0$

- $\lim_{x\to+\infty} \dfrac{x^4+2x}{x-4x^3+2} = \lim_{x\to+\infty} \dfrac{x^4}{-4x^3} = \lim_{x\to+\infty} -\dfrac{x}{4} = -\infty.$

In generale Se $p(x) = a_n x^n + \dots + a_1 x + a_0$ con $a_n \neq 0$
e $q(x) = b_m x^m + \dots + b_1 x + b_0$ con $b_m \neq 0$,

allora:

$$\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \begin{cases} \frac{a_n}{b_n} & \text{se } n = m \\ 0 & \text{se } m > n \\ \infty \text{ con segno da determinare} & \text{se } n > m \end{cases}$$

- $\lim_{x\to-\infty} \dfrac{3x-1}{4x+2} = \dfrac{3}{4}$

- $\lim_{x\to-\infty} \dfrac{x^4+2}{x^2-2x^3} = \lim_{x\to-\infty} \dfrac{x^4}{-2x^3} = \lim_{x\to-\infty} -\dfrac{x^4}{2} = -\infty.$

  ($x\to-\infty \Rightarrow x^4 \to +\infty$)

- $\lim_{x\to+\infty} \dfrac{5x^4+3}{x^8+5} = 0$ $\quad\left(\lim_{x\to+\infty} \frac{5x^4}{x^8} = \lim_{x\to+\infty} \frac{5}{x^4} = 0\right)$

- $\lim_{x \to -\infty} \frac{x}{e^x + 2^x} = \frac{-\infty}{0^+} = -\infty$

Rapporti tra polinomi infinitesimi

- $\lim_{x \to 1} \frac{x^2 - 1}{x^2 + x - 2} = \frac{0}{0}$ f.i.

Idea: fattorizziamo i due polinomi:

$= \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)(x+2)}$

$= \lim_{x \to 1} \frac{x+1}{x+2} = \frac{2}{3}$

$x^2 + x - 2 = 0$

$x_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \begin{cases} 1 \\ -2 \end{cases}$

$x^2 + x - 2 = (x-1)(x+2)$

- $\lim_{x \to 1} \frac{x^2 - 2x + 1}{x^2 + x - 2} = \lim_{x \to 1} \frac{(x-1)^2}{(x-1)(x+2)} = \frac{0}{3} = 0.$

- $\lim_{x \to 1} \frac{x^2 + x - 2}{x^3 - 2x + 1} = \lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)^2} = \lim_{x \to 1} \frac{x+2}{x-1}$

$= \frac{3}{0} \to \text{ne}^- \quad 0^+ \quad \text{ne}^- \quad 0^-$

$\lim_{x \to 1^+} \frac{x+2}{x-1} = \frac{3}{0^+} = +\infty$ ($\lim_{x \to 1}$ non esiste)

$\lim_{x \to 1^-} \frac{x+2}{x-1} = \frac{3}{0^-} = -\infty$

- $\lim_{x \to 1} \frac{x^2 + x - 2}{x^3 - 3x^2 + 3x - 1} = \lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)^3}$

$= \lim_{x \to 1} \frac{x+2}{(x-1)^2} = \frac{3}{0^+} = +\infty.$

- $\lim_{x \to 0} \dfrac{x^2 + x^3 + 4x^4}{x^7 + 2x^2} = \lim_{x \to 0} \dfrac{x^2(1 + x + 4x^2)}{x^2(x^5 + 2)}$

$= \lim_{x \to 0} \dfrac{1 + x + 4x^2}{x^5 + 2} = \dfrac{1}{2}$

Ricordare Per $x \to 0$ nei polinomi contano di più le potenze con esponente più basso.

Attenzione a non cancellare termini importanti

$$\lim_{x \to +\infty} \frac{x^6 + 2x^5 - (x^3+1)(x^3+2)}{x^5 + 3}$$

$$= \lim_{x \to +\infty} \frac{x^6 + 2x^5 - (x^6 + x^3 + 2x^3 + 2)}{x^5 + 3}$$

$$= \lim_{x \to +\infty} \frac{x^6 + 2x^5 - x^6 - 3x^3 - 2}{x^5 + 3}$$

$$= \lim_{x \to +\infty} \frac{2x^5}{x^5} = 2.$$

Gerarchia degli infiniti per $x \to +\infty$

Per $x \to +\infty$, x^n cresce più rapidamente di x^m se $n > m$ $\left(\lim_{x \to +\infty} \dfrac{x^n}{x^m} = +\infty \right)$

Questi ragionamenti si estendono ad altri tipi di funzioni

Per $x \to +\infty$ le quantità: $\log_a x$ con $a>1$

x^α con $\alpha \in \mathbb{R}$, $\alpha > 0$

a^x con $a>1$

x^x

sono tutte infinite. È possibile stabilire una gerarchia degli infiniti:

- $\forall a > 1$ e $\forall \alpha > 0$: $\log_a x$ cresce meno di x^α cioè $\lim_{x\to+\infty} \frac{x^\alpha}{\log_a x} = +\infty$ $\left(\text{o } \lim_{x\to+\infty} \frac{\log_a x}{x^\alpha} = 0\right)$
- Se $0 < \alpha < \beta$: x^β cresce più di x^α.
- Se $a > 1$ e $\alpha > 0$: a^x cresce più di x^α cioè $\lim_{x\to+\infty} \frac{a^x}{x^\alpha} = +\infty$ $\left(\text{o } \lim_{x\to+\infty} \frac{x^\alpha}{a^x} = 0\right)$
- Se $a > 1$, x^x cresce più di a^x. cioè: $\lim_{x\to+\infty} \frac{x^x}{a^x} = +\infty$ $\left(\text{o } \lim_{x\to+\infty} \frac{a^x}{x^x} = 0\right)$

ESEMPI

- $\lim_{x\to+\infty} \frac{e^x - x^{10}}{\pi + x^{14} - e^x} =$

$= \lim_{x\to+\infty} \frac{e^x\left(1 - \frac{x^{10}}{e^x}\right)}{e^x\left(\frac{\pi}{e^x} + \frac{x^{14}}{e^x} - 1\right)} = \frac{1}{-1} = -1$

$\frac{x^{10}}{e^x} \to 0$ per $x\to+\infty$ per la gerarchia.

$\frac{\pi}{e^x} \to \frac{\pi}{+\infty} = 0$

$\frac{x^{14}}{e^x} \to 0$ gerarchia

- $\lim_{x \to +\infty} \underbrace{x^{120}}_{\to +\infty} \underbrace{- \log_2 x}_{\to -\infty} + \underbrace{4^x}_{\to +\infty}$

$$= \lim_{x \to +\infty} 4^x \left(\frac{x^{120}}{4^x} - \frac{\log_2 x}{4^x} + 1 \right) = +\infty \cdot 1 = +\infty.$$

($4^x \to +\infty$, $\frac{x^{120}}{4^x} \to 0$, $\frac{\log_2 x}{4^x} \to 0$)

- $\lim_{x \to +\infty} \sqrt{x} - \ln x$ f.i. $+\infty - (+\infty)$

$$= \lim_{x \to +\infty} x^{\frac{1}{2}} - \ln x = \lim_{x \to +\infty} x^{\frac{1}{2}} \left(1 - \frac{\ln x}{x^{\frac{1}{2}}} \right)$$

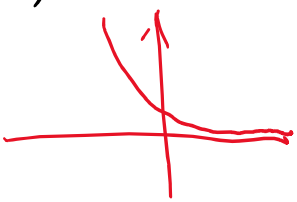
$$= \lim_{x \to +\infty} +\infty \cdot 1 = +\infty.$$

- $\lim_{x \to +\infty} 4^x - 5^x = \lim_{x \to +\infty} 5^x \left(\frac{4^x}{5^x} - 1 \right)$

$$= \lim_{x \to +\infty} 5^x \left(\left(\frac{4}{5}\right)^x - 1 \right)$$

($\left(\frac{4}{5}\right)^x \to 0$)

$$= +\infty \cdot (-1) = -\infty.$$



- $\lim_{x \to -\infty} x^3 - 10^x$ ($10^{-\infty} = 0$)

$$= -\infty - 0$$

$$= -\infty$$

