MATEMATICA – LEZIONE 32 – 28/11/24 14.00 – 16.00 Aula 4

Esercizi di riepilogo sui limiti calcolabili con le formule di Taylor

1) $$\lim_{x\to 0} \frac{\sin(x+2x^2) - \cos x - x + 1}{\ln(1+x) - \sin x}$$ forma indeterminata $\left[\frac{0}{0}\right]$

Determiniamo lo sviluppo di Taylor del denominatore per $x\to 0$

$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$ per $x\to 0$

$\sin x = x - \frac{x^3}{6} + o(x^3)$ per $x\to 0$

$$\ln(1+x) - \sin x = \left(\cancel{x} - \frac{x^2}{2} + o(x^2)\right) - \left(\cancel{x} - \frac{x^3}{6} + o(x^3)\right)$$

$$= -\frac{x^2}{2} + o(x^2) + \frac{x^3}{6} + o(x^3)$$

$$= -\frac{x^2}{2} + o(x^2) + \frac{1}{6}o(x^2) + o(x^3)$$

$\boxed{x^3 = o(x^2) \text{ per } x\to 0}$

$$= -\frac{x^2}{2} + o(x^2) + o(x^2) + o(x^3) = -\frac{x^2}{2} + o(x^2) + o(x^3)$$

$\boxed{\frac{1}{6}o(x^2) = o(x^2) \text{ per } x\to 0}$ $\boxed{o(x^2) + o(x^2) = o(x^2) \text{ per } x\to 0}$

$$= -\frac{x^2}{2} + o(x^2) \quad \text{per } x\to 0$$

Determiniamo lo sviluppo di Taylor del numeratore per $x\to 0$

$\sin y = y + o(y)$ per $y\to 0$

$\sin y = y + o(y^2)$ per $y\to 0$

$y = x + 2x^2 \longrightarrow 0$ per $x\to 0$

$$\sin(x+2x^2) = x + 2x^2 + o\left((x+2x^2)^2\right)$$

$$= x + 2x^2 + o(x^2 + 4x^3 + 4x^4)$$

$$= x + 2x^2 + o(x^2) \quad \text{per } x\to 0$$

$\cos x = 1 - \frac{x^2}{2} + o(x^2)$ per $x\to 0$

$$\sin(x+2x^2) - \cos x - x + 1 = \cancel{x} + 2x^2 + o(x^2) - \left(\cancel{1} - \frac{x^2}{2} + o(x^2)\right) - \cancel{x} + \cancel{1}$$

$$= \frac{5}{2}x^2 + o(x^2) - o(x^2) = \frac{5}{2}x^2 + o(x^2) \quad \text{per } x\to 0$$

ricordiamo che : $f(x) = o(x^n)$ per $x \to 0$ $\Leftrightarrow$ $\lim_{x\to 0} \frac{f(x)}{x^n} = 0$

$$\lim_{x\to 0} \frac{\frac{5}{2}x^2 + o(x^2)}{-\frac{x^2}{2} + o(x^2)} = \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}} \frac{\frac{5}{2} + \overbrace{\frac{o(x^2)}{x^2}}^{\to 0}}{-\frac{1}{2} + \underbrace{\frac{o(x^2)}{x^2}}_{\to 0}}$$

$$= \lim_{x\to 0} \frac{\frac{5}{2} + \frac{o(x^2)}{x^2}}{-\frac{1}{2} + \frac{o(x^2)}{x^2}} = \frac{\frac{5}{2}}{-\frac{1}{2}} = -5$$

2) $\lim_{x\to 0} \frac{\sin(2x) - x - x\sqrt{1-3x^2}}{x(\cos^2 x - e^{x^2})}$ forma indeterminata $\left[\frac{0}{0}\right]$

$e^y = 1 + y + o(y)$ per $y \to 0$ $\qquad \lim_{x\to 0} x^2 = 0$

$e^{x^2} = 1 + x^2 + o(x^2)$

$\cos x = 1 - \frac{x^2}{2} + o(x^3)$ per $x \to 0$

$\cos^2 x = \left(1 - \frac{x^2}{2} + o(x^3)\right)^2 \overset{\uparrow}{=} 1 + \frac{x^4}{4} + (o(x^3))^2 - x^2 + o(x^3) + x^2 o(x^3)$

$$\boxed{(A+B+C)^2 = A^2 + B^2 + C^2 + 2AB + 2AC + 2BC}$$

$= 1 + \frac{x^4}{4} + o(x^6) - x^2 + o(x^3) + o(x^5)$

$= 1 - x^2 + \frac{x^4}{4} + o(x^3) = 1 - x^2 + o(x^3)$ per $x \to 0$

$\cos^2 x - e^{x^2} = \cancel{1} - x^2 + o(x^3) - (\cancel{1} + x^2 + o(x^2)) = -2x^2 + o(x^3) - o(x^2) = -2x^2 + o(x^2)$ per $x \to 0$

$x(\cos^2 x - e^{x^2}) = x(-2x^2 + o(x^2)) = -2x^3 + o(x^3)$ per $x \to 0$

Determiniamo lo sviluppo di Taylor di ordine 3 del numeratore

$\sin y = y - \frac{y^3}{6} + o(y^3)$ per $y \to 0$ $\qquad \lim_{x\to 0} (2x) = 0$

$\sin(2x) = 2x - \frac{(2x)^3}{6} + o((2x)^3)$

$= 2x - \frac{4}{3}x^3 + o(x^3)$ per $x \to 0$

$\sqrt{1+y} = 1 + \frac{y}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} y^2 + o(y^2)$

$= 1 + \frac{y}{2} - \frac{y^2}{8} + o(y^2)$ per $y \to 0$ $\qquad \lim_{x\to 0} (-3x^2) = 0$

$\sqrt{1-3x^2} = 1 + \frac{1}{2}(-3x^2) - \frac{1}{8}(-3x^2)^2 + o((-3x^2)^2)$

$= 1 - \frac{3}{2}x^2 - \frac{9}{8}x^4 + o(x^4) = 1 - \frac{3}{2}x^2 + o(x^3)$ per $x \to 0$

$\sin(2x) - x - x\sqrt{1-3x^2} = 2x - \frac{4}{3}x^3 + o(x^3) - x - x\left(1 - \frac{3}{2}x^2 + o(x^3)\right)$

$= \cancel{2x} - \frac{4}{3}x^3 + o(x^3) - \cancel{2x} + \frac{3}{2}x^3 + o(x^4) = \frac{x^3}{6} + o(x^3)$

$$\lim_{x\to 0} \frac{\sin(2x) - x - x\sqrt{1-3x^2}}{x(\cos^2 x - e^{x^2})} = \lim_{x\to 0} \frac{\frac{x^3}{6} + o(x^3)}{-2x^3 + o(x^3)} = -\frac{1}{12}$$

3) $\lim_{x\to 0} \frac{\ln\left(\frac{1+x}{1-x}\right) - \sin(2x)}{x\ln(1+5x^4)}$ forma indeterminata $\left[\frac{0}{0}\right]$

Determiniamo lo sviluppo del denominatore

$\ln(1+y) = y + o(y)$ per $y \to 0$ $\quad \lim_{x\to 0} 5x^4 = 0$

$\ln(1+5x^4) = 5x^4 + o(x^4)$ per $x\to 0$

$x\ln(1+5x^4) = x(5x^4 + o(x^4)) = 5x^5 + o(x^5)$ per $x \to 0$

per il numeratore otteniamo

$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + o(x^5)$ per $x\to 0$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + o(x^5)$ per $x \to 0$

$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + o(x^5)$ per $x\to 0$

$\sin y = y - \frac{y^3}{6} + \frac{y^5}{120} + o(y^5)$ per $y \to 0$ $\quad \lim_{x\to 0}(2x) = 0$

$\sin(2x) = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + o(x^5)$ per $x \to 0$

$$\ln\left(\frac{1+x}{1-x}\right) - \sin(2x) = \cancel{2x} + \frac{2}{3}x^3 + \frac{2}{5}x^5 + o(x^5) - \left(\cancel{2x} - \frac{4}{3}x^3 + \frac{4}{15}x^5 + o(x^5)\right)$$
$$= 2x^3 + \frac{2}{15}x^5 + o(x^5) \quad \text{per } x\to 0$$

$$\lim_{x\to 0} \frac{\ln\left(\frac{1+x}{1-x}\right) - \sin(2x)}{x\ln(1+5x^4)} = \lim_{x\to 0} \frac{2x^3 + \frac{2}{15}x^5 + o(x^5)}{5x^5 + o(x^5)}$$

$$= \lim_{x\to 0} \frac{2x^3 + o(x^3)}{5x^5 + o(x^5)} = \lim_{x\to 0} \underbrace{\frac{1}{x^2}}_{\to +\infty} \cdot \underbrace{\frac{2 + \frac{o(x^3)}{x^3}}{5 + \frac{o(x^5)}{x^5}}}_{\to \frac{2}{5}} = +\infty$$

4) $\lim_{x\to +\infty} x^3\left(\sin\left(\frac{1}{x}\right) - \arctan\left(\frac{1}{x}\right)\right)$ forma indeterminata $[0\cdot\infty]$

$\sin y = y - \frac{y^3}{6} + o(y^3)$ per $y \to 0$

$\arctan y = y - \frac{y^3}{3} + o(y^3)$ per $y\to 0$

$$\lim_{x\to+\infty} \frac{1}{x} = 0$$

$$\sin\left(\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{6x^3} + o\left(\frac{1}{x^3}\right) \quad \text{per } x\to+\infty$$

$$\arctan\left(\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{3x^3} + o\left(\frac{1}{x^3}\right) \quad \text{per } x\to+\infty$$

$$\sin\left(\frac{1}{x}\right) - \arctan\left(\frac{1}{x}\right) = \cancel{\frac{1}{x}} - \frac{1}{6x^3} + o\left(\frac{1}{x^3}\right) - \left(\cancel{\frac{1}{x}} - \frac{1}{3x^3} + o\left(\frac{1}{x^3}\right)\right)$$

$$= \frac{1}{6x^3} + o\left(\frac{1}{x^3}\right) \quad \text{per } x\to 0$$

$$\lim_{x\to+\infty} x^3\left(\sin\left(\frac{1}{x}\right) - \arctan\left(\frac{1}{x}\right)\right) = \lim_{x\to+\infty} x^3\left(\frac{1}{6x^3} + o\left(\frac{1}{x^3}\right)\right)$$

$$= \lim_{x\to+\infty}\left(\frac{1}{6} + \frac{o\left(\frac{1}{x^3}\right)}{\frac{1}{x^3}}\right) = \frac{1}{6}$$

5) $$\lim_{x\to-\infty} x^3\left(e^{\frac{1}{x}} - 1 - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{3x^3}\right)$$

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + o(y^3) \quad \text{per } y\to 0 \qquad \lim_{x\to-\infty}\frac{1}{x} = 0$$

$$e^{\frac{1}{x}} = 1 + \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} + o\left(\frac{1}{x^3}\right) \quad \text{per } x\to-\infty$$

$$\lim_{x\to-\infty} x^3\left(-\frac{1}{6x^3} + o\left(\frac{1}{x^3}\right)\right) = \lim_{x\to-\infty}\left(-\frac{1}{6} + \underbrace{x^3 o\left(\frac{1}{x^3}\right)}_{\to 0}\right) = -\frac{1}{6}$$

6) $$\lim_{x\to 0} \frac{e^{\arctan x} - 1 - \arctan x}{\left(e^{\arctan x} - 1\right)^2} \underset{\uparrow}{=} \lim_{y\to 0} \frac{e^y - 1 - y}{(e^y - 1)^2}$$

$y = \arctan x$
per $x\to 0$: $y \to \arctan 0 = 0$

per il numeratore

$$e^y = 1 + y + \frac{y^2}{2} + o(y^2) \quad \text{per } y\to 0 \implies e^y - 1 - y = \frac{y^2}{2} + o(y^2) \quad \text{per } y\to 0$$

per il denominatore

$$e^y = 1 + y + o(y) \quad \text{per } y\to 0 \implies e^y - 1 = y + o(y) \quad \text{per } y\to 0$$

$$\implies (e^y - 1)^2 = (y + o(y))^2 = y^2 + o(y^2) \quad \text{per } y\to 0$$

$$\lim_{y\to 0} \frac{e^y - 1 - y}{(e^y - 1)^2} = \lim_{y\to 0} \frac{\frac{y^2}{2} + o(y^2)}{y^2 + o(y^2)} = \frac{1}{2}$$

7) $\lim_{x\to 0^+} \left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{\sin^2 x}}$ forma indeterminata $[1^\infty]$

$$= \lim_{x\to 0^+} \exp\left[\ln\left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{\sin^2 x}}\right]$$

$$= \lim_{x\to 0^+} \exp\left[\frac{1}{\sin^2 x} \ln\left(\frac{\ln(1+x)}{x}\right)\right]$$ forma indeterminata $[\frac{0}{0}]$

$$= \lim_{x\to 0^+} \exp\left[\frac{\ln\left(\frac{\ln(1+x)}{x}\right)}{\sin^2 x}\right]$$

$\sin x = x + o(x)$ per $x\to 0$

$\sin^2 x = (x+o(x))^2 = x^2 + o(x^2)$ per $x\to 0$

$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$ per $x\to 0$

$\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{o(x^2)}{x} = 1 - \frac{x}{2} + o(x)$ per $x\to 0$

$\ln\left(\frac{\ln(1+x)}{x}\right) = \ln\left(1 - \frac{x}{2} + o(x)\right) = \ln(1+y) = y + o(y)$

$\boxed{y = -\frac{x}{2} + o(x) \text{ per } x\to 0}$

$= -\frac{x}{2} + o(x) + o\left(-\frac{x}{2} + o(x)\right)$

$= -\frac{x}{2} + o(x) + o(x) = -\frac{x}{2} + o(x)$ per $x\to 0$

$$\lim_{x\to 0^+} \exp\left(\frac{\ln\left(\frac{\ln(1+x)}{x}\right)}{\sin^2 x}\right) = \lim_{x\to 0^+} \exp\left(\frac{-\frac{x}{2} + o(x)}{x^2 + o(x^2)}\right)$$

$$= \lim_{x\to 0^+} \exp\left(\underbrace{\frac{1}{x}}_{\to +\infty} \cdot \underbrace{\frac{-\frac{1}{2} + o(1)}{1 + o(1)}}_{\to -\frac{1}{2}}\right)$$

$$= \lim_{t\to -\infty} e^t = 0^+$$

cambio di variabile nel limite

8) $\lim_{x\to 0} \frac{\sin(2x) - 2\sin x}{x(\cos(4x) - \cos(3x))}$ forma indeterminata $[\frac{0}{0}]$

$\cos y = 1 - \frac{y^2}{2} + o(y^2)$ per $y\to 0$ $\qquad \lim_{x\to 0} 4x = 0 = \lim_{x\to 0} 3x$

$\cos(4x) = 1 - 8x^2 + o(x^2)$ per $x\to 0$

$\cos(3x) = 1 - \frac{9}{2}x^2 + o(x^2)$ per $x\to 0$

$$x(\cos(4x)-\cos(3x)) = x\cdot\left(\cancel{1}-8x^2+o(x^2)-\cancel{1}+\frac{9}{2}x^2-o(x^2)\right)$$

$$= x\cdot\left(-\frac{7}{2}x^2+o(x^2)\right) = -\frac{7}{2}x^3+o(x^3) \quad \text{per } x\to 0$$

Determiniamo il polinomio di Taylor di ordine 3 del numeratore

$$\sin y = y-\frac{y^3}{6}+o(y^3) \quad \text{per } y\to 0 \qquad \lim_{x\to 0} 2x = 0$$

$$\sin(2x) = 2x-\frac{4}{3}x^3+o(x^3) \quad \text{per } x\to 0$$

$$\sin(2x)-2\sin x = 2\cancel{x}-\frac{4}{3}x^3+o(x^3)-2\left(\cancel{x}-\frac{x^3}{6}+o(x^3)\right)$$

$$= -x^3+o(x^3) \quad \text{per } x\to 0$$

$$\lim_{x\to 0}\frac{\sin(2x)-2\sin x}{x(\cos(4x)-\cos(3x))} = \lim_{x\to 0}\frac{-x^3+o(x^3)}{-\frac{7}{2}x^3+o(x^3)} = \frac{2}{7}$$