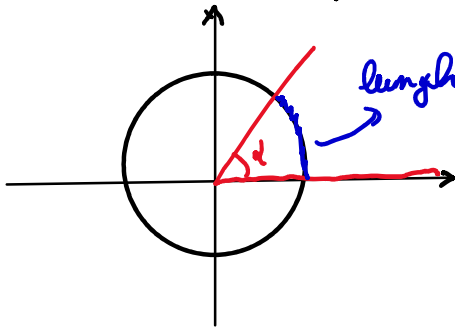


MATEMATICA - LEZIONE 10

venerdì 16 ottobre 2026 11:06

Circonferenza goniometrica:

$$C: x^2 + y^2 = 1 \quad \left(\begin{array}{l} \text{centro } (0,0) \\ \text{e raggio } 1 \end{array} \right)$$



lunghezza: misura in radianti
dell'angolo α
(a meno del segno)

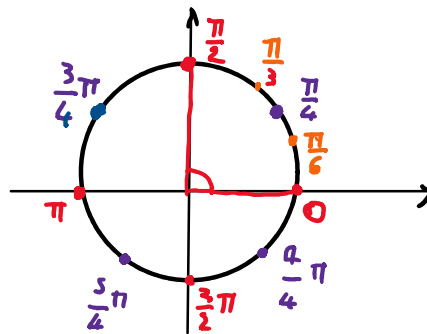
$$\begin{aligned} \alpha = 0^\circ &\longleftrightarrow 0 \\ \alpha = 360^\circ &\longleftrightarrow 2\pi \end{aligned}$$

Le misure in radianti degli altri angoli
si possono ricavare con una proporzione:

$$\text{misura in radianti} = \frac{\pi}{180} \cdot (\text{misura in gradi})$$

Tabelle di conversione

gradi	radianti
0	0
30	$\frac{\pi}{6}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
180°	π
240°	$\frac{3}{2}\pi$
360°	2π



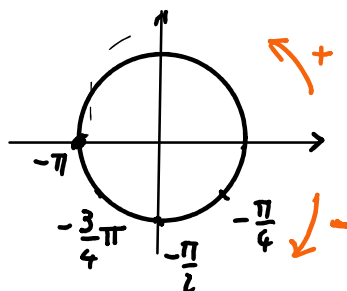
$$135^\circ = 90^\circ + 45^\circ$$

$$\frac{\pi}{2} + \frac{\pi}{4} = \frac{3}{4}\pi$$

Nota:

Misure negative in radianti indicano che l'angolo è ottenuto ruotando in senso orario.

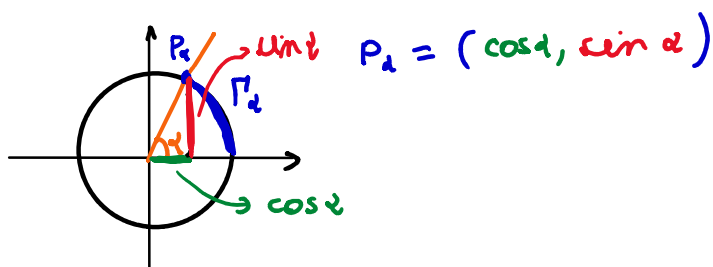
$$-\frac{\pi}{4} \Leftrightarrow -45^\circ$$



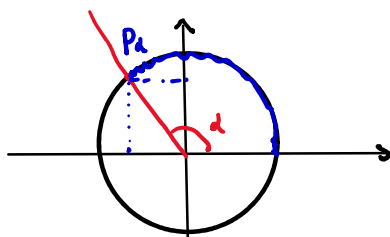
Def: Sia $\alpha \in \mathbb{R}$. Sia Γ_α l'arco sulla circonferenza C che parte dal punto $(1,0)$ e corrisponde ad un angolo di ampiezza (con segno) in radianti pari ad α .

Indichiamo con P_α il punto finale dell'arco Γ_α . Le coordinate di P_α si dicono **COSENO** e **SENO** di α

(e si indicano con **$\cos \alpha$** e **$\sin \alpha$**)



Nota: seno e coseno hanno un segno che dipende dal quadrante in cui si trova il punto P_α :



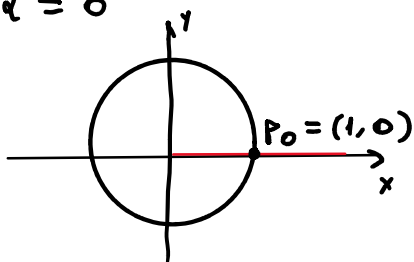
II quadrante:

$$\cos \alpha < 0$$

$$\sin \alpha > 0$$

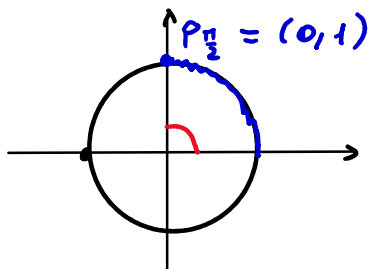
Valori "noti" di seno e coseno

• $\alpha = 0$



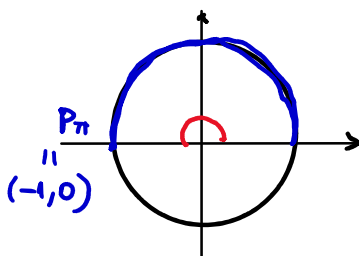
$$\cos 0 = 1$$
$$\sin 0 = 0$$

• $\alpha = \frac{\pi}{2}$
(90°)



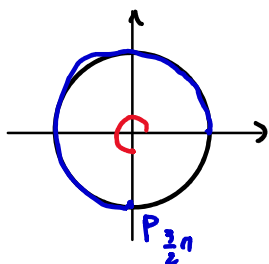
$$\cos \frac{\pi}{2} = 0$$
$$\sin \frac{\pi}{2} = 1$$

• $\alpha = \pi$
(180°)



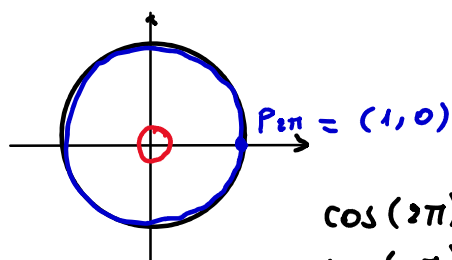
$$\cos \pi = -1$$
$$\sin \pi = 0$$

• $\alpha = \frac{3}{2}\pi$
(270°)



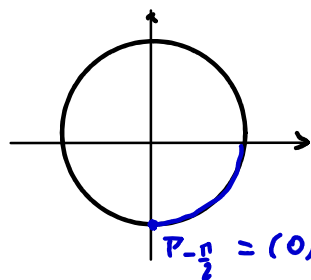
$$\cos \frac{3}{2}\pi = 0$$
$$\sin \frac{3}{2}\pi = -1$$

• $\alpha = 2\pi$
(360°)



$$\cos (2\pi) = 1$$
$$\sin (2\pi) = 0$$

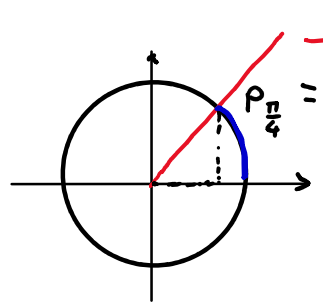
• $\alpha = -\frac{\pi}{2}$
(-90°)



$$\cos \left(-\frac{\pi}{2}\right) = 0$$
$$\sin \left(-\frac{\pi}{2}\right) = -1.$$

$$\alpha = \frac{\pi}{4}$$

(45°)



→ bisettrice del I quadrante

$$\cos \frac{\pi}{4} = \sin \frac{\pi}{4} > 0$$

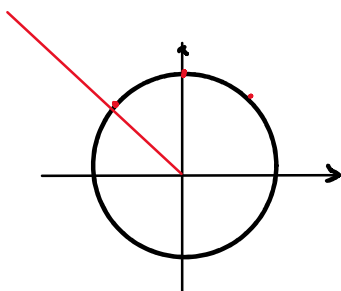
e $\cos^2 \frac{\pi}{4} = \frac{1}{2}$. Poiché quindi:

$$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\alpha = \frac{3}{4}\pi$$

(135°)

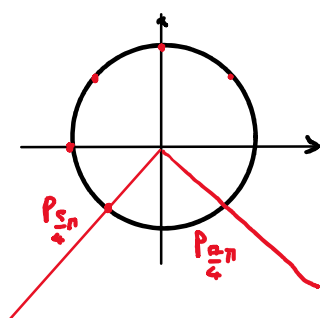


$$\cos\left(\frac{3}{4}\pi\right) = -\frac{1}{\sqrt{2}}$$

$$\sin\left(\frac{3}{4}\pi\right) = \frac{1}{\sqrt{2}}$$

$$\alpha = \frac{5}{4}\pi$$

(225°)



$$\cos\left(\frac{5}{4}\pi\right) = -\frac{1}{\sqrt{2}}$$

$$\sin\left(\frac{5}{4}\pi\right) = -\frac{1}{\sqrt{2}}$$

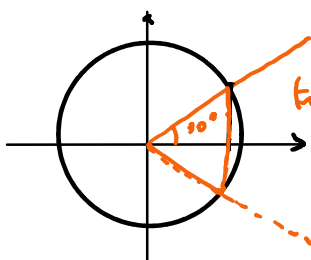
$$\cos\left(\frac{7}{4}\pi\right) = \frac{1}{\sqrt{2}}$$

$$\sin\left(\frac{7}{4}\pi\right) = -\frac{1}{\sqrt{2}}$$

$$\alpha = \frac{\pi}{6} \quad (30^\circ)$$

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\sin \frac{\pi}{6} = \frac{1}{2}$$



triangolo equilatero

$$c^2 + s^2 = 1$$

$$s = \frac{1}{2}$$

$$c^2 + \frac{1}{4} = 1$$

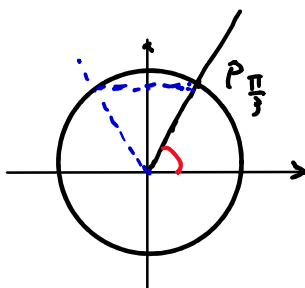
$$c^2 = \frac{3}{4}$$

$$c = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\alpha = \frac{\pi}{3}$$

$$\cos \frac{\pi}{3} = \frac{1}{2}$$

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$



$$\alpha = \frac{7\pi}{6} = \frac{\pi}{6} + \pi$$

$$\cos\left(\frac{7\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

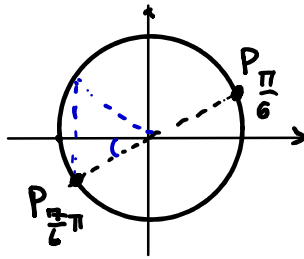
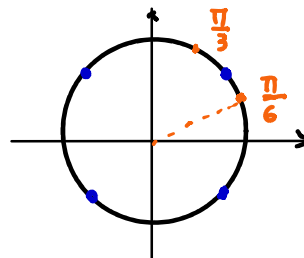
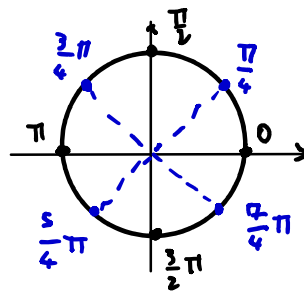
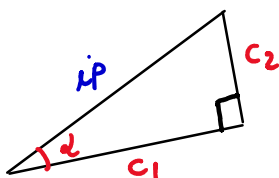


Tabella di riepilogo dei valori noti di seno e coseno.

α	$\cos \alpha$	$\sin \alpha$
0	1	0
$\frac{\pi}{2}$	0	1
π	-1	0
$\frac{3\pi}{2}$	0	-1
2π	1	0
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{3\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\frac{7\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$



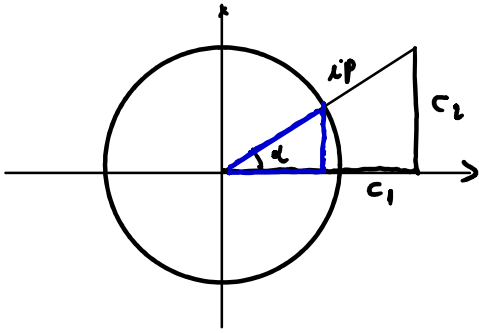
Interpretazione nei triangoli rettangoli.



ip : ipotenusa

c_1 : cateto adiacente ad α

c_2 : cateto opposto ad α



$$\frac{rP}{1} = \frac{c_2}{\sin \alpha}$$

$$c_2 = rP \cdot \sin \alpha$$

$$\frac{rP}{1} = \frac{c_1}{\cos \alpha}$$

$$c_1 = rP \cdot \cos \alpha$$

Conseguenze:

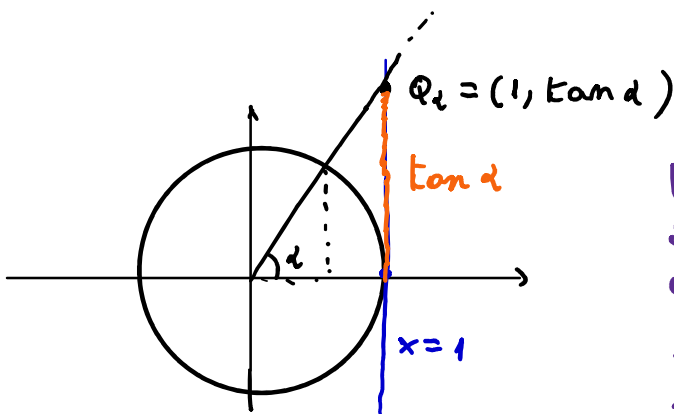
$$1) c_1 = rP \cdot \cos \alpha$$

$$2) c_2 = rP \cdot \sin \alpha$$

$$3) \frac{c_2}{c_1} = \frac{\sin \alpha}{\cos \alpha}$$

TANGENTE DI α
(indicata con $\tan \alpha$, $t_g \alpha$)

Interpretazione grafica della tangente:



$\tan \alpha$ è la seconda coordinata del punto Q_α di intersezione tra la retta che individua α e la retta tangente a C nel punto $(1,0)$

PROPRIETÀ DI SENO E COSENO

$$1) \forall \alpha \in \mathbb{R} : \cos^2 \alpha + \sin^2 \alpha = 1$$

$$2) \forall \alpha \in \mathbb{R} : -1 \leq \cos \alpha \leq 1 \quad \text{e} \quad -1 \leq \sin \alpha \leq 1$$

$$\text{cioè: } |\cos \alpha| \leq 1 \quad \text{e} \quad |\sin \alpha| \leq 1$$

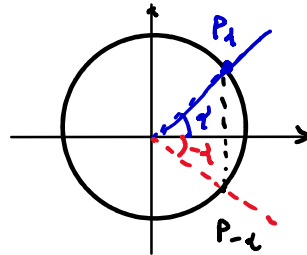
$$3) \forall \alpha \in \mathbb{R} : \quad \cos(\alpha + 2\pi) = \cos \alpha \\ \sin(\alpha + 2\pi) = \sin \alpha.$$

più in generale $\forall k \in \mathbb{Z} :$

$$\cos(\alpha + 2k\pi) = \cos \alpha$$

$$\sin(\alpha + 2k\pi) = \sin \alpha$$

$$4) \forall \alpha \in \mathbb{R} : \quad \cos(-\alpha) = \cos \alpha \\ \sin(-\alpha) = -\sin \alpha.$$

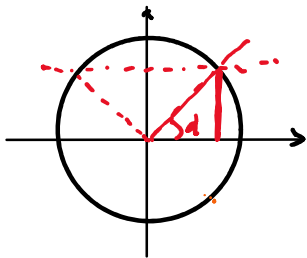


$$5) \forall \alpha, \beta \in \mathbb{R} :$$

$$\cos \alpha = \cos \beta \Leftrightarrow \begin{aligned} \alpha &= \beta + 2k\pi \vee \\ \alpha &= -\beta + 2k\pi \text{ con } k \in \mathbb{Z} \end{aligned}$$

$$6) \forall \alpha, \beta \in \mathbb{R} :$$

$$\sin \alpha = \sin \beta \Leftrightarrow \begin{aligned} \alpha &= \beta + 2k\pi \\ \vee \quad \alpha &= \pi - \beta + 2k\pi \\ \text{con } k &\in \mathbb{Z} \end{aligned}$$



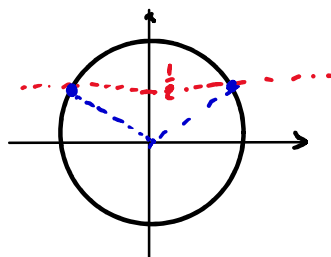
Risolvere equazioni e disequazioni
elementari con seno e coseno.

ESEMPLI

$$2 \sin x = 1$$

$$\sin x = \frac{1}{2}$$

$$\sin x = \sin \frac{\pi}{6}$$



$$x = \frac{\pi}{6} + 2k\pi \quad \vee \quad x = \pi - \frac{\pi}{6} + 2k\pi$$

$$x = \frac{\pi}{6} + 2k\pi \quad \vee \quad x = \frac{5}{6}\pi + 2k\pi.$$

$$2) \quad \cos^2 x = \frac{1}{2}$$

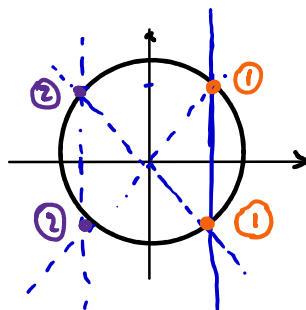
$$\cos x = \pm \frac{1}{\sqrt{2}}$$

$$\textcircled{1} \quad \cos x = \frac{1}{\sqrt{2}} \quad \vee \quad \cos x = -\frac{1}{\sqrt{2}} \quad \textcircled{2}$$

$$\textcircled{1} \quad \cos x = \cos \frac{\pi}{4}$$

$$x = \frac{\pi}{4} + 2k\pi \quad \vee$$

$$x = -\frac{\pi}{4} + 2k\pi$$



$$\textcircled{2} \quad \cos x = -\frac{1}{\sqrt{2}}$$

$$\cos x = \cos\left(\frac{3}{4}\pi\right)$$

$$x = \frac{3}{4}\pi + 2k\pi \quad \vee \quad x = -\frac{3}{4}\pi + 2k\pi.$$

Conclusione:

$$x = \begin{cases} \frac{\pi}{4} + 2k\pi \\ -\frac{\pi}{4} + 2k\pi \\ \frac{3}{4}\pi + 2k\pi \\ \frac{5}{4}\pi + 2k\pi \end{cases}$$

con $k \in \mathbb{Z}$.

oss

$$x = -\frac{\pi}{4} + 2k\pi \quad \text{con } k \in \mathbb{Z}$$

$$= -\frac{\pi}{4} + 2\pi - 2\pi + 2k\pi$$

$$= -\frac{\pi}{4} + 2\pi + 2\pi(k-1)$$

$$= \frac{7}{4}\pi + 2\pi(\underbrace{n-1}_{\in \mathbb{Z}})$$

è equivalente a scrivere

$$x = \frac{7}{4}\pi + 2k\pi.$$

In modo simile

$$x = -\frac{3}{4}\pi + 2k\pi \quad \Leftrightarrow \quad x = \frac{5}{4}\pi + 2k\pi, \quad k \in \mathbb{Z}.$$

Formule utili:

1) FORMULE DI ADDIZIONE E SOTTRAZIONE

$\forall \alpha, \beta \in \mathbb{R}$:

$$\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta.$$

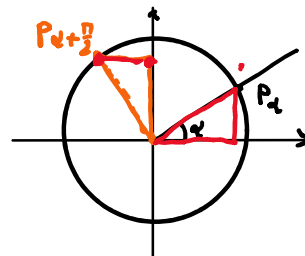
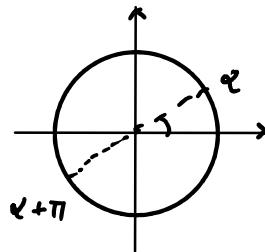
Casi particolari:

$$\cos(\alpha + \pi) = -\cos \alpha$$

$$\sin(\alpha + \pi) = -\sin \alpha$$

$$\cos(\alpha + \frac{\pi}{2}) = -\sin \alpha$$

$$\sin(\alpha + \frac{\pi}{2}) = \cos \alpha$$



2) FORMULE DI DUPLICAZIONE.

$$\forall \alpha \in \mathbb{R}: \quad \cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha$$

$$\sin(2\alpha) = 2 \sin \alpha \cos \alpha$$

OSS: la formula per $\cos 2\alpha$ si può scrivere come:

$$\begin{aligned}\cos(2\alpha) &= \cos^2 \alpha - \sin^2 \alpha \\ &= 1 - 2\sin^2 \alpha \\ &= 2\cos^2 \alpha - 1\end{aligned}$$

$$\left(\begin{array}{l} \cos^2 \alpha + \sin^2 \alpha = 1 \\ \cos^2 \alpha = 1 - \sin^2 \alpha \\ \text{e} \\ \sin^2 \alpha = 1 - \cos^2 \alpha \end{array} \right)$$

OSS 2:

$$\cos(2\alpha) = 2\cos^2 \alpha - 1 \Rightarrow \cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}$$

$$\cos(2\alpha) = 1 - 2\sin^2 \alpha \Rightarrow \sin^2 \alpha = \frac{1 - \cos(2\alpha)}{2}$$

3) FORMULE DI BISEZIONE

$\forall \alpha \in \mathbb{R}$:

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

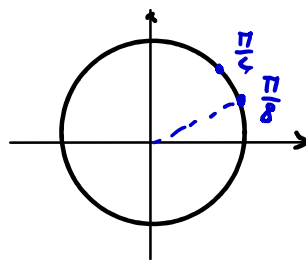
$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

con i segni da determinare in base alla posizione del punto che corrisponde ad $\frac{\alpha}{2}$ su $\mathbb{C}$.

ESEMPLI

• $\cos\left(\frac{\pi}{8}\right)$ che segno avrà $\cos\left(\frac{\pi}{8}\right)$?

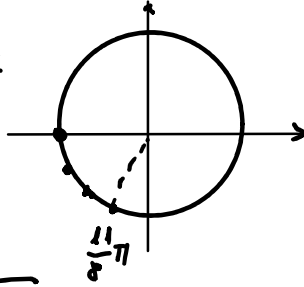
$$\begin{aligned}\cos\left(\frac{\pi}{8}\right) &= \sqrt{\frac{1 + \cos \frac{\pi}{4}}{2}} \\ &= \sqrt{\frac{1 + \frac{1}{\sqrt{2}}}{2}}\end{aligned}$$



I quadranti
 $\cos \frac{\pi}{8} > 0$
 $\sin \frac{\pi}{8} > 0$.

• $\cos\left(\frac{11}{8}\pi\right)$ $\frac{11}{8}$ è la metà di $\frac{11}{4}$

$$\frac{11}{8} = 1 + \frac{3}{8}$$



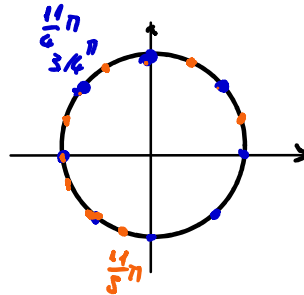
$$\cos\left(\frac{11}{8}\pi\right) < 0$$

$$\cos\left(\frac{11}{8}\pi\right) = -\sqrt{\frac{1 + \cos\left(\frac{11}{4}\pi\right)}{2}}$$

$$= -\sqrt{\frac{1 + \cos\left(\frac{3}{4}\pi\right)}{2}}$$

$$= -\sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{2}}$$

$$\begin{aligned}\frac{11}{4}\pi &= \left(2 + \frac{3}{4}\right)\pi \\ &= 2\pi + \frac{3}{4}\pi\end{aligned}$$



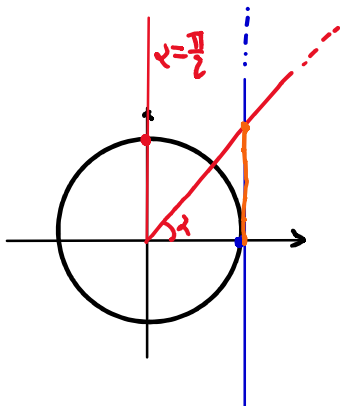
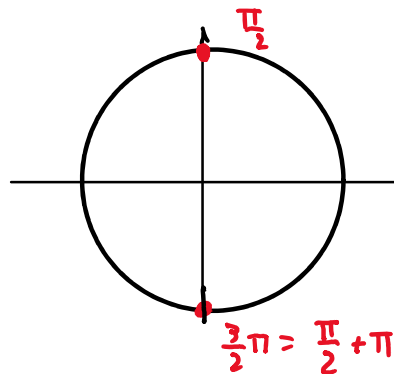
Def: Sia $\alpha \in \mathbb{R}$, $\alpha \neq \frac{\pi}{2} + k\pi$ con $k \in \mathbb{Z}$.

Definiamo **TANGENTE** di α la quantità

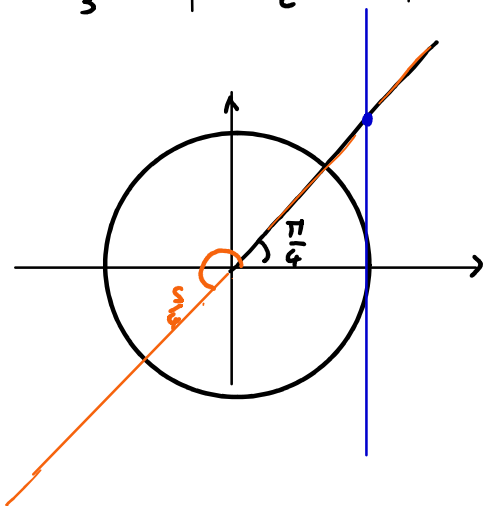
$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$$

oss

$\alpha \neq \frac{\pi}{2} + k\pi$ garantisce
che $\cos \alpha \neq 0$.



α	$\cos \alpha$	$\sin \alpha$	$\tan \alpha$
0	1	0	0
$\frac{\pi}{2}$	0	1	$\frac{1}{0}$ non definita
π	-1	0	$\frac{0}{-1} = 0$
$\frac{3}{2}\pi$	0	-1	n.d.
2π	1	0	0
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{3}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1
$\frac{5}{4}\pi$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
$\frac{7}{4}\pi$	$+\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$



la tangente non cambia se si aggiunge π ad un angolo.

oss

1) $\forall \alpha \in \mathbb{R}, \alpha \neq \frac{\pi}{2} + k\pi$ con $k \in \mathbb{Z}$.

$$\tan(\alpha + \pi) = \tan \alpha.$$

2) $\forall \alpha, \beta \in \mathbb{R}, \alpha, \beta \neq \frac{\pi}{2} + k\pi :$

$$\tan \alpha = \tan \beta \Leftrightarrow \alpha = \beta + k\pi \text{ con } k \in \mathbb{Z}$$

$$\Leftrightarrow \begin{aligned} \alpha &= \beta + 2k\pi \\ \vee \alpha &= \beta + \pi + 2k\pi \end{aligned} \text{ con } k \in \mathbb{Z}.$$