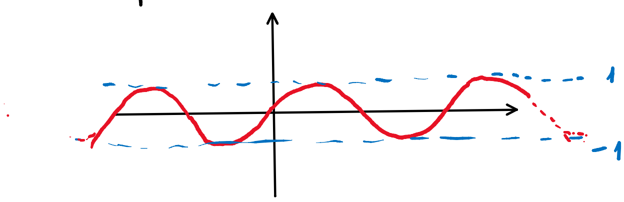
Nella scorsa lezione abbiamo introdotto:

- Ampliamento di $\mathbb{R}$: $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$
- Intorni di $x_0 \in \mathbb{R}^*$
- Punti di accumulazione di un insieme A ($D_r(A)$)
- Definizione di limite:
  Data $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ e dati $x_0 \in D_r(A)$, $\ell \in \mathbb{R}^*$:
  $$\lim_{x \to x_0} f(x) = \ell \iff \forall V \in \mathcal{D}_\ell \ \exists U \in \mathcal{D}_{x_0} \text{ t.c. } f(x) \in V \ \forall x \in U \cap A \setminus \{x_0\}$$
- Punti di accumulazione e limiti da destra / sinistra.
- limiti di funzioni elementari.

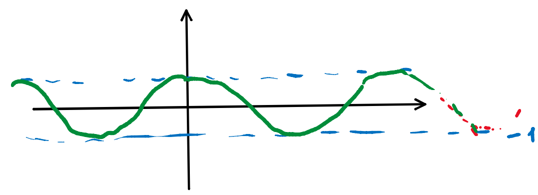
Limiti di funzioni trigonometriche

- **Seno e coseno**

$f(x) = \sin x$



$f(x) = \cos x$

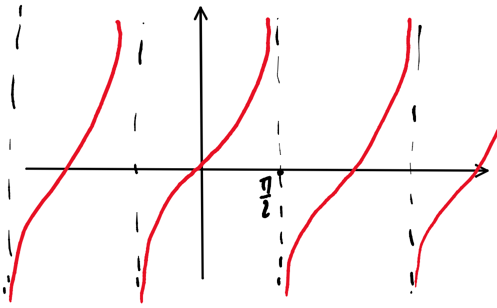


- $\lim_{x \to x_0} \sin x = \sin x_0 \quad \forall x_0 \in \mathbb{R}$
- $\lim_{x \to +\infty} \sin x \ \nexists$
- $\lim_{x \to -\infty} \sin x \ \nexists$

- $\lim_{x \to x_0} \cos x = \cos x_0 \quad \forall x_0$
- $\lim_{x \to +\infty} \cos x \ \nexists$
- $\lim_{x \to -\infty} \cos x \ \nexists$

- Tangente

$f(x) = \tan x$ $\quad$ $Dom(f) = \mathbb{R} \setminus \{ \frac{\pi}{2} + k\pi \mid k \in \mathbb{Z} \}$



- $\lim_{x \to x_0} \tan x = \tan x_0$ se $x_0 \in Dom(f)$

  $(x \neq \frac{\pi}{2} + k\pi$ con $k \in \mathbb{Z})$

- Se $x_0 = \frac{\pi}{2} + k\pi$ con $k \in \mathbb{Z}$

  $\lim_{x \to x_0^+} \tan x = -\infty$

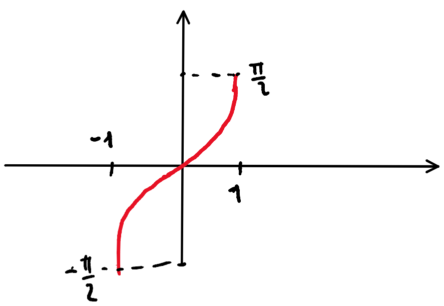
  $\lim_{x \to x_0^-} \tan x = +\infty$

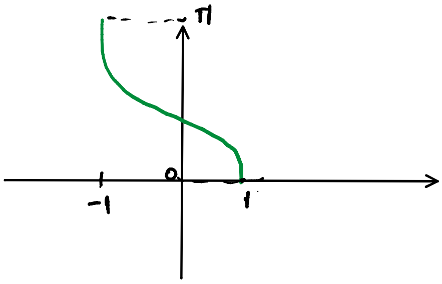
- $\nexists \lim_{x \to +\infty} \tan x$ e $\nexists \lim_{x \to -\infty} \tan x$.

Inverse delle funzioni trigonometriche:

Arcoseno e Arcocoseno

- $f(x) = \arcsin x$ $\quad$ $f(x) = \arccos x$





$Dom(\arcsin) = Dom(\arccos) = [-1, 1]$

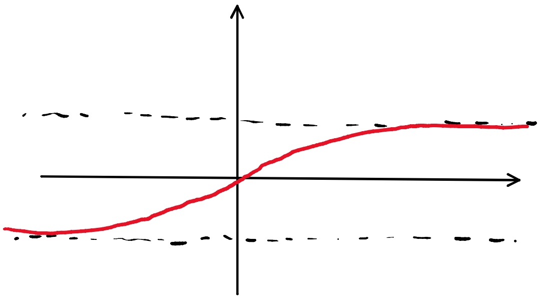
$D_r([-1,1]) = [-1, 1]$

$\forall x_0 \in [-1, 1]$: $\lim_{x \to x_0} \arcsin x = \arcsin x_0$

$\lim_{x \to x_0} \arccos x = \arccos x_0$.

Arcotangente:

$f(x) = \arctan x$



- Se $x_0 \in \mathbb{R}$: $\lim_{x \to x_0} \arctan x = \arctan x_0$
- Se $x_0 = +\infty$:

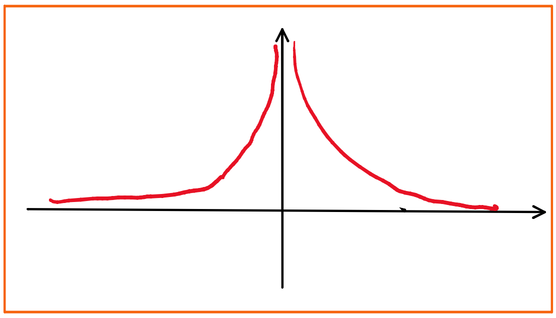
$$\lim_{x \to +\infty} \arctan x = \frac{\pi}{2}$$

- Se $x_0 = -\infty$

$$\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$$

ESEMPI DI RIEPILOGO

- $\lim_{x \to +\infty} x^3 = +\infty$
- $\lim_{x \to -\infty} x^3 = -\infty$
- $\lim_{x \to -\infty} x^{10} = +\infty$
- $\lim_{x \to +\infty} x^{-2} = \lim_{x \to +\infty} \frac{1}{x^2} = 0$ o o o o



- $\lim_{x \to 0} x^{-2} = \lim_{x \to 0} \frac{1}{x^2} = +\infty$
- $\lim_{x \to 0} \frac{1}{x^3} \nexists$ perché $\lim_{x \to 0^+} \frac{1}{x^3} = +\infty$ e $\lim_{x \to 0^-} \frac{1}{x^3}$.
- $\lim_{x \to +\infty} e^x = +\infty$
- $\lim_{x \to -\infty} e^x = 0$ o o o o o o

$$\boxed{e^x = e^{-|x|} = \frac{1}{e^{|x|}} \quad \text{se } x < 0}$$

- $\lim_{x \to +\infty} \left(\frac{1}{2}\right)^x = \lim_{x \to +\infty} \frac{1}{2^x} = 0$

Teorema (operazioni tra limiti)

Sia $A \subseteq \mathbb{R}$, sia $x_0 \in D(A)$ e siano $f_1, f_2 : A \longrightarrow \mathbb{R}$ due funzioni. Supponiamo

$$\lim_{x \to x_0} f_1(x) = L_1 \in \mathbb{R} \quad \text{e} \quad \lim_{x \to x_0} f(x) = L_2 \in \mathbb{R}$$

Allora:

1) $\lim_{x \to x_0} (f_1(x) + f_2(x)) = L_1 + L_2.$

2) $\lim_{x \to x_0} c f_1(x) = c L_1 \quad \forall c \in \mathbb{R}$

3) $\lim_{x \to x_0} (f_1(x) - f_2(x)) = L_1 - L_2$

4) $\lim_{x \to x_0} f_1(x) \cdot f_2(x) = L_1 \cdot L_2$

5) Se $L_2 \neq 0$, allora

$$\lim_{x \to x_0} \frac{f_1(x)}{f_2(x)} = \frac{L_1}{L_2}$$

6) Se $L_1 > 0$ allora $\lim_{x \to x_0} f_1(x)^{f_2(x)} = L_1^{L_2}$.

ESEMPIO

- $\lim_{x \to 0} (2e^x + x) = 2e^0 + 0 = 2 \cdot 1 + 0 = 2$
- $\lim_{x \to \pi} x^3 \cos x = \pi^3 \cos \pi = -\pi^3.$
- $\lim_{x \to 2} \frac{3^x}{x-3} = \frac{3^2}{2-3} = \frac{9}{-1} = -9.$

OSS In molti casi, le regole del teorema precedente si applicano anche quando L_1, L_2 sono $\pm\infty$.

Per la somma:

- Se $L_1 \in \mathbb{R}$, $L_2 = +\infty$: $L_1 + L_2 = +\infty.$
- Se $L_1 = +\infty$ e $L_2 \in \mathbb{R}$: $L_1 + L_2 = +\infty$
- Se $L_1 = -\infty$ e $L_2 \in \mathbb{R}$: $L_1 + L_2 = -\infty$
- Se $L_1 \in \mathbb{R}$ e $L_2 = -\infty$: $L_1 + L_2 = -\infty.$
- Se $L_1 = +\infty$ e $L_2 = +\infty$: $L_1 + L_2 = +\infty$
- Se $L_1 = -\infty$ e $L_2 = -\infty$: $L_1 + L_2 = -\infty.$

Attenzione

Non c'è una regola fissa per i casi $+\infty + (-\infty)$ e $-\infty + (+\infty)$. Si dice che queste operazioni sono FORME INDETERMINATE.

Per il prodotto per una costante:

- Sia $c \in \mathbb{R}$ e $L_1 = +\infty$:

$$\lim_{x \to x_0} c f_1(x) = \begin{cases} +\infty & \text{se } c > 0 \\ -\infty & \text{se } c < 0 \\ 0 & \text{se } c = 0 \end{cases}$$

- Se $L_1 = -\infty$.

$$\lim_{x \to x_0} c f_1(x) = \begin{cases} -\infty & \text{se } c > 0 \\ +\infty & \text{se } c < 0 \\ 0 & \text{se } c = 0. \end{cases}$$

- Per la differenza: I casi si ricavano da quelli per la somma e $+\infty - (+\infty)$ e $-\infty - (-\infty)$ sono forme indeterminate.

- Per il prodotto di due funzioni:
  - $L_1 = +\infty$ e $L_2 \in \mathbb{R}^* \setminus \{0\}$:

$$\lim_{x \to x_0} f_1(x) \cdot f_2(x) = \begin{cases} +\infty & \text{se } L_2 > 0 \\ -\infty & \text{se } L_2 < 0. \end{cases}$$

  - $L_1 = -\infty$ e $L_2 \in \mathbb{R}^? \setminus \{0\}$

$$\lim_{x \to x_0} f_1(x) \cdot f_2(x) = \begin{cases} -\infty & \text{se } L_2 > 0 \\ +\infty & \text{se } L_2 < 0 \end{cases}$$

Attenzione: $+\infty \cdot 0$, $-\infty \cdot 0$, $0 \cdot (+\infty)$, $0 \cdot (-\infty)$ sono forme indeterminate.

Per il quoziente:

- Se $L_1 = +\infty$ e $L_2 \in \mathbb{R} \setminus \{0\}$ allora:

$$\lim_{x \to x_0} \frac{f_1(x)}{f_2(x)} = \begin{cases} +\infty & \text{se } L_2 > 0 \\ -\infty & \text{se } L_2 < 0 \end{cases}$$

- Se $L_1 = -\infty$ e $L_2 \in \mathbb{R} \setminus \{0\}$

$$\lim_{x \to x_0} \frac{f_1(x)}{f_2(x)} = \begin{cases} -\infty & \text{se } L_2 > 0 \\ +\infty & \text{se } L_2 < 0. \end{cases}$$

- Se $L_1 \in \mathbb{R}$ e $L_2 \in \{+\infty, -\infty\}$

$$\lim_{x \to x_0} \frac{f_1(x)}{f_2(x)} = 0$$

Forme indeterminate: $\frac{+\infty}{+\infty}$, $\frac{-\infty}{-\infty}$, $\frac{+\infty}{-\infty}$, $\frac{-\infty}{+\infty}$, $\frac{0}{0}$

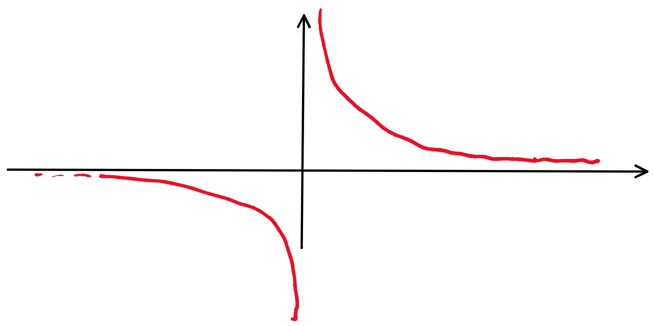
Attenzione: Se $L_1 = \mathbb{R}^* \setminus \{0\}$ e $L_2 = 0$. In questo caso il risultato può dipendere dal segno di numeratore e denominatore.

- $\lim_{x \to 0} \frac{1}{x^2} = +\infty \quad \left(\frac{1}{0}\right)$
- $\lim_{x \to 0} \frac{1}{x} \nexists \quad \left(\frac{1}{0}\right)$

Def (Limite per eccesso). Sia $A \subseteq \mathbb{R}$, sia $x_0 \in D(A)$ e sia $f: A \to \mathbb{R}$. Scriveremo che $\lim_{x \to x_0} f(x) = 0^+$ se $\lim_{x \to x_0} f(x) = 0$ e $\exists\, U$ intorno di x_0 t.c. $f(x) > 0 \quad \forall x \in U \cap A \setminus \{x_0\}$.

Def (Limite per difetto): Sia $A \subseteq \mathbb{R}$, sia $x_0 \in D(A)$ e sia $f: A \to \mathbb{R}$. Scriveremo $\lim_{x \to x_0} f(x) = 0^-$ se $\lim_{x \to x_0} f(x) = 0$ e $\exists\, U$ intorno di x_0 t.c. $f(x) < 0 \quad \forall x \in U \cap A \setminus \{x_0\}$.

- $\lim_{x \to 0} x^2 = 0^+$
- $\lim_{x \to +\infty} \frac{1}{x} = 0^+$
- $\lim_{x \to -\infty} \frac{1}{x} = 0^-$



Oss: Nei limiti dei quozienti: se $L_1 \in \mathbb{R}^* \setminus \{0\}$.

- Se $L_2 = 0^+$, allora

$$\lim_{x \to x_0} \frac{f_1(x)}{f_2(x)} = \begin{cases} +\infty & \text{se } L_1 > 0 \\ -\infty & \text{se } L_1 < 0 \end{cases}$$

- Se $l_2 = 0^-$:

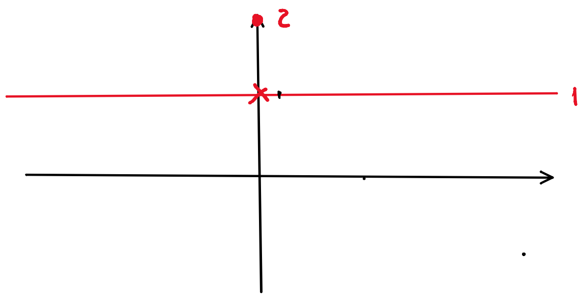
$$\lim_{x \to x_0} \frac{f_1(x)}{f_2(x)} = \begin{cases} -\infty & \text{se } l_1 > 0 \\ +\infty & \text{se } l_1 < 0. \end{cases}$$

- $\lim_{x \to 0} \frac{1}{x^2} = \frac{1}{0^+} = +\infty$
- $\lim_{x \to 0} \frac{1}{x^{20}} = \frac{1}{0^+} = +\infty$
- $\lim_{x \to 0} \frac{1}{x} \nexists$ perché $\lim_{x \to 0^+} \frac{1}{x} = \frac{1}{0^+} = +\infty$ ma $\lim_{x \to 0^-} \frac{1}{x} = \frac{1}{0^-} = -\infty$.
- $\lim_{x \to 4} \frac{1}{|x-4|} = \frac{1}{0^+} = +\infty$.
- $\lim_{x \to 4} \frac{x-5}{|x-4|} = \frac{-1}{0^+} = -\infty$
- $\lim_{x \to 0} \frac{1}{1-\sqrt{1+x^2}} = \frac{1}{0^-} = -\infty$. ……

$\boxed{\begin{array}{l} \sqrt{1+x^2} > 1 \quad \forall x \in \mathbb{R} \setminus \{0\} \\ 1-\sqrt{1+x^2} < 0 \quad \forall x \in \mathbb{R} \setminus \{0\} \end{array}}$

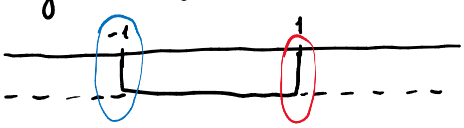
$$f(x) = \begin{cases} 1 & \text{se } x \neq 0 \\ 2 & \text{se } x = 0 \end{cases}$$

$\lim_{x \to 0} f(x) = 1$



- $\lim_{x \to 1} \frac{1}{1-x^2}$ $\left(\frac{1}{0}\right)$

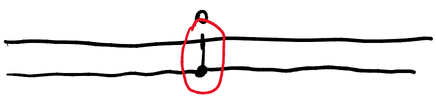
Segno di $1-x^2$



$\lim_{x \to 1^+} \frac{1}{1-x^2} = \frac{1}{0^-} = -\infty$

$\lim_{x \to 1^-} \frac{1}{1-x^2} = \frac{1}{0^+} = +\infty$

$x^2 > 0$



$1-x^2 > 0 \iff -1 < x < 1$

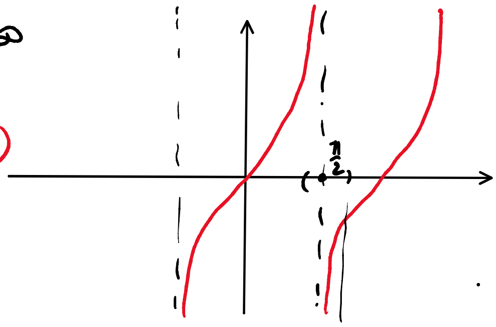
- $\lim_{x \to -1^+} \frac{1}{1-x^2} = \frac{1}{0^+} = +\infty$

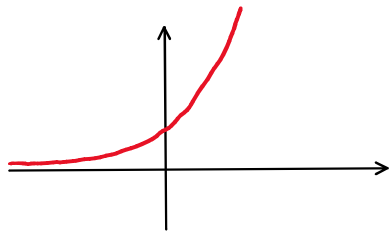
  $\lim_{x \to -1^-} \frac{1}{1-x^2} = \frac{1}{0^-} = -\infty.$

- $\lim_{x \to +\infty} \frac{1}{x} - 3x = 0 - \infty = -\infty$

- $\lim_{x \to +\infty} x e^x = +\infty \cdot +\infty = +\infty$

- $\lim_{x \to \left(\frac{\pi}{2}\right)^+} \frac{x}{\tan x} = \frac{\frac{\pi}{2}}{-\infty} = 0 \ (0^-)$





- $\lim_{x \to 0} 2^{\frac{1}{x}}$

  $\lim_{x \to 0^+} 2^{\frac{1}{x}} = \text{"}2^{+\infty}\text{"} = +\infty$ e $\lim_{x \to 0^-} 2^{\frac{1}{x}} = \text{"}2^{-\infty}\text{"} = 0$

  se $x<0$: $2^{\frac{1}{x}} = 2^{-\left|\frac{1}{x}\right|} = \frac{1}{2^{\left|\frac{1}{x}\right|}} = \frac{1}{+\infty} = 0$

Alcune forme indeterminate facili:

- Polinomi:

  $\lim_{x \to +\infty} (x^3 - x^4 + 2)$ $\quad$ $\text{"}\underbrace{+\infty - \infty}_{\text{f.i.}} + 2\text{"}$

  Si raccoglie il termine di grado più alto:

  $\lim_{x \to +\infty} x^4 \left(\frac{1}{x} - 1 + \frac{2}{x^4}\right) = +\infty\,(0 - 1 + 0) = +\infty \cdot (-1) = -\infty.$

  Questo metodo funziona per $\lim_{x \to +\infty}$ o $\lim_{x \to -\infty}$ di polinomi o rapporti tra polinomi.

- $\lim\limits_{x \to -\infty} 1 - x^3 + x^2 = 1 + \infty + \infty = +\infty$

- $\lim\limits_{x \to -\infty} 1 + 2x^3 + x^2 = \lim\limits_{x \to -\infty} x^3 \left(\frac{1}{x^3} + 2 + \frac{1}{x} \right) = -\infty \cdot 2 = -\infty.$

  $\frac{1}{x^3} \to 0$, $\frac{1}{x} \to 0$

- $\lim\limits_{x \to +\infty} \dfrac{x^2 + 1}{2x^2 + 3}$ $\qquad \dfrac{+\infty}{+\infty}$

$$= \lim_{x \to +\infty} \frac{\cancel{x^2}\left(1 + \frac{1}{x^2}\right)}{\cancel{x^2}\left(2 + \frac{3}{x^2}\right)} = \lim_{x \to +\infty} \frac{1 + \frac{1}{x^2}}{2 + \frac{3}{x^2}} = \frac{1}{2}.$$

$\frac{1}{x^2} \to 0$, $\frac{3}{x^2} \to 0$

- $\lim\limits_{x \to -\infty} \dfrac{x^3 + x + 1}{x^6 + 2x + 1} = \lim\limits_{x \to -\infty} \dfrac{\cancel{x^3}\left(1 + \frac{1}{x^2} + \frac{1}{x^3}\right)}{x^{\cancel{6}\,3}\left(1 + \frac{2}{x^5} + \frac{1}{x^6}\right)}$

$$= \lim_{x \to -\infty} \frac{1 + \frac{1}{x^2} + \frac{1}{x^3}}{x^3\left(1 + \frac{2}{x^3} + \frac{1}{x^6}\right)} = \frac{1}{(-\infty)\cdot 1} = \frac{1}{-\infty} = 0.$$