MATEMATICA - LEZIONE 24 - 14/11/24 14.00 - 16.00 Aula 4

Esempi (derivata della funzione composta)

1) $D(\ln(\cos x))$

$\ln(\cos x) = g(f(x))$ $\qquad g(x) = \ln x$, $f(x) = \cos x$

$\qquad g'(x) = \frac{1}{x}$, $f'(x) = -\sin x$

$$D(\ln(\cos x)) = g'(f(x)) \cdot f'(x)$$
$$= \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)} = \frac{-\sin x}{\cos x} = -\tan x$$

Più in generale $\boxed{D \ln(f(x)) = \frac{f'(x)}{f(x)}}$

2) $D \ln(x^3+3x^2+6x+4) = \frac{3x^2+6x+6}{x^3+3x^2+6x+4} = \frac{3(x^2+2x+2)}{x^3+3x^2+6x+4}$

3) $D \ln\left(\frac{x^2-x}{x+1}\right) = \frac{1}{\frac{x^2-x}{x+1}} \cdot D\left(\frac{x^2-x}{x+1}\right)$

$$= \frac{x+1}{x^2-x} \cdot \frac{D(x^2-x)(x+1) - (x^2-x)D(x+1)}{(x+1)^2} = \frac{\cancel{x+1}}{x^2-x} \frac{(2x-1)(x+1)-(x^2-x)}{(x+1)^{\cancel{2}}}$$

$$= \frac{2x^2+2x-x-1-x^2+x}{(x^2-x)(x+1)} = \frac{x^2+2x-1}{(x^2-x)(x+1)} = \frac{x^2+2x-1}{x(x-1)(x+1)}$$

4) $D(\sin\sqrt{x})$

$\sin\sqrt{x} = g(f(x))$ $\qquad g(x) = \sin x$, $f(x) = \sqrt{x}$

$\qquad g'(x) = \cos x$, $f'(x) = \frac{1}{2\sqrt{x}}$

$$D(\sin\sqrt{x}) = g'(f(x))\, f'(x)$$
$$= \cos(f(x)) \cdot f'(x) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

In generale

$$\boxed{\begin{aligned} D \sin(f(x)) &= f'(x) \cdot \cos f(x) \\ D \cos(f(x)) &= -f'(x) \cdot \sin f(x) \\ D \tan(f(x)) &= \frac{f'(x)}{\cos^2 f(x)} = f'(x)\,(1+\tan^2 f(x)) \end{aligned}}$$

5) $D\left((\ln x)^3 \right)$

$$(\ln x)^3 = g(f(x)) \qquad f(x) = \ln x \,, \quad g(x) = x^3$$
$$f'(x) = \frac{1}{x} \,, \quad g'(x) = 3x^2$$

$$D\,(\ln x)^3 = g'(f(x)) \cdot f'(x)$$
$$= 3\,(f(x))^2 \cdot f'(x)$$
$$= 3\,(\ln x)^2 \cdot \frac{1}{x} = \frac{3\,(\ln x)^2}{x}$$

6) $$D\left(\sqrt[5]{\sin x} \right) = \frac{\cos x}{5\sqrt[5]{\sin^4 x}}$$

In generale $$\boxed{D\,(f(x))^{\alpha} = \alpha\,(f(x))^{\alpha-1}\, f'(x)}$$

Teorema (derivata della funzione inversa)

Siano $I, J \subseteq \mathbb{R}$ intervalli e $f: I \to J$ biettiva, denotiamo con $f^{-1}: J \to I$ la funzione inversa di f. Se f è derivabile in x_0 e $f'(x_0) \neq 0$ allora f^{-1} è derivabile in $y_0 = f(x_0)$ e $$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Osservazione Possiamo riscrivere la formula precedente

$$\frac{df^{-1}}{dx}(x) = \frac{1}{\left.\frac{df}{dy}(y)\right|_{y=f^{-1}(x)}}$$

Esempi (derivata della funzione inversa)

1) Calcoliamo la derivata della funzione $\log_a$

$$\frac{d}{dx}\log_a x = \frac{1}{\left.\frac{d}{dy}a^y\right|_{y=\log_a x}} = \frac{1}{\left.\ln a \cdot a^y\right|_{y=\log_a x}}$$
$$= \frac{1}{\ln a \cdot x}$$

$$a^{\log_a x} = x$$

2) Calcoliamo la derivata di $\sqrt[n]{x}$

$$\frac{d}{dx}\sqrt[n]{x} = \frac{1}{\frac{d}{dy}y^n \Big|_{y=\sqrt[n]{x}}} = \frac{1}{ny^{n-1}\Big|_{y=\sqrt[n]{x}}} = \frac{1}{n(\sqrt[n]{x})^{n-1}}$$

$$= \frac{1}{n\sqrt[n]{x^{n-1}}}$$

3) Calcoliamo la derivata di $\arctan x$

$$\frac{d}{dx}\arctan x = \frac{1}{\frac{d}{dy}\tan y\Big|_{y=\arctan x}} = \frac{1}{1+\tan^2 y\Big|_{y=\arctan x}}$$

$$\underset{\uparrow}{=} \frac{1}{1+x^2} \implies \boxed{D\arctan x = \frac{1}{1+x^2}}$$

$\tan(\arctan x) = x$

4) Calcoliamo la derivata di $\arcsin x$

$$\frac{d}{dx}\arcsin x = \frac{1}{\frac{d}{dy}\sin y\Big|_{y=\arcsin x}} = \frac{1}{\cos y\Big|_{y=\arcsin x}}$$

$\arcsin : [-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\cos^2 y + \sin^2 y = 1 \quad \forall y \in \mathbb{R} \implies \cos y = \pm\sqrt{1-\sin^2 y}$

$y = \arcsin x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, sappiamo che la funzione coseno è positiva in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, pertanto otteniamo

$$\cos(\arcsin x) = +\sqrt{1-\sin^2(\arcsin x)} \underset{\uparrow}{=} \sqrt{1-x^2}$$

$\sin(\arcsin x) = x$

Concludiamo che

$$\boxed{D(\arcsin x) = \frac{1}{\sqrt{1-x^2}}} \qquad x \in (-1,1)$$

Analogamente si prova che $\boxed{D(\arccos x) = -\frac{1}{\sqrt{1-x^2}}} \qquad x \in (-1,1)$

Utilizzando le derivate delle funzioni trigonometriche inverse abbiamo che

$$\boxed{\begin{aligned} D \arctan f(x) &= \frac{f'(x)}{1+f^2(x)} \\ D \arcsin f(x) &= \frac{f'(x)}{\sqrt{1-f^2(x)}} \\ D \arccos f(x) &= \frac{-f'(x)}{\sqrt{1-f^2(x)}} \end{aligned}}$$

Esercizi di riepilogo sul calcolo delle derivate

1) $f(x) = \cos(3x)\cdot e^{2x} + \sin x \cdot e^{-x}$

$$\begin{aligned} f'(x) &= D(\cos(3x)\, e^{2x}) + D(\sin x \cdot e^{-x}) \\ &= D(\cos(3x))\cdot e^{2x} + \cos(3x)\, D(e^{2x}) + D(\sin x) e^{-x} + \sin x\, D(e^{-x}) \\ &= -3\sin(3x)\cdot e^{2x} + \cos(3x)\cdot 2e^{2x} + \cos x\, e^{-x} + \sin x \cdot e^{-x}\cdot(-1) \\ &= [-3\sin(3x) + 2\cos(3x)]\, e^{2x} + [\cos x - \sin x]\, e^{-x} \end{aligned}$$

2) $f(x) = \arctan(e^{x^3})$

regola della catena per la composizione di tre funzioni:

$$\begin{aligned} D(h \circ g \circ f) &= D(h \circ (g \circ f)) = (D(h) \circ (g \circ f)) \cdot D(g \circ f) \\ &= (D(h) \circ g \circ f) \cdot (D(g) \circ f) \cdot D(f) \end{aligned}$$

quindi

$$\boxed{D(h \circ g \circ f) = (D(h) \circ g \circ f) \cdot (D(g) \circ f) \cdot D(f)}$$

$$f'(x) = \frac{1}{1+(e^{x^3})^2} \cdot D(e^{x^3}) = \frac{1}{1+e^{2x^3}} \cdot e^{x^3} \cdot (3x^2) = \frac{3x^2 e^{x^3}}{1+e^{2x^3}}$$

3) $f(x) = \sqrt{\ln(x^2+1)}$

$$f'(x) = \frac{1}{2\sqrt{\ln(x^2+1)}} \cdot \frac{1}{x^2+1} \cdot 2x = \frac{x}{(x^2+1)\sqrt{\ln(x^2+1)}}$$

4) $f(x) = \underbrace{e^{\pi}}_{\text{costante}} + \sin(4x) + \cos(x^3+2) + \arcsin(1-x)$

$$f'(x) = D(\sin(4x)) + D(\cos(x^3+2)) + D(\arcsin(1-x))$$
$$= 4\cos(4x) - 3x^2\sin(x^3+2) - \frac{1}{\sqrt{1-(1-x)^2}}$$

5) $f(x) = \arcsin(x^2) + \arctan(x^3)$

$$f'(x) = D(\arcsin(x^2)) + D(\arctan(x^3))$$
$$= \frac{2x}{\sqrt{1-x^4}} + \frac{3x^2}{1+x^6}$$

6) $f(x) = \cos(5x)\sin\left(\frac{x}{2}\right) + e^{\cos x}$

$$f'(x) = D(\cos(5x))\sin\left(\frac{x}{2}\right) + \cos(5x)\,D\left(\sin\left(\frac{x}{2}\right)\right) + D\left(e^{\cos x}\right)$$
$$= -5\sin(5x)\sin\left(\frac{x}{2}\right) + \frac{1}{2}\cos(5x)\cos\left(\frac{x}{2}\right) - e^{\cos x}\sin x$$

7) $f(x) = \dfrac{e^x}{x^2+1}$

$$f'(x) = \frac{D(e^x)(x^2+1) - e^x \cdot D(x^2+1)}{(x^2+1)^2} = \frac{e^x(x^2+1) - e^x \cdot 2x}{(x^2+1)^2}$$
$$= \frac{e^x(x^2-2x+1)}{(x^2+1)^2} = \frac{e^x(x-1)^2}{(x^2+1)^2}$$

8) $f(x) = x^2 \cdot \sin(5x) \cdot e^x$

Regola di Leibniz per il prodotto di tre funzioni

$$D(f\cdot g\cdot h) = D(f\cdot(g\cdot h)) = D(f)\cdot(g\cdot h) + f\cdot D(g\cdot h)$$
$$= D(f)\cdot g\cdot h + f\cdot[D(g)\cdot h + g\cdot D(h)]$$
$$= D(f)\cdot g\cdot h + f\cdot D(g)\cdot h + f\cdot g\cdot D(h)$$

riassumendo $\boxed{D(f\cdot g\cdot h) = D(f)\cdot g\cdot h + f\cdot D(g)\cdot h + f\cdot g\cdot D(h)}$

$$f'(x) = 2x\cdot\sin(5x)\,e^x + 5x^2\cos(5x)\cdot e^x + x^2\sin(5x)\,e^x$$
$$= xe^x\left[2\sin(5x) + 5x\cos(5x) + x\sin(5x)\right]$$

9) $f(x) = \arcsin x + \arccos x$

$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$

10) $f(x) = \arctan x + \arctan\left(\frac{1}{x}\right)$

$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} - \frac{1}{\frac{x^2+1}{x^2}} \cdot \frac{1}{x^2}$

$= \frac{1}{1+x^2} - \frac{x^2}{1+x^2} \cdot \frac{1}{x^2} = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$

11) $f(x) = 2^x + \log_2(x^3-x)$

$f'(x) = \ln 2 \cdot 2^x + \frac{1}{\ln 2} \cdot \frac{3x^2-1}{x^3-x}$

12) $f(x) = x^4 - x^3 + 2x + \frac{1}{x} + \sqrt{x} + \frac{1}{\sqrt[3]{x}} + x^\pi$

$f'(x) = 4x^3 - 3x^2 + 2 - \frac{1}{x^2} + \frac{1}{2\sqrt{x}} - \frac{1}{3\sqrt[3]{x^4}} + \pi x^{\pi-1}$

13) $f(x) = \tan(x^3+4x)$

$f'(x) = \left(1 + \tan^2(x^3+4x)\right) \cdot (3x^2+4)$

14) $f(x) = e^{\arctan x} + \arctan(e^x)$

$f'(x) = \frac{e^{\arctan x}}{1+x^2} + \frac{e^x}{1+e^{2x}}$

15) $f(x) = x^3 + 3^x$

$f'(x) = 3x^2 + \ln 3 \cdot 3^x$

16) $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$f'(x) = \frac{D(e^x - e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})\,D(e^x + e^{-x})}{(e^x + e^{-x})^2}$

$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{(e^x + e^{-x} + e^x - e^{-x})(e^x + e^{-x} - e^x + e^{-x})}{(e^x + e^{-x})^2}$

$= \frac{2e^x \cdot 2e^{-x}}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}$

17) $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

$$f'(x) = \frac{1}{\frac{1+x}{1-x}} \cdot \frac{(1-x)-(1+x)(-1)}{(1-x)^2} = \frac{\cancel{1-x}}{1+x} \cdot \frac{1\cancel{-x}+1\cancel{+x}}{(1-x)^{\cancel{2}}} = \frac{2}{(1+x)(1-x)}$$

18) $f(x) = \sqrt{\frac{x^2-1}{x+3}}$

$$f'(x) = \frac{1}{2\sqrt{\frac{x^2-1}{x+3}}} \cdot \frac{2x(x+3)-(x^2-1)}{(x+3)^2} = \frac{1}{2}\sqrt{\frac{x+3}{x^2-1}} \cdot \frac{x^2+6x+1}{(x+3)^2}$$

19) $f(x) = \sqrt[3]{1+\sqrt[4]{1+\sqrt[5]{x}}}$

$$f'(x) = \frac{1}{3\sqrt[3]{\left(1+\sqrt[4]{1+\sqrt[5]{x}}\right)^2}} \cdot D\left(1+\sqrt[4]{1+\sqrt[5]{x}}\right)$$

$$= \frac{1}{3\sqrt[3]{\left(1+\sqrt[4]{1+\sqrt[5]{x}}\right)^2}} \cdot \frac{1}{4\sqrt[4]{(1+\sqrt[5]{x})^3}} \cdot \underbrace{D(1+\sqrt[5]{x})}_{\frac{1}{5\sqrt[5]{x^4}}}$$

$$= \frac{1}{60\sqrt[3]{\left(1+\sqrt[4]{1+\sqrt[5]{x}}\right)^2} \cdot \sqrt[4]{(1+\sqrt[5]{x})^3} \cdot \sqrt[5]{x^4}}$$

20) $f(x) = \ln\left(\frac{3\sin x+1}{2\cos x+5}\right)$

$$f'(x) = \frac{1}{\frac{3\sin x+1}{2\cos x+5}} \cdot \frac{3\cos x(2\cos x+5)+(3\sin x+1)2\sin x}{(2\cos x+5)^2}$$

$$= \frac{\cancel{2\cos x+5}}{3\sin x+1} \cdot \frac{6\cos^2 x+15\cos x+6\sin^2 x+2\sin x}{(2\cos x+5)^{\cancel{2}}}$$

$$= \frac{6+15\cos x+2\sin x}{(3\sin x+1)(2\cos x+5)}$$

21) $f(x) = x^{\frac{1}{x}} = e^{\frac{1}{x}\cdot\ln x} = e^{\frac{\ln x}{x}}$

$$\uparrow$$

$$\boxed{(f(x))^{g(x)} = e^{g(x)\ln f(x)}}$$

$$f'(x) = e^{\frac{\ln x}{x}} \cdot \frac{\frac{1}{x}\cdot x-\ln x}{x^2} = e^{\frac{\ln x}{x}} \frac{1-\ln x}{x^2} = x^{\frac{1}{x}} \cdot \frac{1-\ln x}{x^2}$$