MATEMATICA - LEZIONE 34

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- $\int 1\, dx = x + c$
- $\int \alpha\, dx = \alpha x + c$
- $\int x\, dx = \frac{1}{2}x^2 + c$
- $\int x^2\, dx = \frac{1}{3}x^3 + c$
- $\int x^\alpha\, dx = \frac{1}{1+\alpha} x^{1+\alpha} + c$ se $\alpha \neq -1$
- $\int \frac{1}{x}\, dx = \ln|x| + c$
- $\int e^x\, dx = e^x + c$
- $\int \sin x\, dx = -\cos x + c$
- $\int \cos x\, dx = \sin x + c$
- $\int \frac{1}{\cos^2 x}\, dx = \tan x + c$
- $\int \frac{1}{1+x^2}\, dx = \arctan x + c$.
- $\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + c$
- $\int -\frac{1}{\sqrt{1-x^2}}\, dx = \arccos x + c$

PROPRIETÀ

1) $\forall \alpha, \beta \in \mathbb{R}$ allora

$$\int (\alpha f(x) + \beta g(x))\, dx = \alpha \int f(x)\, dx + \beta \int g(x)\, dx$$

2) Se F è una primitiva di f, allora $\forall\, a, b \in \mathbb{R}$ con $a \neq 0$:

$$\int f(ax+b)\, dx = F(ax+b) \cdot \frac{1}{a} + c$$

3) $$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$$

4) $$\int f(x)^\alpha f'(x)\, dx = \frac{1}{1+\alpha} f(x)^{1+\alpha} + c$$

5) Se f è continua in $[a,b]$ e F è una sua primitiva:

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

ESEMPI

1) $\int \frac{x-1}{x}\,dx = \int 1 - \frac{1}{x}\,dx = \int 1\,dx - \int \frac{1}{x}\,dx$

$= x - \ln|x| + c.$

2) $\int \cos(2x)\,dx = \sin(2x)\cdot\frac{1}{2} + c$

3) $\int \frac{1}{3x-1}\,dx = \ln|3x-1|\cdot\frac{1}{3} + c$

$\int \frac{1}{3x-1}\,dx = \int \frac{1}{3}\cdot\frac{3}{3x-1}\,dx = \frac{1}{3}\int\frac{3}{3x-1}\,dx = \frac{1}{3}\ln|3x-1| + c.$

4) $\int \frac{6x-1}{3x^2-x}\,dx = \ln|3x^2-x| + c$

$\int \frac{1}{3x^2-x} = \, ?$ Nella prossima lezione.

5) $\int \sin^2 x\,dx$

$= \int \frac{1-\cos 2x}{2}\,dx$

$= \int \frac{1}{2} - \frac{\cos(2x)}{2}\,dx$

$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + c.$

$\cos(2x) = \cos^2 x - \sin^2 x$
$= 1 - 2\sin^2 x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

Formula di integrazione per parti

Def: Sia $I \subseteq \mathbb{R}$ un intervallo e sia $f: I \to \mathbb{R}$.
Si dice che f è di CLASSE C^1 IN I se f è derivabile in I e f' è continua in I. L'insieme delle funzioni di classe C^1 in I si indica con $C^1(I)$.

Def: Sia $I \subseteq \mathbb{R}$ un intervallo e sia $f: I \longrightarrow \mathbb{R}$. Dato $k \in \mathbb{N}$ $k \geq 1$, si dice che f è di CLASSE C^k IN I ($f \in C^k(I)$) se f è derivabile k volte in I e le derivate di f fino all'ordine k sono continue.

TEOREMA (FORMULA DI INTEGRAZIONE PER PARTI)

Siano $a, b \in \mathbb{R}$, $a < b$ e siano $f, g \in C^1([a,b])$.

Allora:

1) $$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx$$

2) $$\int_a^b f'(x)\, g(x) = \underbrace{f(x)\, g(x) \Big|_a^b}_{f(b)g(b) - f(a)g(a).} - \int_a^b f(x)\, g'(x)\, dx$$

DIM

1) $$(f(x)\, g(x))' = f'(x)\, g(x) + f(x)\, g'(x)$$

$$f'(x)\, g(x) = (f(x)\, g(x))' - f(x)\, g'(x)$$

$$\int f'(x)\, g(x)\, dx = \int (f(x)\, g(x))' dx - \int f(x)\, g'(x)\, dx$$

$$= f(x)\, g(x) - \int f(x)\, g'(x)\, dx$$

2) Uguale ma con gli integrali definiti.

ESEMPIO

Calcolare $\int \underbrace{x}_{g}\, \underbrace{e^x}_{f'}\, dx$

$$\boxed{\begin{array}{l} f'(x) = e^x \Rightarrow f(x) = \int e^x dx = e^x + c \\ g(x) = x \Rightarrow g'(x) = 1 \end{array}}$$

per parti:

$$= e^x x - \int e^x \cdot 1\, dx$$

$$= e^x x - \int e^x\, dx$$

$$= e^x x - e^x + c.$$

2) $$\int x\, \underbrace{\cos(2x)}_{f'}\, dx = \underbrace{\sin(2x) \cdot \frac{1}{2}}_{f} \cdot x - \int \sin(2x) \cdot \frac{1}{2} \cdot 1\, dx$$

$$= \frac{1}{2} x \sin(2x) - \frac{1}{2} \int \sin(2x)\, dx$$

$$= \frac{1}{2} x \sin(2x) - \frac{1}{2} \left(-\cos(2x) \cdot \frac{1}{2} \right) + C$$

$$= \frac{1}{2} x \sin(2x) + \frac{1}{4} \cos(2x) + C .$$

3) $\int x^2 \underbrace{e^{-x}}_{f'} dx$

$$\int e^{-x} dx = \frac{e^{-x}}{-1} + C = -e^{-x} + C$$

$$f'(x) = e^{-x} , \quad f(x) = -e^{-x}$$

per parti:

$$= \underbrace{-e^{-x}}_{f(x)} x^2 - \int (-e^{-x}) \cdot 2x \, dx$$

$$= -e^{-x} x^2 + 2 \int \underbrace{e^{-x}}_{f'} \cdot x \, dx$$

$$= -e^{-x} x^2 + 2 \left(-e^{-x} \cdot x - \int (-e^{-x}) \cdot 1 \, dx \right)$$

$$= -e^{-x} x^2 + 2 \left(-e^{-x} x + \int e^{-x} dx \right)$$

$$= -e^{-x} x^2 - 2 e^{-x} x + 2 \int e^{-x} dx$$

$$= -e^{-x} x^2 - 2 e^{-x} x - 2 e^{-x} + C .$$

4) $\int \sin x \underbrace{\cos x}_{f'} dx = \underbrace{\sin x}_{f} \cdot \sin x - \int \sin x \cdot \cos x \, dx$

$$= \sin^2 x - \int \sin x \cos x \, dx$$

Quindi

$$\int \sin x \cos x \, dx = \sin^2 x - \int \sin x \cos x \, dx$$

e

$$\int \sin x \cos x \, dx + \int \sin x \cos x = \sin^2 x + C$$

$$2 \int \sin x \cos x \, dx = \sin^2 x + C$$

$$\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + C_2 = \frac{C}{2}$$

5) $$\int \ln x \, dx = \int \underbrace{1}_{f'} \cdot \ln x \, dx$$
$$= x \ln x - \int x \cdot \frac{1}{x} \, dx$$
$$= x \ln x - \int 1 \, dx$$
$$= x \ln x - x + C$$

6) $$\int \arctan x \, dx = \int \underbrace{1}_{f'} \cdot \arctan x \, dx$$
$$= x \arctan x - \int x \frac{1}{1+x^2} \, dx$$
$$= x \arctan x - \int \frac{x}{1+x^2} \, dx$$
$$= x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$
$$= x \arctan x - \frac{1}{2} \ln(1+x^2) + C.$$

7) $$\int \underbrace{x^7}_{f'} \ln x \, dx = \underbrace{\frac{1}{8} x^8}_{f} \ln x - \int \frac{1}{8} x^8 \cdot \frac{1}{x} \, dx$$
$$= \frac{1}{8} x^8 \ln x - \frac{1}{8} \int x^7 \, dx$$
$$= \frac{1}{8} x^8 \ln x - \frac{1}{64} x^8 + C.$$

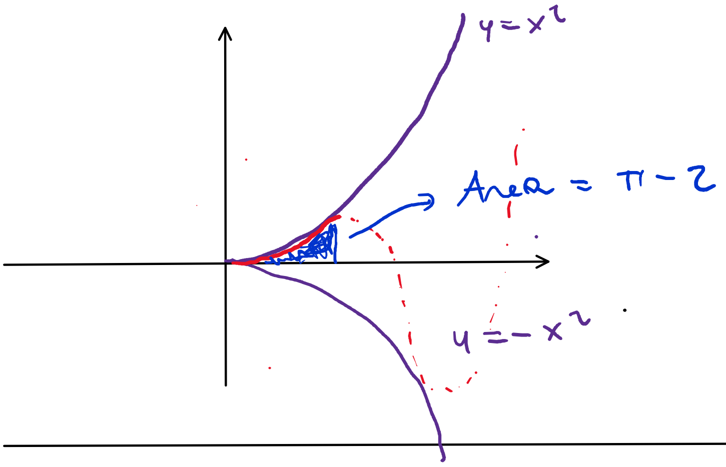
8) $$\int_0^{\frac{\pi}{2}} x^2 \underbrace{\sin x}_{f'} \, dx = -\cos x \cdot x^2 \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x) \cdot 2x$$
$$= -0 - (-0) + 2 \int_0^{\frac{\pi}{2}} \cos x \cdot x \, dx$$
$$= 2 \int_0^{\frac{\pi}{2}} \underbrace{\cos x}_{f'} \cdot x \, dx$$
$$= 2 \left(\sin x \cdot x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \cdot 1 \, dx \right)$$

$$= 2\left(1\cdot\frac{\pi}{2} - 0 - \int_0^{\frac{\pi}{2}} \sin x \, dx\right)$$

$$= \pi - 2\int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= \pi - 2\left(-\cos x \Big|_0^{\frac{\pi}{2}}\right)$$

$$= \pi - 2\left(0-(-1)\right) = \pi - 2.$$



Integrazione per sostituzione.

Idea: Se F è una primitiva di f e φ è una seconda funzione:

$$\left(F(\varphi(x))\right)' = F'(\varphi(x))\cdot\varphi'(x)$$
$$= f(\varphi(x))\cdot\varphi'(x)$$

FORMULA DI INTEGRAZIONE PER SOSTITUZIONE

$$\int f(\varphi(x))\,\varphi'(x)\,dx = F(\varphi(x)) + C.$$

Formalmente si può sostituire y al posto di $\varphi(x)$ e dy al posto di $\varphi'(x)\,dx$

$$\int f(\underbrace{\varphi(x)}_{y=\varphi(x)})\,\overbrace{\varphi'(x)\,dx}^{dy} = \int f(y)\,dy = F(y) + C$$
$$= F(\varphi(x)) + C.$$

Notazione

$y = \varphi(x) \qquad \varphi'(x) = \dfrac{d\varphi(x)}{dx} = \dfrac{dy}{dx} \implies dy = \varphi'(x)\,dx$

La formula si può usare in diversi modi:

1) $\int f(\varphi(x))\,\varphi'(x)\,dx$

$= \int f(y)\,dy$

$y = \varphi(x)$
$dy = \varphi'(x)\,dx$

2) $\int f(x)\,dx$

$= \int f(\varphi(t))\,\varphi'(t)\,dt.$

$x = \varphi(t)$
$dx = \varphi'(t)\,dt$

ESEMPIO

$$\int e^{x^2} \cdot x\,dx = \frac{1}{2}\int e^{x^2}\underbrace{2x\,dx}_{dy}$$

$y = x^2$
$dy = 2x\,dx$

$$= \frac{1}{2}\int e^y\,dy$$

$$= \frac{1}{2}e^y + c$$

$$= \frac{1}{2}e^{x^2} + c$$

TEOREMA (INTEGRAZIONE PER SOSTITUZIONE NEGLI INTEGRALI DEFINITI)

Siano $\alpha, \beta \in \mathbb{R}$ con $\alpha < \beta$ e sia $\varphi: [\alpha, \beta] \longrightarrow \mathbb{R}$ t.c. $\varphi(\alpha) = a$ e $\varphi(\beta) = b$. Allora:

$$\int_\alpha^\beta f(\varphi(x))\,\varphi'(x)\,dx = \int_a^b f(y)\,dy.$$

$y = \varphi(x) \qquad a = \varphi(\alpha) \qquad b = \varphi(\beta)$

Anche gli estremi vanno sostituiti!

$$\int_0^4 \frac{1}{1+\sqrt{x}}\,dx$$

$$= \int_0^4 \frac{1}{1+\sqrt{x}}\,\frac{2\sqrt{x}}{2\sqrt{x}}\,dx$$

$y = \sqrt{x} + 1$
$dy = \dfrac{1}{2\sqrt{x}}\,dx$
$\sqrt{x} = y - 1$

Estremi:

$x = 0 \implies y = \sqrt{0} + 1 = 1$
$x = 4 \implies y = \sqrt{4} + 1 = 3$

$$= \int_1^3 \frac{1}{y} \, 2(y-1)\, dy = 2\int_1^3 \frac{y-1}{y}\, dy = 2\int_1^3 1 - \frac{1}{y}\, dy$$

$$= 2\left(y - \ln y \Big|_1^3 \right)$$

$$= 2\left(3 - \ln 3 - (1 - \underbrace{\ln 1}_{=0}) \right) = 2(2 - \ln 3)$$

$$= 4 - 2\ln 3.$$

Metodo alternativo: Calcolare la primitiva in x

$$\int_0^4 \frac{1}{1+\sqrt{x}}\, dx$$

Calcoliamo $\int \frac{1}{1+\sqrt{x}}\, dx$ $\qquad$ $y = 1+\sqrt{x}$, $\quad dy = \frac{1}{2\sqrt{x}}\, dx$

$$= 2\int 1 - \frac{1}{y}\, dy$$

$$= 2\left(y - \ln|y| \right) + C$$

$$= 2y - 2\ln|y| + C$$

$$= 2(1+\sqrt{x}) - 2\ln(1+\sqrt{x}) + C$$

Quindi:

$$\int_0^4 \frac{1}{1+\sqrt{x}}\, dx = \left(2(1+\sqrt{x}) - 2\log(1+\sqrt{x}) \right)\Big|_0^4$$

$$= 6 - 2\ln 3 - (2 - \ln 1)$$

$$= 4 - 2\ln 3$$

Nota

$$y = \sqrt{x} + 1$$

$$dy = \frac{1}{2\sqrt{x}}\, dx$$

$$dx = 2\sqrt{x}\, dy = 2(y-1)\, dy$$

$$\int \frac{1}{1+\sqrt{x}}\, dx = \int \frac{1}{y}\, 2(y-1)\, dy.$$

ESEMPIO

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx \qquad x = \sin t \qquad dx = \cos t \, dt$$

$x = 1 \Rightarrow t = \frac{\pi}{2}$

$x = -1 \Rightarrow t = -\frac{\pi}{2}$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\ \cos t \, dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2 t}\ \cos t \, dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt$$

$$\boxed{\begin{aligned} \cos 2t &= \cos^2 t - \sin^2 t \\ &= 2\cos^2 t - 1 \\ \cos^2 t &= \frac{1+\cos 2t}{2} \end{aligned}}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2} \, dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos 2t \, dt$$

$$= \frac{1}{2} t + \frac{1}{4} \sin 2t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4} + \frac{1}{4}\cdot 0 - \left(-\frac{\pi}{4} + 0\right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

ESEMPIO

- $$\int \sin^2 x \, \underbrace{\cos x \, dx}_{dy} \qquad y = \sin x \qquad dy = \cos x \, dx$$

$$= \int y^2 \cdot dy = \int y^2 \, dy = \frac{1}{3} y^3 + C = \frac{1}{3}\sin^3 x + C.$$

- $$\int \frac{\ln(\ln x)}{x} \, dx \qquad y = \ln x \qquad dy = \frac{1}{x}\, dx$$

$$= \int \ln(\ln x) \overbrace{\frac{1}{x}\, dx}^{dy} = \int \ln y \, dy$$

$$= y \ln y - y + C$$

$$= \ln x \ln(\ln x) - \ln x + C.$$

$y = \sqrt{1+x} \Rightarrow y^2 = 1+x \Rightarrow x = y^2 - 1$

$dy = \frac{1}{2\sqrt{1+x}}\, dx \Rightarrow \frac{1}{\sqrt{1+x}}\, dx = 2\, dy.$

- $\int \frac{x+2}{\sqrt{1+x}}\, dx$

$$= \int (x+2) \cdot \underbrace{\frac{1}{\sqrt{1+x}}\, dx}$$

$$= \int (y^2 - 1 + 2) \cdot 2\, dy$$

$$= 2 \int (y^2 + 1)\, dy$$

$$= 2\left(\frac{1}{3} y^3 + y\right) + C$$

$$= \frac{2}{3} y^3 + 2y + C$$

$$= \frac{2}{3}\left(\sqrt{1+x}\right)^3 + 2\sqrt{1+x} + C.$$