

Esercitazione sullo studio di funzione

1) $f(x) = \frac{x^4}{x^3+1}$

C.E. $x^3+1 \neq 0 \Rightarrow x^3 \neq -1 \Rightarrow x \neq -1$

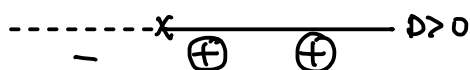
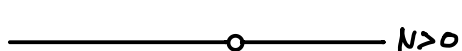
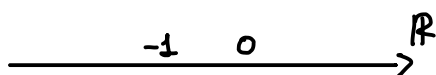
$D_f = \mathbb{R} \setminus \{-1\} = (-\infty, -1) \cup (-1, +\infty)$

zeri di f : $f(x) = 0 \Leftrightarrow \frac{x^4}{x^3+1} = 0 \Leftrightarrow \begin{cases} x^4 = 0 \\ x \in D_f \end{cases}$
 $\Leftrightarrow \begin{cases} x = 0 \\ x \in D_f \end{cases} \Leftrightarrow x = 0$

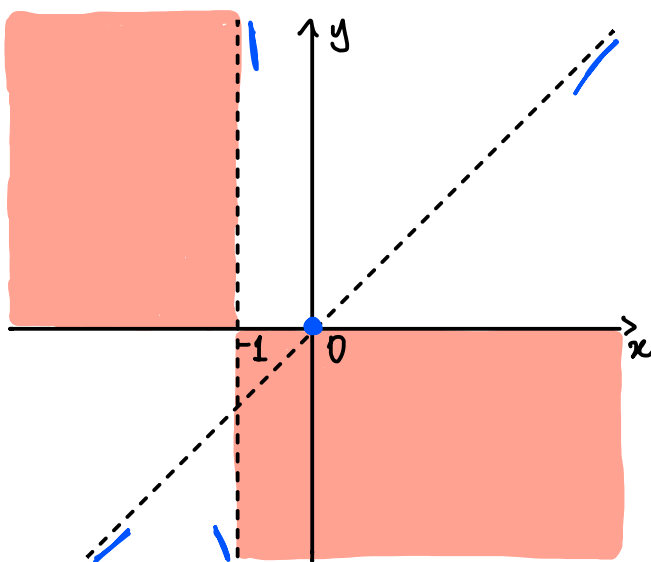
segno di f : $f(x) > 0 \Leftrightarrow \frac{x^4}{x^3+1} > 0$

$N > 0$: $x^4 > 0 \Leftrightarrow x \neq 0$

$D > 0$: $x^3+1 > 0 \Leftrightarrow x^3 > -1 \Leftrightarrow x > \sqrt[3]{-1} = -1$



$f(x) > 0 \Leftrightarrow x > -1 \wedge x \neq 0$



$\lim_{x \rightarrow +\infty} \frac{x^4}{x^3+1} \stackrel{[\frac{\infty}{\infty}]}{=} \lim_{x \rightarrow +\infty} x = +\infty$

asintoto obliquo in $x \rightarrow +\infty$

$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{x^3}{x^3+1} = 1$

$\lim_{x \rightarrow +\infty} [f(x) - x] = \lim_{x \rightarrow +\infty} \left(\frac{x^4}{x^3+1} - x \right)$

$= \lim_{x \rightarrow +\infty} \frac{x^4 - x(x^3+1)}{x^3+1}$

$= \lim_{x \rightarrow +\infty} \frac{x^4 - x^4 - x}{x^3+1} = \lim_{x \rightarrow +\infty} \frac{-x}{x^3+1}$

$= \lim_{x \rightarrow +\infty} -\frac{1}{x^2} = 0^-$

$y = x$ è asintoto obliquo in $x \rightarrow +\infty$

$\lim_{x \rightarrow -\infty} \frac{x^4}{x^3+1} \stackrel{[\frac{\infty}{\infty}]}{=} \lim_{x \rightarrow -\infty} x = -\infty$

ripetendo gli stessi calcoli troviamo che $y = x$ è asintoto obliquo anche per $x \rightarrow -\infty$

$$\lim_{x \rightarrow -1^-} \frac{x^4}{x^3+1} = -\infty, \quad \lim_{x \rightarrow -1^+} \frac{x^4}{x^3+1} = +\infty$$

$$f'(x) = \frac{4x^3(x^3+1) - x^4 \cdot 3x^2}{(x^3+1)^2} = \frac{4x^3(x^3+1) - 3x^6}{(x^3+1)^2}$$

$$= \frac{x^3(4x^3+4-3x^3)}{(x^3+1)^2} = \frac{x^3(x^3+4)}{(x^3+1)^2}$$

$$D_{f'} = D_f$$

punti stazionari: $f'(x) = 0 \Leftrightarrow \frac{x^3(x^3+4)}{(x^3+1)^2} = 0$

$$\Leftrightarrow \begin{cases} x^3(x^3+4) = 0 \\ x \in D_{f'} \end{cases} \Leftrightarrow \begin{cases} x^3 = 0 \vee x^3+4 = 0 \\ x \in D_{f'} \end{cases}$$

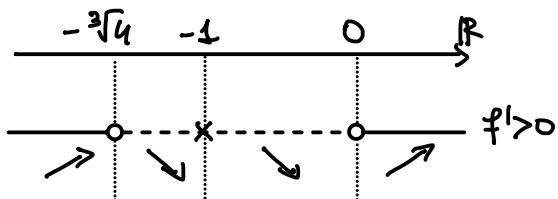
$$\Leftrightarrow \begin{cases} x = 0 \vee x = -\sqrt[3]{4} \\ x \in D_{f'} \end{cases} \Leftrightarrow x = 0 \vee x = -\sqrt[3]{4}$$

segno di f' :

$$f'(x) > 0 \Leftrightarrow \frac{x^3(x^3+4)}{(x^3+1)^2} > 0 \Leftrightarrow \begin{cases} x^3(x^3+4) > 0 \\ x \in D_{f'} \end{cases}$$

$$t = x^3 \quad t(t+4) > 0 \Leftrightarrow t < -4 \vee t > 0 \Leftrightarrow x^3 < -4 \vee x^3 > 0$$

$$\Leftrightarrow x < -\sqrt[3]{4} \vee x > 0 \quad \sqrt[3]{4} \simeq 1.58$$

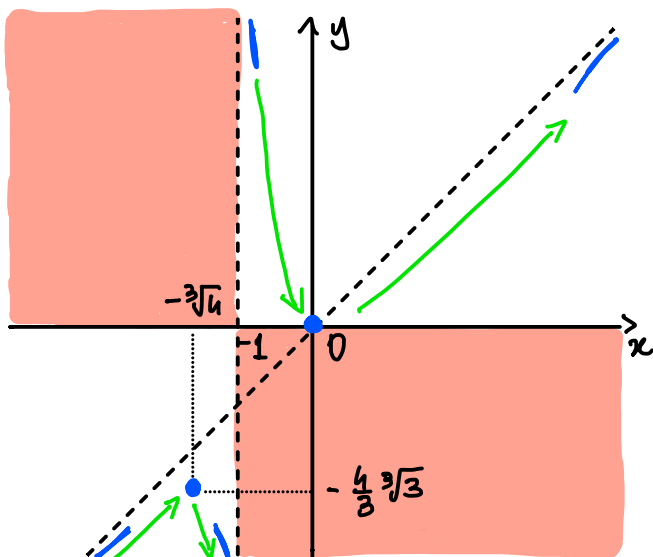


$$f(-\sqrt[3]{4}) = -\frac{4}{3}\sqrt[3]{4} \simeq -2.12$$

$$f(0) = 0$$

$x = -\sqrt[3]{4}$ è punto di massimo relativo

$x = 0$ è punto di minimo relativo



f crescente in $(-\infty, -\sqrt[3]{4})$

f decrescente in $(-\sqrt[3]{4}, -1)$

f decrescente in $(-1, 0)$

f crescente in $(0, +\infty)$

$$f'(x) = \frac{x^3(x^3+4)}{(x^3+1)^2}$$

$$f''(x) = \frac{3x^2(x^3+4+x^3) \cdot (x^3+1)^2 - x^3(x^3+4) \cdot 2(x^3+1) \cdot 3x^2}{(x^3+1)^4}$$

$$= \frac{3x^2(x^3+1) [(2x^3+4)(x^3+1) - 2x^3(x^3+4)]}{(x^3+1)^4}$$

$$= \frac{6x^2 \cancel{(x^3+1)} [(x^3+2)(x^3+1) - x^3(x^3+4)]}{(x^3+1)^{\cancel{4}}_3}$$

$$= \frac{6x^2 [x^6+3x^3+2 - x^6-4x^3]}{(x^3+1)^3} = \frac{6x^2(2-x^3)}{(x^3+1)^3}$$

$$D_{f''} = D_f$$

punti di flesso: $f''(x) = 0 \Leftrightarrow \begin{cases} 6x^2(2-x^3) = 0 \\ x \in D_{f''} \end{cases}$

$$\Leftrightarrow \begin{cases} x^2=0 \vee x^3=2 \\ x \in D_{f''} \end{cases} \Leftrightarrow \begin{cases} x=0 \vee x=\sqrt[3]{2} \\ x \in D_{f''} \end{cases}$$

$$\Leftrightarrow x=0 \vee x=\sqrt[3]{2}$$

$\uparrow$
non è punto di flesso perché è punto di minimo relativo

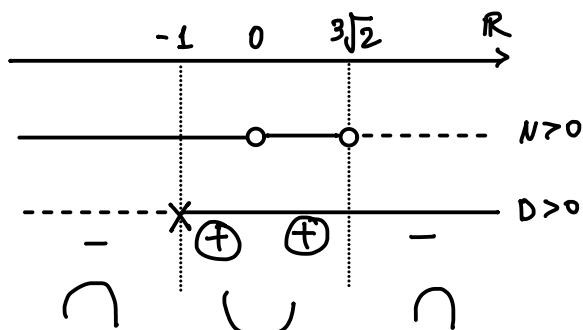
segno di f'' :

$$f''(x) > 0 \Leftrightarrow \frac{6x^2(2-x^3)}{(x^3+1)^3} > 0$$

$$N > 0 : 6x^2(2-x^3) > 0 \Leftrightarrow x \neq 0 \wedge 2-x^3 > 0$$

$$\Leftrightarrow x \neq 0 \wedge x < \sqrt[3]{2} \quad \sqrt[3]{2} \simeq 1.26$$

$$D > 0 : (x^3+1)^3 > 0 \Leftrightarrow x^3+1 > 0 \Leftrightarrow x > -1$$



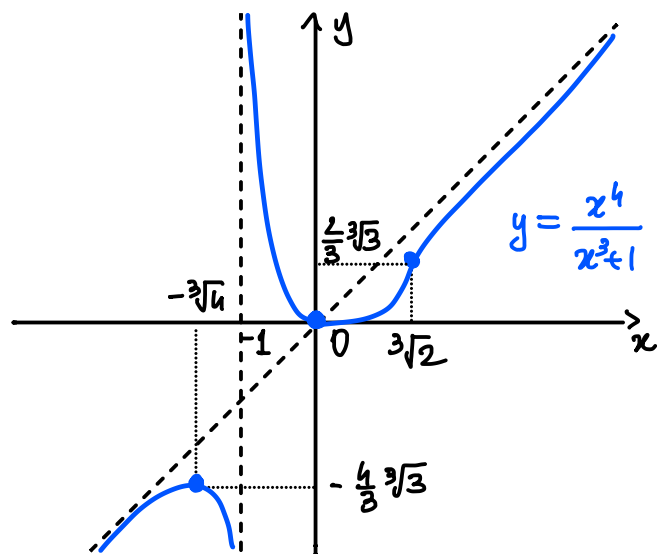
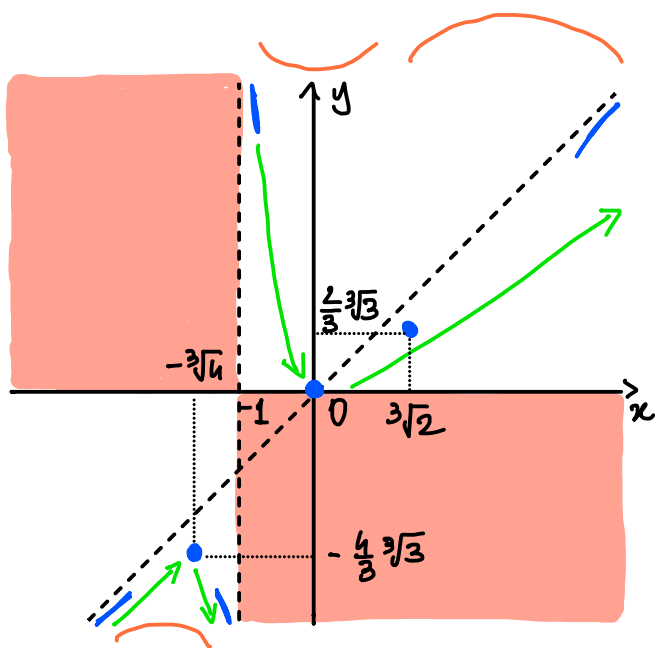
$x = \sqrt[3]{2}$ punto di flesso

$$f(\sqrt[3]{2}) = \frac{2\sqrt[3]{2}}{3} \simeq 0.84$$

f concava in $(-\infty, -1)$

f convessa in $(-1, \sqrt[3]{2})$

f concava in $(\sqrt[3]{2}, +\infty)$

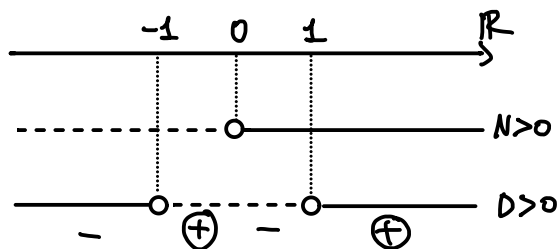


2) $f(x) = \ln\left(\frac{x}{x^2-1}\right)$

C.E. $\frac{x}{x^2-1} > 0$

$N > 0 : x > 0$

$D > 0 : x^2 - 1 > 0 \Leftrightarrow x < -1 \vee x > 1$



$D_f = (-1, 0) \cup (1, +\infty)$

zeri di f : $f(x) = 0 \Leftrightarrow \ln\left(\frac{x}{x^2-1}\right) = 0 = \ln 1$

$\Leftrightarrow \begin{cases} \frac{x}{x^2-1} = 1 \\ x \in D_f \end{cases} \Leftrightarrow \begin{cases} x^2 - x - 1 = 0 \\ x \in D_f \end{cases}$

$x^2 - x - 1 = 0 \quad \Delta = (-1)^2 - 4 \cdot (-1) \cdot 1 = 5 \quad x_{1/2} = \frac{1 \pm \sqrt{5}}{2}$

$x = \frac{1 + \sqrt{5}}{2} \approx 1.618$ è accettabile perché $\frac{1 + \sqrt{5}}{2} \in (1, +\infty)$

$x = \frac{1 - \sqrt{5}}{2} \approx -0.618$ è accettabile perché $\frac{1 - \sqrt{5}}{2} \in (-1, 0)$

$f(x) = 0 \Leftrightarrow x = \frac{1 \pm \sqrt{5}}{2}$

segno di f : $f(x) > 0 \Leftrightarrow \ln\left(\frac{x}{x^2-1}\right) > 0 = \ln 1$

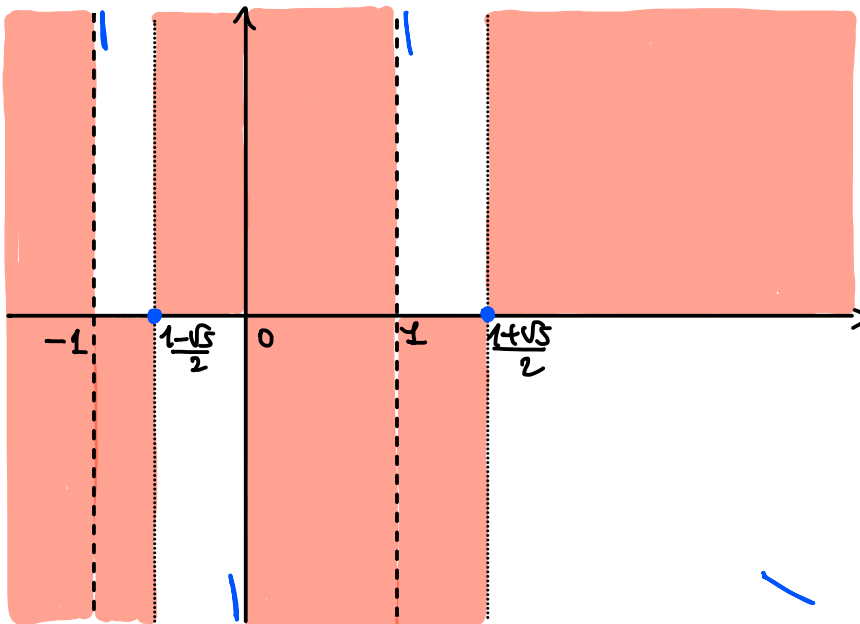
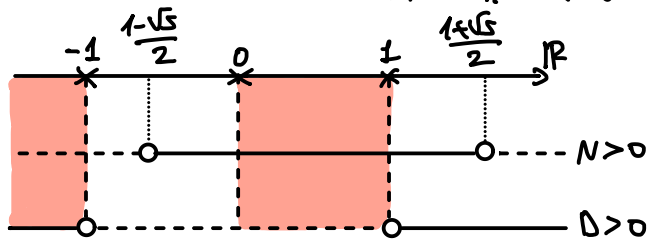
$\Leftrightarrow \begin{cases} \frac{x}{x^2-1} > 1 \\ x \in D_f \end{cases}$

$$\Leftrightarrow \begin{cases} \frac{x}{x^2-1} - 1 > 0 \\ x \in D_f \end{cases} \Leftrightarrow \begin{cases} \frac{x-(x^2-1)}{x^2-1} > 0 \\ x \in D_f \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{-x^2+x+1}{x^2-1} > 0 \\ x \in D_f \end{cases}$$

$$N > 0 : -x^2+x+1 > 0 \Leftrightarrow x^2-x-1 < 0 \Leftrightarrow \frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}$$

$$D > 0 : x^2-1 > 0 \Leftrightarrow x < -1 \vee x > 1$$



$$\begin{aligned} \lim_{x \rightarrow -1^+} \ln\left(\frac{x}{x^2-1}\right) &= \lim_{t \rightarrow +\infty} \ln t = +\infty & x = -1 \text{ è asintoto verticale dx} \\ \lim_{x \rightarrow 0^-} \ln\left(\frac{x}{x^2-1}\right) &= \lim_{t \rightarrow 0^+} \ln t = -\infty & x = 0 \text{ è asintoto verticale sx} \\ \lim_{x \rightarrow 1^+} \ln\left(\frac{x}{x^2-1}\right) &= \lim_{t \rightarrow +\infty} \ln t = +\infty & x = 1 \text{ è asintoto verticale dx} \\ \lim_{x \rightarrow +\infty} \ln\left(\frac{x}{x^2-1}\right) &= \lim_{t \rightarrow 0^+} \ln t = -\infty \end{aligned}$$

$$f'(x) = \frac{1}{\frac{x}{x^2-1}} \cdot \frac{x^2-1 - x \cdot 2x}{(x^2-1)^2} = \frac{x^2-1}{x} \cdot \frac{-x^2-1}{(x^2-1)^2} = -\frac{x^2+1}{x(x^2-1)}$$

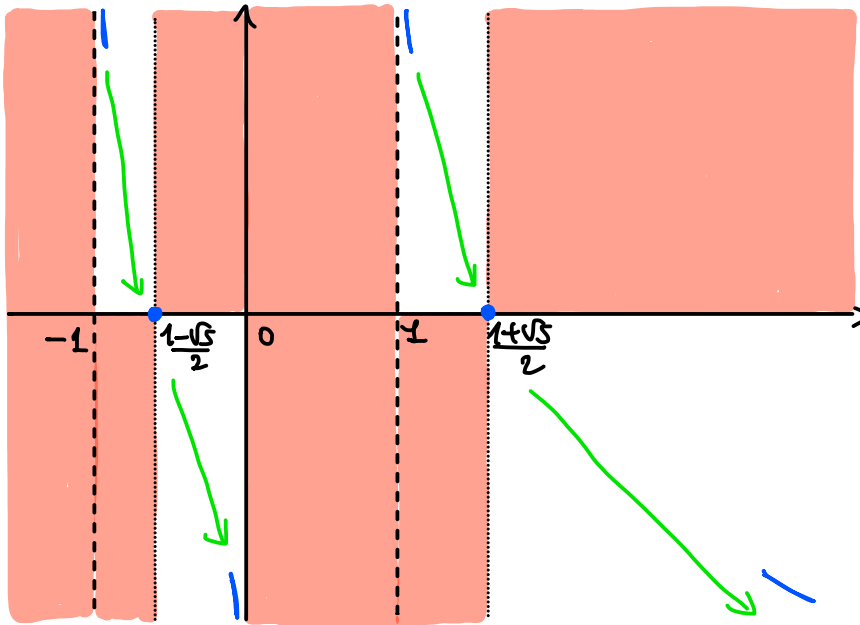
$$D_{f'} = D_f$$

punti stazionari: $f'(x) = 0 \Leftrightarrow \begin{cases} -\frac{x^2+1}{x(x^2-1)} = 0 \\ x \in D_{f'} \end{cases}$ non ci sono soluzioni

$$\text{segno di } f' : f'(x) > 0 \Leftrightarrow \begin{cases} -\frac{x^2+1}{x(x^2-1)} > 0 \\ x \in D_{f'} \end{cases} \Leftrightarrow \begin{cases} \frac{x^2+1}{x(x^2-1)} < 0 \\ x \in D_{f'} \end{cases}$$

$$\Leftrightarrow \begin{cases} x(x^2-1) < 0 & \text{non è mai verificata in } D_{f'} \\ x \in D_{f'} \end{cases}$$

$$\forall x \in D_f : f'(x) < 0$$



$$\begin{aligned} f'(x) &= -\frac{x^2+1}{x(x^2-1)} \\ f''(x) &= -\frac{2x^2(x^2-1) - (x^2+1)(3x^2-1)}{x^2(x^2-1)^2} = -\frac{2x^4 - 2x^2 - (3x^4 + 2x^2 - 1)}{x^2(x^2-1)^2} \\ &= -\frac{2x^4 - 2x^2 - 3x^4 - 2x^2 + 1}{x^2(x^2-1)^2} = -\frac{-x^4 - 4x^2 + 1}{x^2(x^2-1)^2} \\ &= \frac{x^4 + 4x^2 - 1}{x^2(x^2-1)^2} \end{aligned}$$

$$D_{f''} = D_f$$

$$\text{punti di flesso : } f''(x) = 0 \Leftrightarrow \begin{cases} \frac{x^4 + 4x^2 - 1}{x^2(x^2-1)^2} = 0 \\ x \in D_{f''} \end{cases}$$

$$\Leftrightarrow \begin{cases} x^4 + 4x^2 - 1 = 0 \\ x \in D_{f''} \end{cases}$$

$$t = x^2 \quad t^2 + 4t - 1 = 0 \quad \Delta = 2^2 - 1 \cdot (-1) = 5 \quad t_{1/2} = -2 \pm \sqrt{5}$$

$$x^2 = -2 \pm \sqrt{5} \quad (\Leftrightarrow) \quad x^2 = -2 - \sqrt{5} \quad \vee \quad x^2 = \sqrt{5} - 2$$

non ha
soluzioni

$$(\Leftrightarrow) \quad x = \pm \sqrt{\sqrt{5} - 2} \quad \sqrt{\sqrt{5} - 2} \approx 0.48$$

le soluzioni $x = \sqrt{\sqrt{5} - 2}$ non è accettabile perché $\notin D_f$

le soluzioni $x = -\sqrt{\sqrt{5} - 2}$ è accettabile

Segno di f'' : $f''(x) > 0 \quad (\Leftrightarrow) \quad \begin{cases} \frac{x^4 + 4x^2 - 1}{x^2(x^2 - 1)^2} > 0 \\ x \in D_{f''} \end{cases} \quad (\Leftrightarrow) \quad \begin{cases} x^4 + 4x^2 - 1 > 0 \\ x \in D_{f''} \end{cases}$

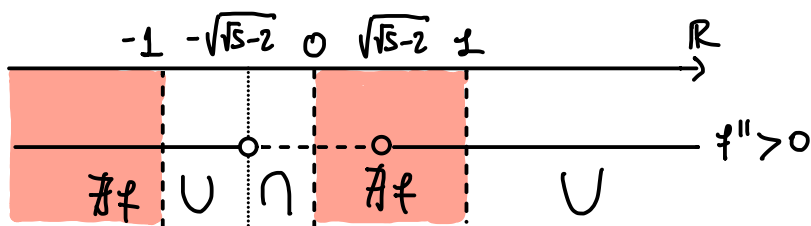
$$t = x^2 \quad t^2 + 4t - 1 > 0 \quad (\Leftrightarrow) \quad t < -2 - \sqrt{5} \quad \vee \quad t > \sqrt{5} - 2$$

$$(\Leftrightarrow) \quad x^2 < -2 - \sqrt{5} \quad \vee \quad x^2 > \sqrt{5} - 2$$

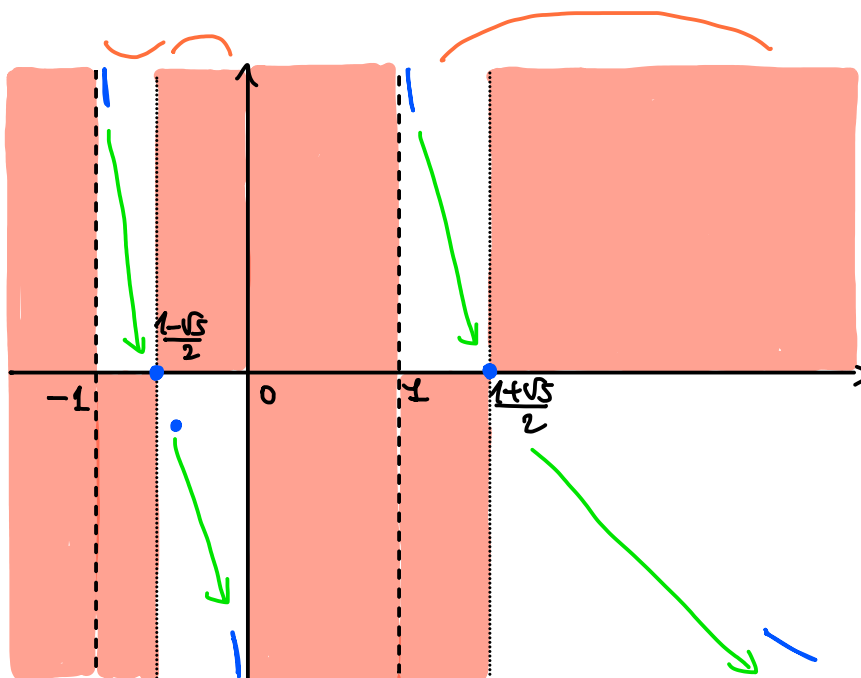
non ammette soluzioni

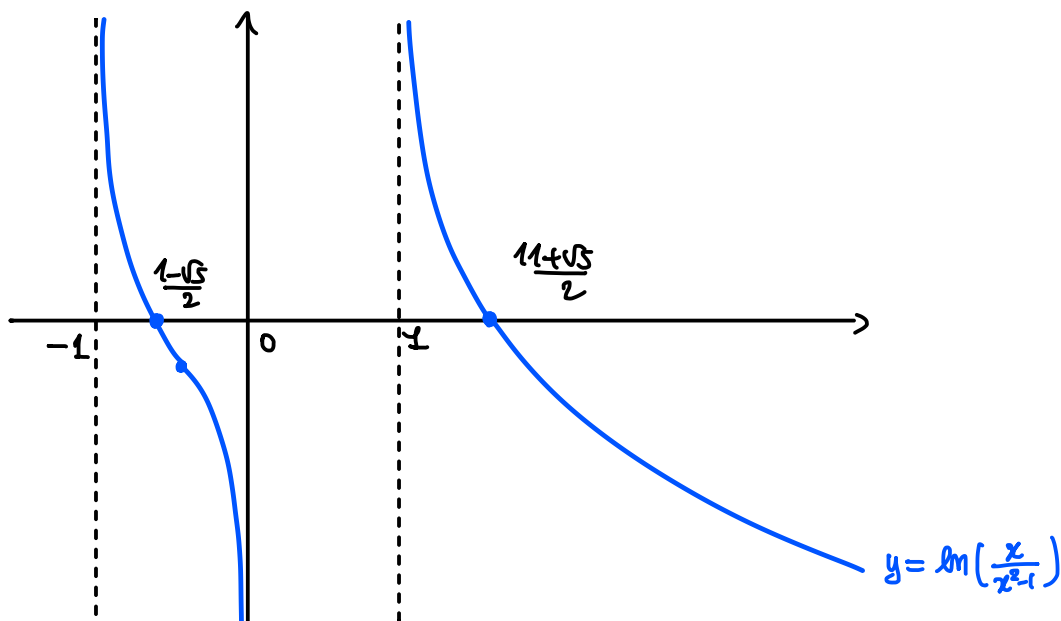
$$(\Leftrightarrow) \quad x^2 > \sqrt{5} - 2$$

$$(\Leftrightarrow) \quad x < -\sqrt{\sqrt{5} - 2} \quad \vee \quad x > \sqrt{\sqrt{5} - 2}$$



$x = -\sqrt{\sqrt{5} - 2}$
punto di flesso





3) $f(x) = x^2 e^{-\frac{1}{x}}$

C.E. $x \neq 0$

$D_f = (-\infty, 0) \cup (0, +\infty)$

zeri f : $f(x) = 0 \Leftrightarrow x^2 = 0 \vee e^{-\frac{1}{x}} = 0$

$\Leftrightarrow x = 0$ non accettabile

segno di f : $f(x) > 0 \quad \forall x \in D_f$

$\lim_{x \rightarrow -\infty} x^2 e^{-\frac{1}{x}} = +\infty$

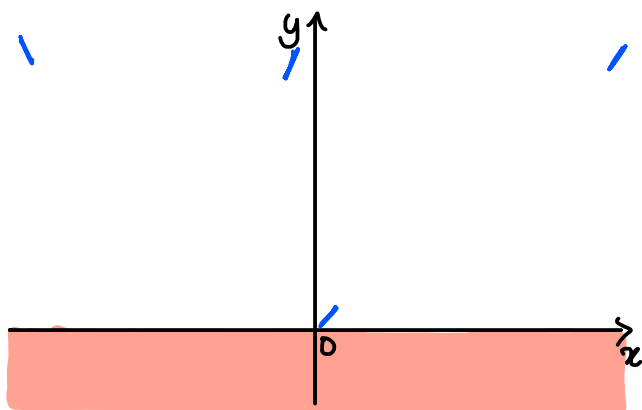
perché $\lim_{x \rightarrow -\infty} e^{-\frac{1}{x}} = \lim_{t \rightarrow 0^+} e^t = 1$

$\lim_{x \rightarrow +\infty} x^2 e^{-\frac{1}{x}} = +\infty$

$\lim_{x \rightarrow 0^-} x^2 e^{-\frac{1}{x}} \stackrel{[0, \infty]}{=} \lim_{x \rightarrow 0^-} \frac{e^{-\frac{1}{x}}}{\frac{1}{x^2}} = \lim_{t \rightarrow +\infty} \frac{e^t}{t^2} \stackrel{\text{gerarchia degli infiniti}}{=} +\infty$
 $t = -\frac{1}{x}$ per $x \rightarrow 0^- : t \rightarrow +\infty$

$\lim_{x \rightarrow 0^+} x^2 e^{-\frac{1}{x}} = 0$

$x = 0$ è asintoto verticale



Osserviamo che non ci sono

asintoti obliqui perché

$\lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \lim_{x \rightarrow \pm\infty} x e^{-\frac{1}{x}} = \pm\infty$

$$f'(x) = 2x e^{-\frac{1}{x}} + x^2 \cdot \left(-\frac{(-1)}{x^2}\right) \cdot e^{-\frac{1}{x}} = (2x+1)e^{-\frac{1}{x}}$$

$$D_{f'} = D_f$$

Osserviamo che $\lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^+} (2x+1)e^{-\frac{1}{x}} = 0^+$

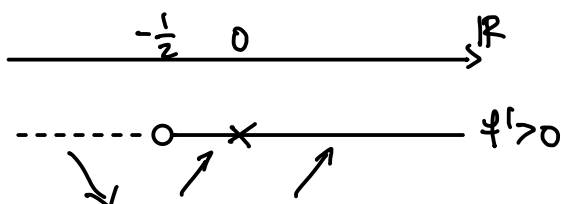
Punti stazionari: $f'(x) = 0 \Leftrightarrow (2x+1)e^{-\frac{1}{x}} = 0$

$$\Leftrightarrow \begin{cases} 2x+1=0 \\ x \in D_{f'} \end{cases} \vee e^{-\frac{1}{x}} = 0 \Leftrightarrow \begin{cases} x = -\frac{1}{2} \\ x \in D_{f'} \end{cases} \Leftrightarrow x = -\frac{1}{2}$$

Studio del segno di f' :

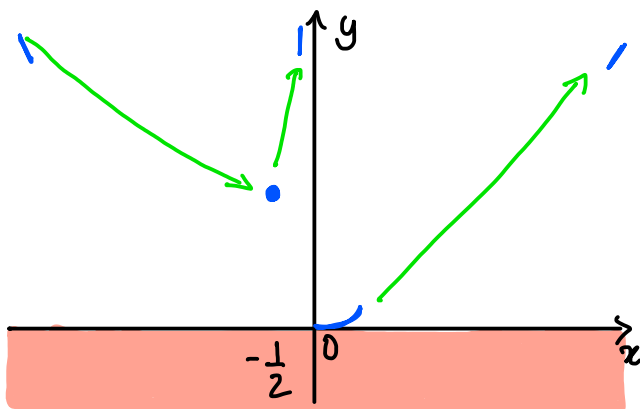
$$f'(x) > 0 \Leftrightarrow (2x+1)e^{-\frac{1}{x}} > 0 \Leftrightarrow \begin{cases} 2x+1 > 0 \\ x \in D_{f'} \end{cases}$$

$$\Leftrightarrow \begin{cases} x > -\frac{1}{2} \\ x \neq 0 \end{cases} \Leftrightarrow x \in (-\frac{1}{2}, 0) \cup (0, +\infty)$$



$x = -\frac{1}{2}$ punto di minimo relativo

$$f(-\frac{1}{2}) = \frac{e^2}{4} \simeq 1.85$$



$$f'(x) = (2x+1)e^{-\frac{1}{x}}$$

$$f''(x) = 2 \cdot e^{-\frac{1}{x}} + (2x+1) \cdot \frac{1}{x^2} e^{-\frac{1}{x}} = \frac{2x^2+2x+1}{x^2} e^{-\frac{1}{x}}$$

$$D_{f''} = D_f$$

Punti di flesso: $f''(x) = 0 \Leftrightarrow \frac{2x^2+2x+1}{x^2} e^{-\frac{1}{x}} = 0$

$$\Leftrightarrow \begin{cases} 2x^2+2x+1=0 \\ x \in D_{f''} \end{cases} \quad \Delta = 1 - 2 \cdot 1 < 0$$

non ci sono soluzioni

$\forall x \in D_{f''} : f''(x) > 0$ perché $2x^2 + 2x + 1 > 0$ (essendo $\Delta < 0$)

Quindi f è convessa in $(-\infty, 0)$ e in $(0, +\infty)$

