

Esercizio sullo studio di funzione (parte II)

4) $f(x) = x \sqrt{4-x^2}$

C.E. $4-x^2 \geq 0 \Leftrightarrow x^2-4 \leq 0 \Leftrightarrow -2 \leq x \leq 2$

$D_f = [-2, 2]$

$f(-x) = (-x)\sqrt{4-(-x)^2} = -x\sqrt{4-x^2} = -f(x)$

f è una funzione dispari

zeri di f : $f(x) = 0 \Leftrightarrow x=0 \vee \sqrt{4-x^2} = 0$

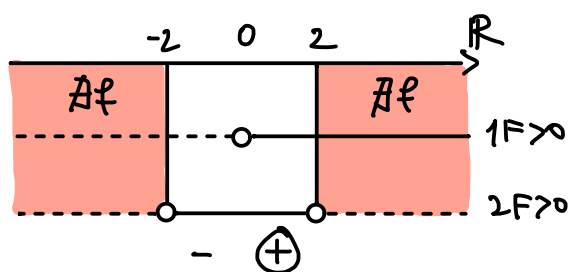
$\Leftrightarrow \begin{cases} x=0 \vee 4-x^2=0 \\ x \in D_f \end{cases} \Leftrightarrow \begin{cases} x=0 \vee x=\pm 2 \\ x \in D_f \end{cases}$

$\Leftrightarrow x=0 \vee x=\pm 2$

Segno di f : $f(x) > 0 \Leftrightarrow \begin{cases} x \sqrt{4-x^2} > 0 \\ x \in D_f \end{cases}$

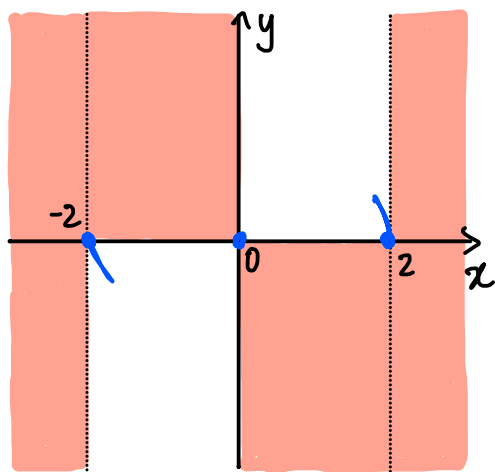
$1^{\circ} f > 0 : x > 0$

$2^{\circ} f > 0 : \sqrt{4-x^2} > 0 \Leftrightarrow -2 < x < 2$



$f(x) > 0 \Leftrightarrow 0 < x < 2$

$f(x) < 0 \Leftrightarrow -2 < x < 0$



f è continua in D_f

$\lim_{x \rightarrow -2^+} f(x) = f(-2) = 0^-$

$\lim_{x \rightarrow 2^-} f(x) = f(2) = 0^+$

$$f(x) = x \sqrt{4-x^2}$$

$$f'(x) = \sqrt{4-x^2} + x \cdot \frac{1}{2\sqrt{4-x^2}} \cdot (-2x) = \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}$$

$$= \frac{(\sqrt{4-x^2})^2 - x^2}{\sqrt{4-x^2}} = \frac{4-x^2-x^2}{\sqrt{4-x^2}} = \frac{2(2-x^2)}{\sqrt{4-x^2}}$$

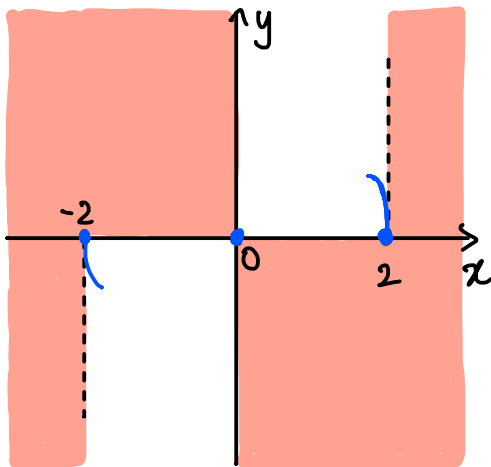
$$D_{f'} = D_f \setminus \{\pm 2\}$$

$$\lim_{x \rightarrow -2^+} f'(x) = \lim_{x \rightarrow -2^+} \frac{2(2-x^2)}{\sqrt{4-x^2}} = -\infty$$

perché $\lim_{x \rightarrow -2^+} 2(2-x^2) = -4$ e $\lim_{x \rightarrow -2^+} \sqrt{4-x^2} = 0^+$

analogamente $\lim_{x \rightarrow 2^-} f'(x) = -\infty$

$x = \pm 2$ sono punti di tangente verticale



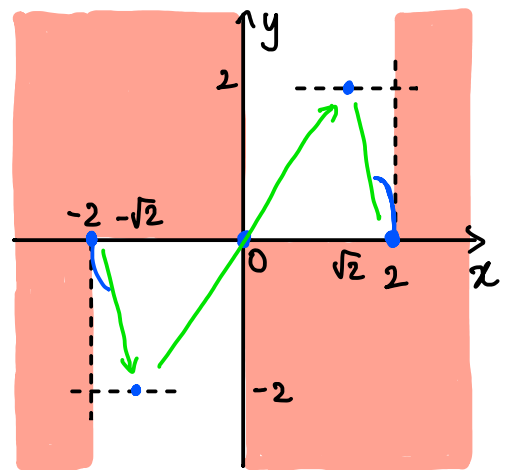
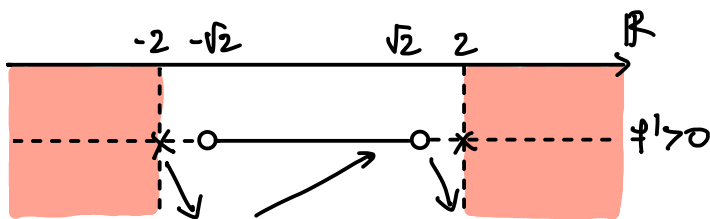
punti stazionari: $f'(x) = 0 \Leftrightarrow \frac{2(2-x^2)}{\sqrt{4-x^2}} = 0$

$$\Leftrightarrow \begin{cases} 2-x^2=0 \\ x \in D_{f'} \end{cases} \Leftrightarrow \begin{cases} x = \pm\sqrt{2} \\ x \in D_{f'} \end{cases} \Leftrightarrow x = \pm\sqrt{2}$$

$$f(\pm\sqrt{2}) = \pm f(\sqrt{2}) = \pm \sqrt{2} \cdot \sqrt{4-(\sqrt{2})^2} = \pm \sqrt{2} \cdot \sqrt{2} = \pm 2$$

segno di f' : $f'(x) > 0 \Leftrightarrow \frac{2(2-x^2)}{\sqrt{4-x^2}} > 0 \Leftrightarrow \begin{cases} 2-x^2 > 0 \\ x \in D_{f'} \end{cases}$

$$\Leftrightarrow \begin{cases} -\sqrt{2} < x < \sqrt{2} \\ -2 < x < 2 \end{cases} \Leftrightarrow -\sqrt{2} < x < \sqrt{2}$$



$x = -\sqrt{2}$ punto di minimo assoluto

$x = \sqrt{2}$ punto di massimo assoluto

$x = -2$ punto di massimo relativo

$x = 2$ punto di minimo relativo

$$f'(x) = \frac{2(2-x^2)}{\sqrt{4-x^2}} \quad \text{calcoliamo la derivata seconda}$$

$$f''(x) = 2 \frac{-2x \cdot \sqrt{4-x^2} - (2-x^2) \frac{-x}{\sqrt{4-x^2}}}{(\sqrt{4-x^2})^2}$$

$$= 2 \frac{\frac{(2-x^2)x}{\sqrt{4-x^2}} - 2x\sqrt{4-x^2}}{(\sqrt{4-x^2})^2} = 2 \frac{(2-x^2)x - 2x(\sqrt{4-x^2})^2}{(\sqrt{4-x^2})^3}$$

$$= 2 \frac{(2-x^2)x - 2x(4-x^2)}{(\sqrt{4-x^2})^3} = \frac{2x(2-x^2-8+2x^2)}{(\sqrt{4-x^2})^3}$$

$$= \frac{2x(x^2-6)}{(\sqrt{4-x^2})^3}$$

$$D_{f''} = D_{f'} = (-2, 2)$$

$$\text{punti di flesso : } f''(x) = 0 \Leftrightarrow \frac{2x(x^2-6)}{(\sqrt{4-x^2})^3} = 0$$

$$\Leftrightarrow \begin{cases} 2x=0 & \vee & x^2-6=0 \\ x \in D_{f''} \end{cases} \Leftrightarrow \begin{cases} x=0 & \vee & x=\pm\sqrt{6} \\ x \in (-2, 2) \end{cases} \quad \text{non accettabili}$$

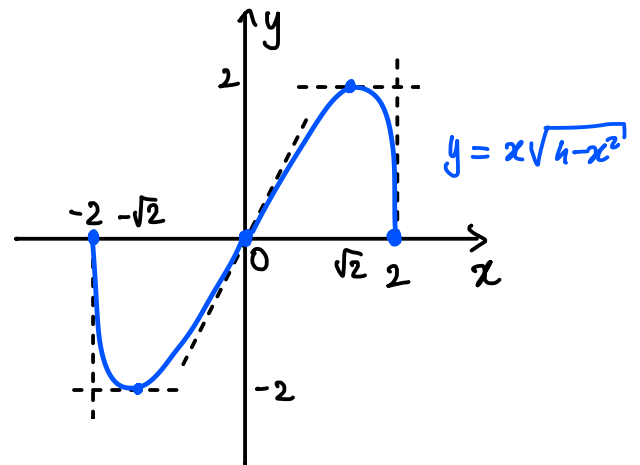
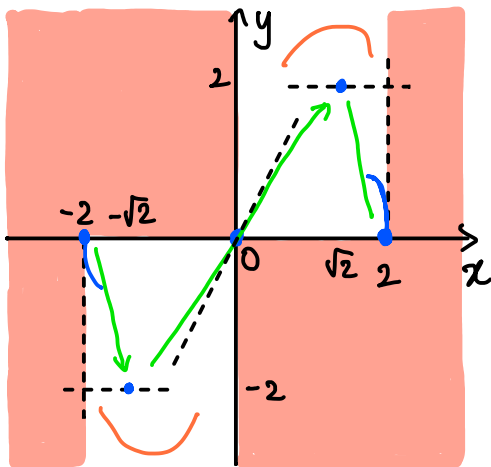
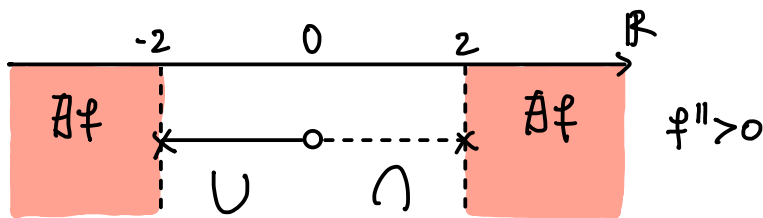
$$\Leftrightarrow x=0$$

$f(0) = 0$, $f'(0) = 2$ coefficiente angolare delle rette tangenti nel punto di flesso $x=0$

$$\text{segno } f'' : f''(x) > 0 \Leftrightarrow \frac{2x(x^2-6)}{(\sqrt{4-x^2})^3} > 0$$

osserviamo che il fattore x^2-6 è positivo per $x < -\sqrt{6}$ o $x > \sqrt{6}$ e pertanto è sempre negativo in Df'' , mentre $(\sqrt{4-x^2})^3$ è sempre positivo in $(-2,2)$. Concludiamo che

$$f''(x) > 0 \Leftrightarrow \begin{cases} 2x < 0 \\ x \in Df'' \end{cases} \Leftrightarrow \begin{cases} x < 0 \\ x \in (-2,2) \end{cases} \Leftrightarrow -2 < x < 0$$



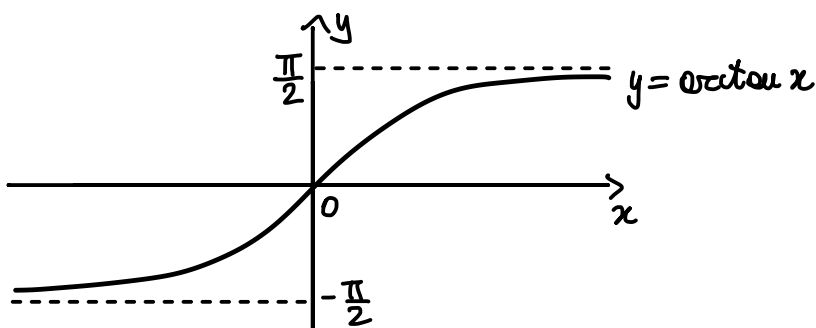
$$f([-2,2]) = [-2,2]$$

$$5) f(x) = \arctan\left(\frac{1}{x^3}\right)$$

$$\text{c.e. } x^3 \neq 0 \Leftrightarrow x \neq 0$$

$$Df = (-\infty, 0) \cup (0, +\infty)$$

ricordiamo il grafico della funzione arcotangente

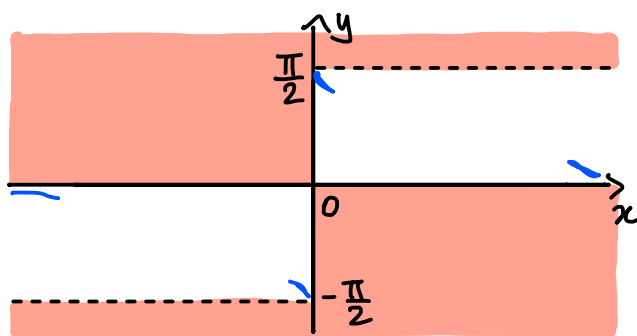


$$f(-x) = \arctan\left(\frac{1}{(-x)^3}\right) = \arctan\left(\frac{1}{-x^3}\right) = \arctan\left(-\frac{1}{x^3}\right) \\ = -\arctan\left(\frac{1}{x^3}\right)$$

f è una funzione dispari

zeri di f : $f(x) = 0 \Leftrightarrow \arctan\left(\frac{1}{x^3}\right) = 0 \Leftrightarrow \frac{1}{x^3} = 0$ impossibile

segno di f : $f(x) > 0 \Leftrightarrow \arctan\left(\frac{1}{x^3}\right) > 0$
 $\Leftrightarrow \frac{1}{x^3} > 0 \Leftrightarrow x^3 > 0 \Leftrightarrow x > 0$



$$\lim_{x \rightarrow +\infty} \arctan\left(\frac{1}{x^3}\right) = \arctan 0 = 0^+$$

$$\lim_{x \rightarrow -\infty} \arctan\left(\frac{1}{x^3}\right) = 0^-$$

$$\lim_{x \rightarrow 0^+} \arctan\left(\frac{1}{x^3}\right) = \lim_{t \rightarrow +\infty} \arctan t = \frac{\pi}{2}^-$$

$$\lim_{x \rightarrow 0^-} \arctan\left(\frac{1}{x^3}\right) = \lim_{t \rightarrow -\infty} \arctan t = -\frac{\pi}{2}^+$$

In particolare, i limiti precedenti ci dicono che

$y=0$ è asintoto orizzontale per $x \rightarrow \pm\infty$

$x=0$ è un punto di discontinuità con salto

$$f'(x) = \frac{1}{1 + \left(\frac{1}{x^3}\right)^2} \cdot \left(-\frac{3}{x^4}\right) = -\frac{3}{\left(1 + \frac{1}{x^6}\right) \cdot x^4} = -\frac{3}{\frac{x^6+1}{x^6} \cdot x^4} \\ = -\frac{3}{\frac{x^6+1}{x^2}} = -\frac{3x^2}{x^6+1}$$

$$D_{f'} = D_f = (-\infty, 0) \cup (0, +\infty)$$

punti stazionari : $f'(x) = 0 \Leftrightarrow \begin{cases} -\frac{3x^2}{1+x^6} = 0 \\ x \in D_f \end{cases}$

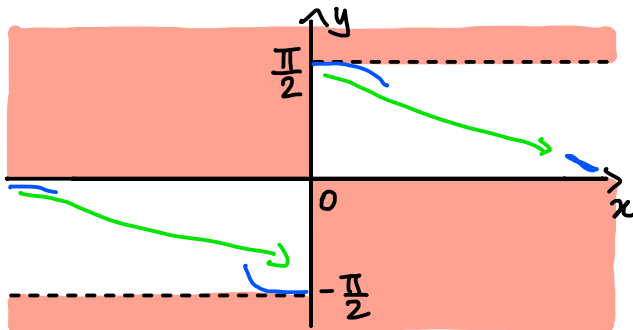
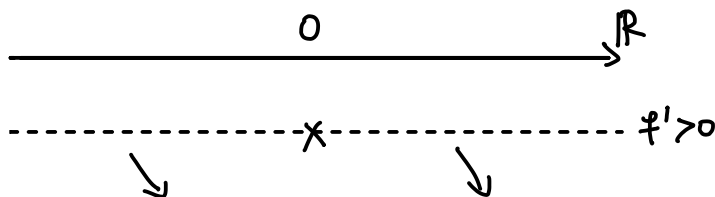
$\Leftrightarrow \begin{cases} -3x^2 = 0 \\ x \in D_f \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ x \in D_f \end{cases}$ non accettabile

$\lim_{x \rightarrow 0^-} f'(x) = \lim_{x \rightarrow 0^-} -\frac{3x^2}{1+x^6} = 0^-$

$\lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^+} -\frac{3x^2}{1+x^6} = 0^-$

segno f' : $f'(x) > 0 \Leftrightarrow \begin{cases} -\frac{3x^2}{1+x^6} > 0 \text{ impossibile} \\ x \in D_f \end{cases}$

$\forall x \in D_f : f'(x) < 0$



$f'(x) = -\frac{3x^2}{1+x^6}$

$f''(x) = -3 \frac{d}{dx} \left(\frac{x^2}{1+x^6} \right) = -3 \frac{2x(1+x^6) - x^2 \cdot 6x^5}{(1+x^6)^2}$

$= -6x \frac{1+x^6 - 3x^6}{(1+x^6)^2} = -6x \frac{1-2x^6}{(1+x^6)^2} = \frac{6x(2x^6-1)}{(1+x^6)^2}$

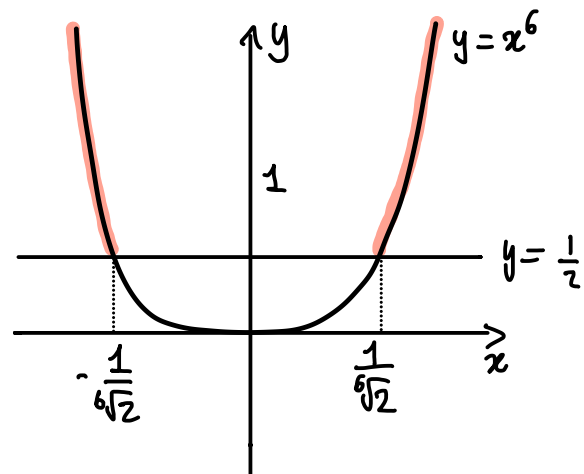
$D_{f''} = D_f = (-\infty, 0) \cup (0, +\infty)$

punti di flesso : $f''(x) = 0 \Leftrightarrow \begin{cases} \frac{6x(2x^6-1)}{(1+x^6)^2} = 0 \\ x \in D_{f''} \end{cases}$

$\Leftrightarrow \begin{cases} 6x = 0 \vee 2x^6 - 1 = 0 \\ x \in D_{f''} \end{cases} \Leftrightarrow \begin{cases} x = 0 \vee x = \pm \frac{1}{\sqrt[6]{2}} \\ x \in D_{f''} \end{cases}$

$\Leftrightarrow x = \pm \frac{1}{\sqrt[6]{2}} \simeq \pm 0.89$

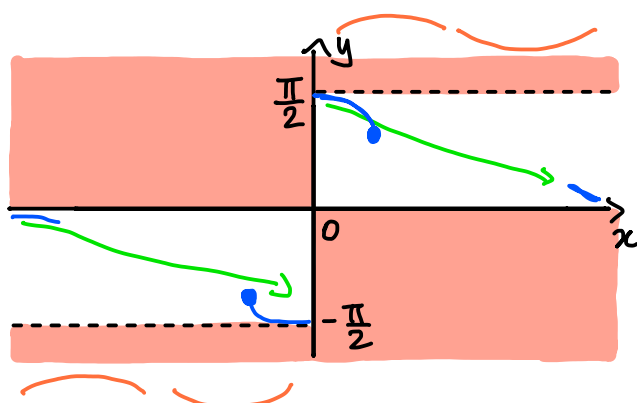
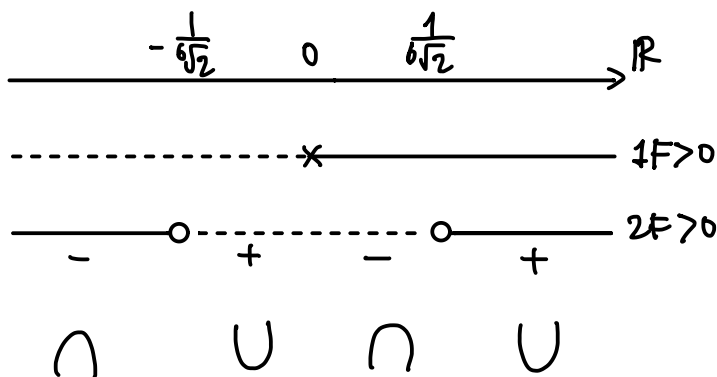
$f\left(\pm \frac{1}{\sqrt[6]{2}}\right) = \pm \arctan\left(\left(\frac{1}{\sqrt[6]{2}}\right)^3\right)$
 $= \pm \arctan(\sqrt[3]{2}) \simeq \pm 0.99$

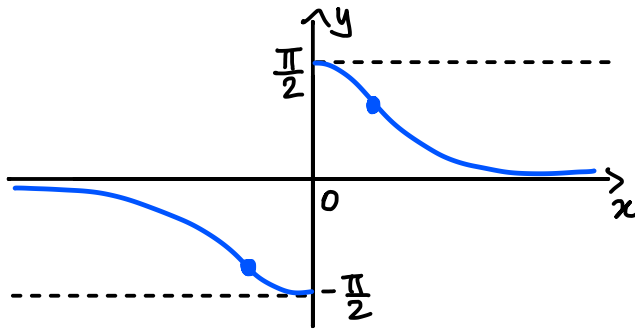


segno f'' : $f''(x) > 0 \Leftrightarrow \frac{6x(2x^6-1)}{(1+x^6)^2} > 0 \Leftrightarrow \begin{cases} 6x(2x^6-1) > 0 \\ x \in D_{f''} \end{cases}$

$1F > 0 : 6x > 0 \Leftrightarrow x > 0$

$2F > 0 : 2x^6 - 1 > 0 \Leftrightarrow x^6 > \frac{1}{2} \Leftrightarrow x < -\frac{1}{\sqrt[6]{2}} \vee x > \frac{1}{\sqrt[6]{2}}$





$$f(\mathbb{R} \setminus \{0\}) = (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$$

6) $f(x) = (|x|-1)e^{-x}$

$$D_f = \mathbb{R} = (-\infty, +\infty)$$

dalla definizione di valore assoluto segue che

$$f(x) = \begin{cases} (x-1)e^{-x} & \text{se } x \geq 0 \\ -(x+1)e^{-x} & \text{se } x \leq 0 \end{cases}$$

zeri di f : $f(x) = 0 \Leftrightarrow (|x|-1)e^{-x} = 0$

$$\Leftrightarrow |x|-1 = 0 \vee e^{-x} = 0 \text{ impossibile}$$

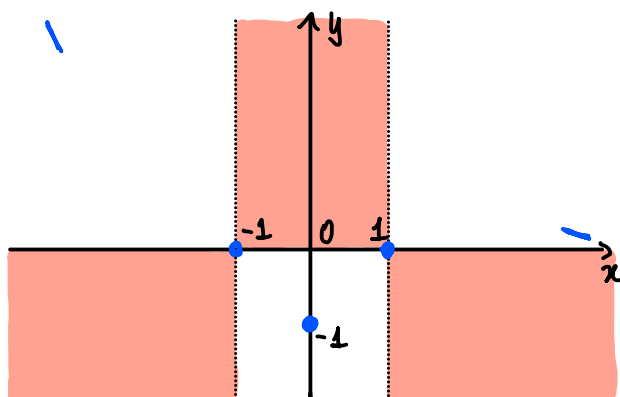
$$\Leftrightarrow |x| = 1 \Leftrightarrow x = \pm 1$$

segno di f : $f(x) > 0 \Leftrightarrow (|x|-1)e^{-x} > 0$

$$\Leftrightarrow |x|-1 > 0 \Leftrightarrow |x| > 1$$

$$\Leftrightarrow x < -1 \vee x > 1$$

calcoliamo l'ordinata del punto di intersezione tra il grafico di f e l'asse y $f(0) = (|0|-1)e^{-0} = -1$



$$\lim_{x \rightarrow -\infty} (|x|-1) e^{-x} = +\infty$$

perché $\lim_{x \rightarrow -\infty} (|x|-1) = +\infty$ e $\lim_{x \rightarrow -\infty} e^{-x} = +\infty$

$$\lim_{x \rightarrow +\infty} (|x|-1) e^{-x} \stackrel{[0, \infty]}{=} \lim_{x \rightarrow +\infty} \frac{x-1}{e^x} \stackrel{[\frac{\infty}{\infty}]}{=} 0^+$$

per le gerarchie doppi infinite

Calcoliamo f' :

per $x > 0$ $f(x) = (x-1) e^{-x}$

$$f'(x) = e^{-x} + (x-1) e^{-x} (-1) = (2-x) e^{-x}$$

per $x < 0$ $f(x) = -(x+1) e^{-x}$

$$f'(x) = -e^{-x} - (x+1) e^{-x} (-1) = x e^{-x}$$

Proviamo che $x=0$ è un punto di non derivabilità

$$\lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^+} (2-x) e^{-x} = 2$$

$$\lim_{x \rightarrow 0^-} f'(x) = \lim_{x \rightarrow 0^-} x e^{-x} = 0$$

Riassumendo
$$f'(x) = \begin{cases} (2-x) e^{-x} & \text{per } x > 0 \\ x e^{-x} & \text{per } x < 0 \end{cases}$$

e $x=0$ è un punto angoloso perché $f'_+(0) = 2 \neq 0 = f'_-(0)$

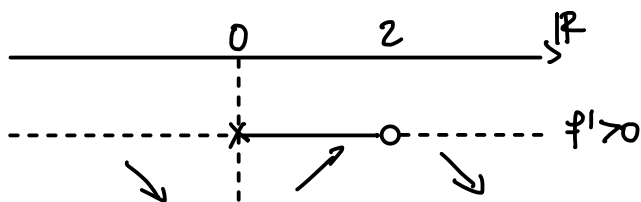
punti stazionari: $f'(x) = 0 \Leftrightarrow \begin{cases} e^{-x} (2-x) = 0 \\ x > 0 \end{cases} \vee \begin{cases} e^{-x} x = 0 \\ x < 0 \end{cases}$

$$\Leftrightarrow \begin{cases} x=2 \\ x > 0 \end{cases} \vee \begin{cases} x=0 \text{ non accettabile} \\ x < 0 \end{cases} \Leftrightarrow x=2$$

$$f(2) = (2-1) e^{-2} = e^{-2} \simeq 0.13$$

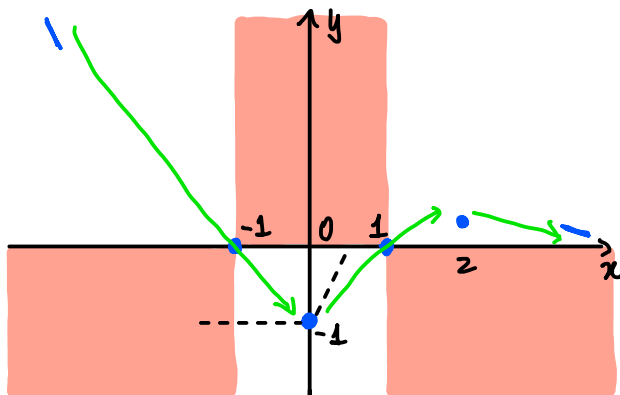
segno f' : $f'(x) > 0 \Leftrightarrow \begin{cases} e^{-x} (2-x) > 0 \\ x > 0 \end{cases} \vee \begin{cases} e^{-x} x > 0 \\ x < 0 \end{cases}$

$$\Leftrightarrow \begin{cases} 2-x > 0 \\ x > 0 \end{cases} \vee \begin{cases} x > 0 \\ x < 0 \end{cases} \text{ impossibile} \Leftrightarrow \begin{cases} x < 2 \\ x > 0 \end{cases} \Leftrightarrow 0 < x < 2$$



$x=0$ punto angoloso di minimo assoluto

$x=2$ punto di massimo relativo



$$f'(x) = \begin{cases} (2-x)e^{-x} & \text{per } x > 0 \\ xe^{-x} & \text{per } x < 0 \end{cases}$$

$$\begin{aligned} f''(x) &= \begin{cases} -e^{-x} + (2-x)e^{-x}(-1) & \text{per } x > 0 \\ e^{-x} + xe^{-x}(-1) & \text{per } x < 0 \end{cases} \\ &= \begin{cases} (x-3)e^{-x} & \text{per } x > 0 \\ (1-x)e^{-x} & \text{per } x < 0 \end{cases} \end{aligned}$$

$$\text{punti di flesso: } f''(x) = 0 \Leftrightarrow \begin{cases} (x-3)e^{-x} = 0 \\ x > 0 \end{cases} \vee \begin{cases} (1-x)e^{-x} = 0 \\ x < 0 \end{cases}$$

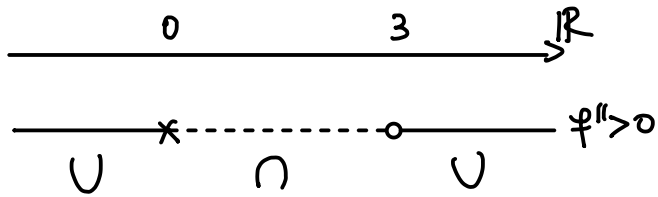
$$\Leftrightarrow \begin{cases} x=3 & \text{accettabile} \\ x > 0 \end{cases} \vee \begin{cases} x=1 & \text{non accettabile} \\ x < 0 \end{cases}$$

$$\Leftrightarrow x=3 \quad f(3) = (3-1)e^{-3} = 2e^{-3} \simeq 0.1$$

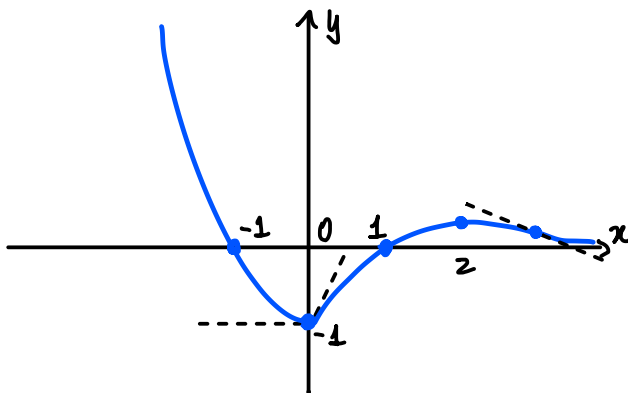
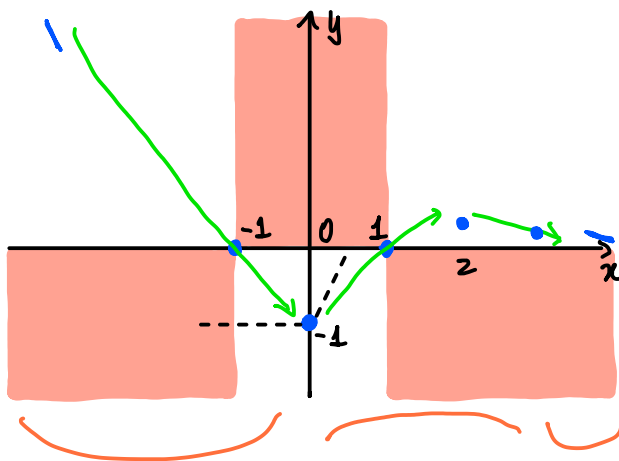
$$\text{segno } f'': \quad f''(x) > 0 \Leftrightarrow \begin{cases} (x-3)e^{-x} > 0 \\ x > 0 \end{cases} \vee \begin{cases} (1-x)e^{-x} > 0 \\ x < 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x-3 > 0 \\ x > 0 \end{cases} \vee \begin{cases} 1-x > 0 \\ x < 0 \end{cases} \Leftrightarrow \begin{cases} x > 3 \\ x > 0 \end{cases} \vee \begin{cases} x < 1 \\ x < 0 \end{cases}$$

$$\Leftrightarrow x > 3 \vee x < 0$$



$x = 3$ punto di flesso



$$\varphi(\mathbb{R}) = [-1, +\infty)$$