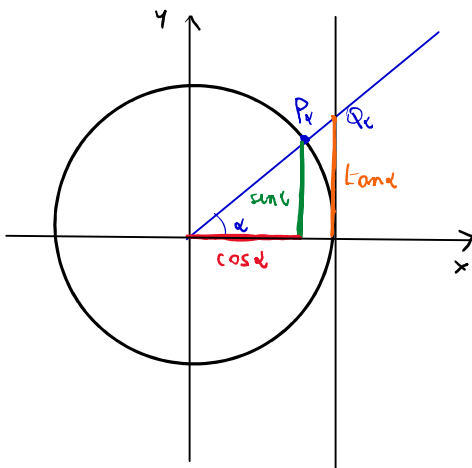


MATEMATICA - LEZIONE 11

lunedì 19 ottobre 2026 09:07



$$P_\alpha = (\cos \alpha, \sin \alpha)$$

$$Q_\alpha = (1, \tan \alpha)$$

α	$\cos \alpha$	$\sin \alpha$	$\tan \alpha$
0	1	0	0
$\frac{\pi}{2}$	0	1	$\frac{1}{0}$ non definita
π	-1	0	$\frac{0}{-1} = 0$
$\frac{3}{2}\pi$	0	-1	n.d.
2π	1	0	0
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{3}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1
$\frac{5}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
$\frac{7}{4}\pi$	$+\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$

Formule utili:

1) FORMULE DI ADDIZIONE E SOTTRAZIONE

$$\forall \alpha, \beta \in \mathbb{R} :$$

$$\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta.$$

2) FORMULE DI DUPLICAZIONE

$$\forall \alpha \in \mathbb{R} : \cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha = 1 - 2\sin^2 \alpha = 2\cos^2 \alpha - 1$$

$$\sin(2\alpha) = 2\sin \alpha \cos \alpha$$

3) FORMULE DI BISEZIONE

$\forall \alpha \in \mathbb{R} :$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

con i segni da determinare in base alla posizione del punto che corrisponde ad $\frac{\alpha}{2}$ su $\mathbb{C}$.

4) FORMULE PARAMETRICHE

Sia $\alpha \in \mathbb{R}$ e sia $t = \tan \frac{\alpha}{2}$, $\alpha \neq \pi + 2k\pi$.

Allora

$$\sin \alpha = \frac{2t}{1+t^2}, \quad \cos \alpha = \frac{1-t^2}{1+t^2} \quad \text{e} \quad \tan \alpha = \frac{2t}{1-t^2} \quad (\text{se } t \neq \pm 1)$$

DIM

$$\sin \alpha = \sin \left(2 \frac{\alpha}{2} \right) = 2 \sin \left(\frac{\alpha}{2} \right) \cos \left(\frac{\alpha}{2} \right) = \frac{2 \sin \left(\frac{\alpha}{2} \right) \cos \left(\frac{\alpha}{2} \right)}{1}$$

$$= \frac{2 \sin \left(\frac{\alpha}{2} \right) \cos \left(\frac{\alpha}{2} \right)}{\cos^2 \left(\frac{\alpha}{2} \right) + \sin^2 \left(\frac{\alpha}{2} \right)}$$

$$= \frac{2 \sin \left(\frac{\alpha}{2} \right) \cancel{\cos \left(\frac{\alpha}{2} \right)}}{\cancel{\cos \left(\frac{\alpha}{2} \right)} \left(1 + \frac{\sin^2 \left(\frac{\alpha}{2} \right)}{\cos^2 \left(\frac{\alpha}{2} \right)} \right)} = \frac{2 \frac{\sin \left(\frac{\alpha}{2} \right)}{\cos \left(\frac{\alpha}{2} \right)}}{1 + \frac{\sin^2 \left(\frac{\alpha}{2} \right)}{\cos^2 \left(\frac{\alpha}{2} \right)}}$$

$$= \frac{2t}{1+t^2}$$

$$\cos \alpha = \cos \left(2 \frac{\alpha}{2} \right) = \cos^2 \left(\frac{\alpha}{2} \right) - \sin^2 \left(\frac{\alpha}{2} \right)$$

$$= \frac{\cos^2 \left(\frac{\alpha}{2} \right) - \sin^2 \left(\frac{\alpha}{2} \right)}{\cos^2 \left(\frac{\alpha}{2} \right) + \sin^2 \left(\frac{\alpha}{2} \right)} = \frac{1-t^2}{1+t^2}$$

$$\tan 2 = \frac{\sin 2}{\cos 2} = \frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}} = \frac{2t}{1+t^2} \cdot \frac{1+t^2}{1-t^2} = \frac{2t}{1-t^2}$$

Equazioni elementari con $\sin / \cos / \tan$

1) $\sin x = c$ con $c \in \mathbb{R}$.

• Se $c > 1$ o $c < -1$ $\nexists x \in \mathbb{R}$ tale che $\sin x = c$.

• Se $c \in [-1, 1]$ $\exists \theta \in \mathbb{R}$ t.c. $\sin \theta = c$ e le soluzioni di $\sin x = c$ sono

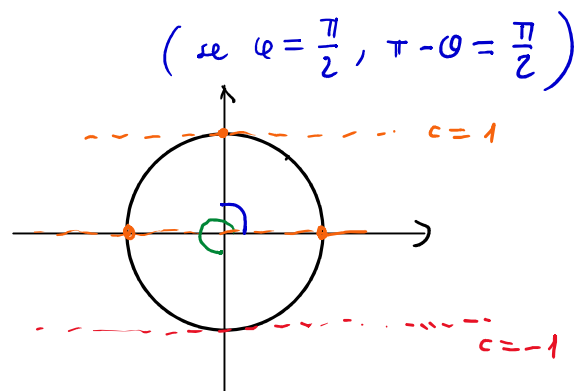
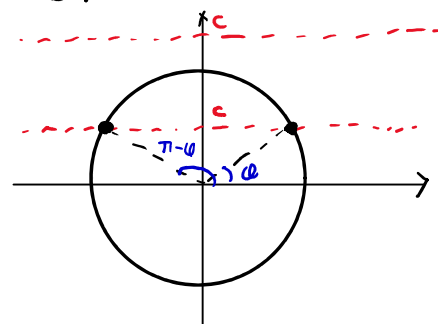
$$x = \theta + 2k\pi \quad \text{o} \quad x = \pi - \theta + 2k\pi \quad \text{con } k \in \mathbb{Z}$$

oss Tre casi particolari:

• $c = 1$: $x = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$

• $c = -1$: $x = -\frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$.

• $c = 0$: $x = 2k\pi \vee x = \pi + 2k\pi$
cioè $x = k\pi$ con $k \in \mathbb{Z}$.



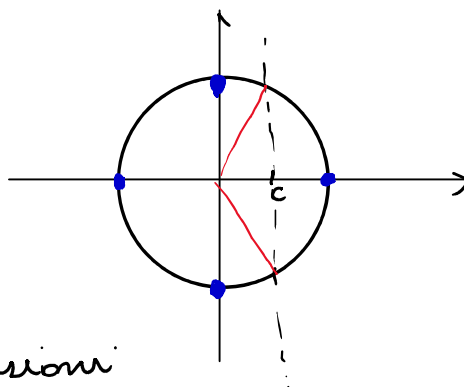
2) $\cos x = c$ con $c \in \mathbb{R}$

• Se $c > 1$ o $c < -1$ non ci sono soluzioni.

• Se $c \in [-1, 1]$, allora esiste una soluzione θ e tutte le soluzioni sono del tipo:

$$x = \theta + 2k\pi \quad \vee \quad x = -\theta + 2k\pi$$

$$(\text{o } x = 2\pi - \theta + 2k\pi)$$



OSS

• $c = 1$: $x = 2k\pi$ con $k \in \mathbb{Z}$

• $c = -1$ $x = \pi + 2k\pi$ con $k \in \mathbb{Z}$.

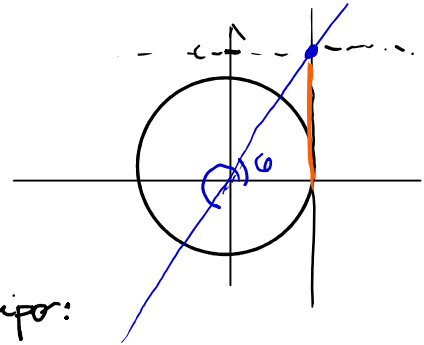
• $c = 0$ $x = \frac{\pi}{2} + 2k\pi \vee x = \frac{3}{2}\pi + 2k\pi$

coe: $x = \frac{\pi}{2} + k\pi$.

3) $\tan x = c$ con $c \in \mathbb{R}$:

Qualunque sia il valore di c $\exists$ almeno una soluzione e tutte le soluzioni sono del tipo:

$x = \theta + k\pi$ con $k \in \mathbb{Z}$.



ESEMPLI

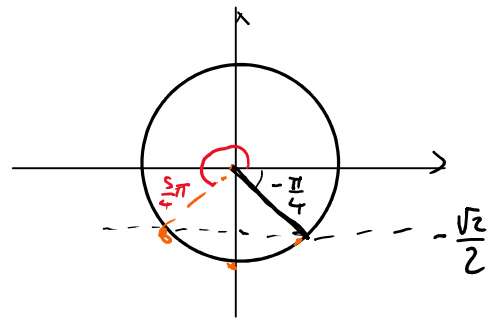
1) $\sin x = -\frac{\sqrt{2}}{2}$ ($\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = \sin(\frac{\pi}{4})$)

$\sin x = \sin(-\frac{\pi}{4})$

$x = -\frac{\pi}{4} + 2k\pi \vee x = \pi - (-\frac{\pi}{4}) + 2k\pi$ con $k \in \mathbb{Z}$

$x = -\frac{\pi}{4} + 2k\pi \vee x = \frac{5}{4}\pi + 2k\pi$.

(o $x = \frac{7}{4}\pi + 2k\pi \vee x = \frac{5}{4}\pi + 2k\pi$)

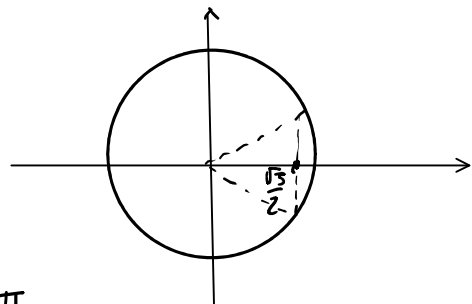


2) $\cos x = \frac{\sqrt{3}}{2}$

$\cos x = \cos(\frac{\pi}{6})$

$x = \frac{\pi}{6} + 2k\pi \vee x = -\frac{\pi}{6} + 2k\pi$

$x = \frac{\pi}{6} + 2k\pi \vee x = \frac{11}{6}\pi + 2k\pi$



$$3) \quad \tan x = 1$$

$$\tan x = \tan \frac{\pi}{4}$$

$$x = \frac{\pi}{4} + k\pi \quad \text{con } k \in \mathbb{Z}.$$

$$4) \quad \sin(3x) = 1$$

$$y = 3x$$

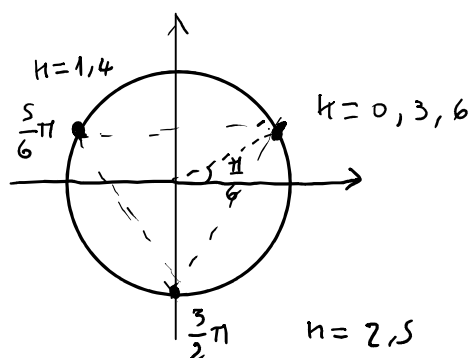
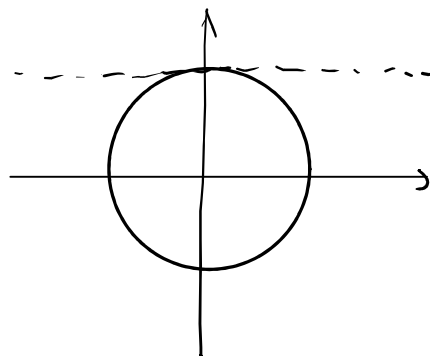
$$\sin y = 1$$

$$\sin y = \sin \frac{\pi}{2}$$

$$y = \frac{\pi}{2} + 2k\pi \quad k \in \mathbb{Z}$$

$$3x = \frac{\pi}{2} + 2k\pi \quad k \in \mathbb{Z}$$

$$x = \frac{\pi}{6} + \frac{2k\pi}{3} \quad k \in \mathbb{Z}.$$



$$5) \quad \sin x + \sin(2x) = 0$$

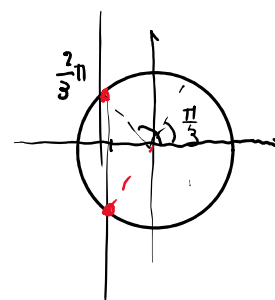
$$\sin x + 2\sin x \cos x = 0$$

$$\sin x (1 + 2\cos x) = 0$$

$$\sin x = 0 \quad \vee \quad 1 + 2\cos x = 0$$

$$x = k\pi \quad \text{con } k \in \mathbb{Z} \quad \vee \quad \cos x = -\frac{1}{2} = \cos\left(\frac{2}{3}\pi\right)$$

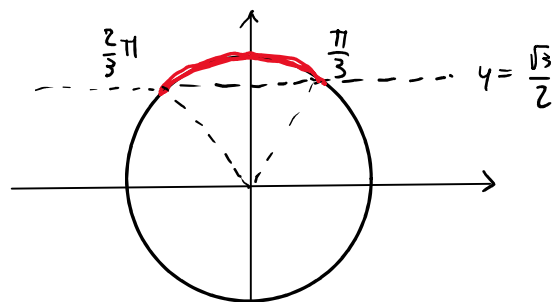
$$x = k\pi \quad \vee \quad x = \frac{2}{3}\pi + 2k\pi \quad \vee \quad x = -\frac{2}{3}\pi + 2k\pi. \quad k \in \mathbb{Z}.$$



$$6) \sin x \geq \frac{\sqrt{3}}{2}$$

$$\frac{\pi}{3} + 2k\pi \leq x \leq \frac{2}{3}\pi + 2k\pi$$

con $k \in \mathbb{Z}$.

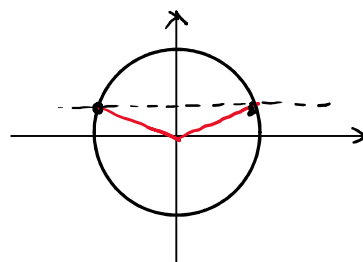


$$7) \sin x = 4 \quad \text{impossibile.}$$

$$\sin x \leq 4 \quad \text{vero } \forall x \in \mathbb{R}$$

$$\sin x > 4 \quad \text{impossibile.}$$

$$\sin x = \frac{1}{3} \quad x = ?$$



Se conosciamo un angolo

θ tale che $\sin \theta = \frac{1}{3}$, le soluzioni
sarebbero $x = \theta + 2k\pi$ $\vee$ $x = \pi - \theta + 2k\pi$.

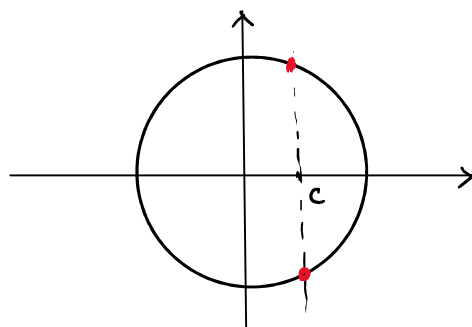
Def: Sia $c \in [-1, 1]$. Si definisce **ARCOSENO** di c
l'unico $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ tale che $\sin x = c$
(si indica con **arcsin** c).

$$\sin x = \frac{1}{3}$$

$$x = \arcsin \frac{1}{3} + 2k\pi \quad \vee \quad x = \pi - \arcsin \frac{1}{3} + 2k\pi.$$

Def: Sia $c \in [-1, 1]$, definiamo
ARCOSENO di c (**arccos** c)

l'unico $x \in [0, \pi]$ tale che
 $\cos x = c$.



$$\cos x = c \in [-1, 1]$$

$$x = \arccos c + 2k\pi$$

$$\vee x = -\arccos c + 2k\pi \text{ con } k \in \mathbb{Z}.$$

Alcuni valori noti di arcsin e arccos

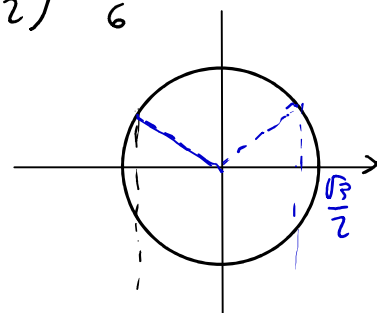
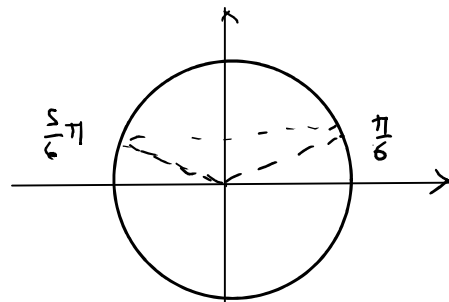
$$\bullet \arcsin(1) = \frac{\pi}{2}$$

$$\bullet \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

$$\bullet \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\bullet \arcsin\left(\sin\left(\frac{5}{6}\pi\right)\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\bullet \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5}{6}\pi$$



Def: Sia $c \in \mathbb{R}$, definiamo **ARCOTANGENTE** di c come l'unico $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ tale che $\tan x = c$.
(si indica con **arctan** c / **arctg**)

Ricordare

$$1) \arctan 0 = 0$$

$$2) \arctan 1 = \frac{\pi}{4}$$

$$3) \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$4) \arctan(\sqrt{3}) = \frac{\pi}{3}$$

$$s) \operatorname{arctan}(-x) = -\operatorname{arctan} x$$

$$\operatorname{arctan}(-1) = -\operatorname{arctan} 1 = -\frac{\pi}{4}.$$

Coordinate polari nel piano cartesiano.

Sia $P \in \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

$P = (x, y)$ con $x, y \in \mathbb{R}$.

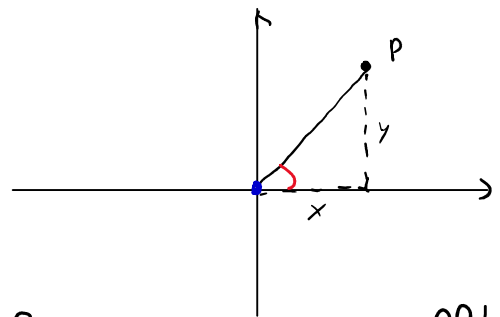
x e y si dicono **COORDINATE CARTESIANE** di P .

P può essere individuato anche da:

- la sua distanza dall'origine

$$\rho = \sqrt{x^2 + y^2}.$$

- l'angolo φ che il segmento che congiunge P all'origine forma con il semiasse positivo delle x .



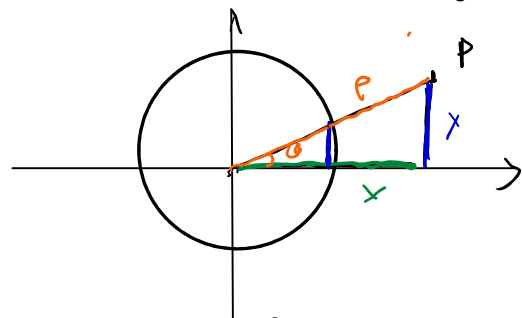
Def: (ρ, φ) vengono dette **COORDINATE POLARI** di P .

- φ è definito solo se $P \neq (0,0)$ e in tal caso è definito a meno di multipli interi di 2π .

- Se conosciamo ρ e φ possiamo ricostruire (x, y) :

$$x = \rho \cos \varphi$$

$$y = \rho \sin \varphi$$



- Viceversa se conosciamo (x, y) , allora:

$$\rho = \sqrt{x^2 + y^2}$$

• θ è un qualsiasi angolo tale che

$$\cos \theta = \frac{x}{\rho} \quad , \quad \sin \theta = \frac{y}{\rho} \quad (\text{se } \rho \neq 0).$$

Formule alternative per θ

$$\theta = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{se } x > 0 \\ \pi + \arctan\left(\frac{y}{x}\right) & \text{se } x < 0 \\ \frac{\pi}{2} & \text{se } x = 0 \text{ e } y > 0 \\ -\frac{\pi}{2} & \text{se } x = 0, \text{ e } y < 0. \end{cases}$$

ESEMPI

$P = (3, 3)$ cerchiamo le coordinate polari:

$$\bullet \rho = \sqrt{9+9} = \sqrt{18} = \sqrt{2 \cdot 9} = \sqrt{2} \cdot \sqrt{9} = 3\sqrt{2}$$

$$\bullet \cos \theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \quad \text{e} \quad \sin \theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4} \text{ (+ } 2k\pi \text{)}$$

$$\rho = 3\sqrt{2}, \quad \theta = \frac{\pi}{4} \text{ (+ } 2k\pi \text{)}$$

$$\bullet P = (1, 2)$$

$$\rho = \sqrt{1+4} = \sqrt{5}$$

$$\cos \theta = \frac{1}{\sqrt{5}} \quad \text{e} \quad \sin \theta = \frac{2}{\sqrt{5}}$$

$$\theta = \arctan\left(\frac{2}{1}\right) = \arctan 2.$$

Coordinate sferiche in $\mathbb{R}^3$.

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \\ = \{ (x, y, z) \mid x, y, z \in \mathbb{R} \}$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

Si dimostra che è possibile scrivere ρ nella forma:

$$\rho = (x, y, z) \text{ con}$$

$$x = \rho \cos \varphi \cos \vartheta$$

$$y = \rho \cos \varphi \sin \vartheta$$

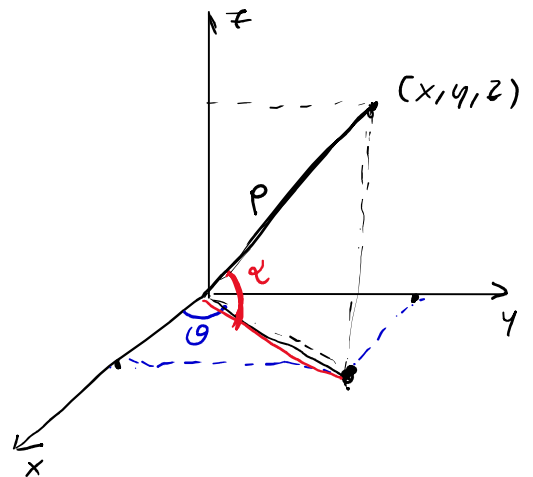
$$z = \rho \sin \varphi$$

$$\text{con } \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

LATITUDE

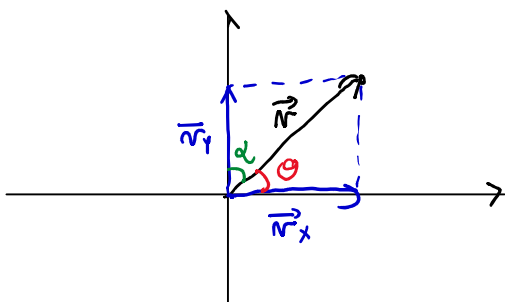
$$\vartheta \in [-\pi, \pi]$$

LONGITUDE



Formule simili si usano per i vettori.

Nel piano:



Se $\vartheta \in [0, \frac{\pi}{2}]$, allora

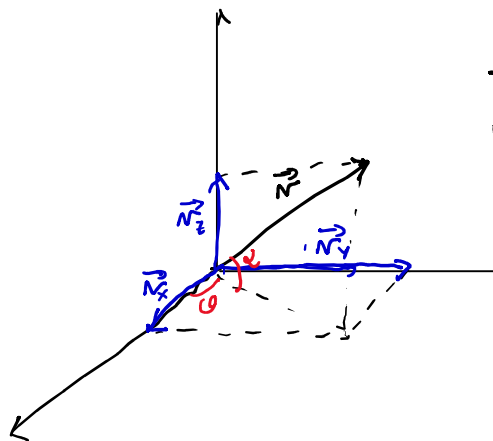
$$|\vec{r}_x| = |\vec{r}| \cos \vartheta$$

$$|\vec{r}_y| = |\vec{r}| \sin \vartheta.$$

$$|\vec{r}_y| = |\vec{r}| \cos \varphi$$

$$|\vec{r}_x| = |\vec{r}| \sin \varphi.$$

Nello spazio:



Supponendo per semplicità
 $\alpha, \theta \in [0, \frac{\pi}{2}]$:

$$|\vec{r}_x| = |\vec{r}| \cos \alpha \cos \theta$$

$$|\vec{r}_y| = |\vec{r}| \cos \alpha \sin \theta$$

$$|\vec{r}_z| = |\vec{r}| \sin \alpha$$

Numeri complessi.

Introduciamo un nuovo numero i (non reale) che soddisfa $i^2 = -1$.

Def: Un numero complesso è un oggetto della forma $x + i y$ con $x, y \in \mathbb{R}$.

• Se $x \in \mathbb{R}$: $x^2 = -1$ è impossibile.

• Se z è un numero complesso:

$$z^2 = -1$$

$$z^2 = i^2$$

$$z^2 - i^2 = 0 \quad \text{se valgono le regole per le operazioni}$$

$$(z - i)(z + i) = 0$$

$$z - i = 0 \quad \vee \quad z + i = 0$$

$$z = i \quad \vee \quad z = -i$$

• Più in generale, se $a \in \mathbb{R}$, $a < 0$.

$$z^2 = a$$

$$z^2 = -|a|$$

$$z^2 = i^2 |a|$$

$$z^2 = i^2 (\sqrt{|a|})^2$$

$$z^2 - i^2(\sqrt{|\Delta|})^2 = 0$$

$$z = i\sqrt{|\Delta|} \quad \vee \quad z = -i\sqrt{|\Delta|}$$

• $az^2 + bz + c = 0$ con $a, b, c \in \mathbb{R}$.

$$\Delta = b^2 - 4ac < 0$$

$$(i\sqrt{|\Delta|})^2 = i^2 |\Delta| = -|\Delta| = \Delta. \text{ Si può dimostrare}$$

che l'equazione ha due soluzioni

$$z_{1,2} = \frac{-b \pm i\sqrt{|\Delta|}}{2a}.$$