MATEMATICA – LEZIONE 18 – 31/10/24 14.00 - 16.00 Aula 4

Sia $z = |z|(\cos\theta + i\sin\theta)$ θ è un argomento di z

Se $n \in \mathbb{N}$, $n \geq 2$

$$z^n = |z|^n(\cos(n\theta) + i\sin(n\theta)) \quad \text{Formula di De Moivre}$$

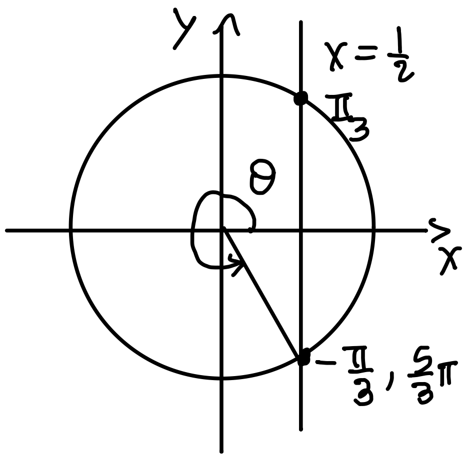
$$\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right) \quad k = 0, 1, \dots, n-1 \ (\text{se } z \neq 0)$$

Esempio: Sia $z = 1 - \sqrt{3}i$ calcoliamo le radici quadrate complesse di z

$$|z| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$$

cerchiamo l'argomento principale $\theta \in [0, 2\pi)$ tale che

$$\begin{cases} \cos\theta = \dfrac{\operatorname{Re} z}{|z|} = \dfrac{1}{2} \\ \sin\theta = \dfrac{\operatorname{Im} z}{|z|} = -\dfrac{\sqrt{3}}{2} \end{cases}$$



$$\theta = \frac{5}{3}\pi$$

$z = 2\left(\cos\left(\frac{5}{3}\pi\right) + i\sin\left(\frac{5}{3}\pi\right)\right)$ forma trigonometrica

Siano w_0, w_1 le radici quadrate complesse di z

$$w_k = \sqrt{2}\left(\cos\left(\frac{\frac{5}{3}\pi + 2k\pi}{2}\right) + i\sin\left(\frac{\frac{5}{3}\pi + 2k\pi}{2}\right)\right)$$

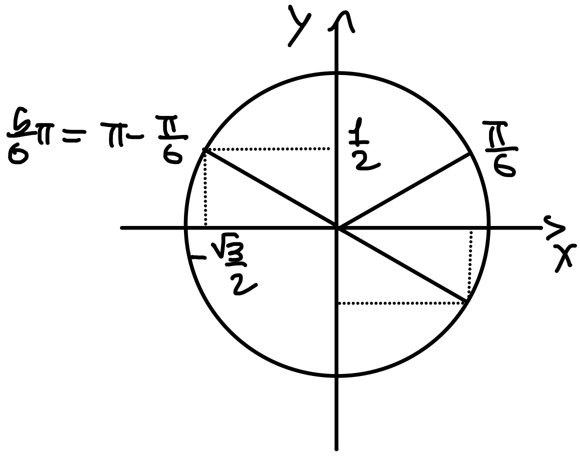
$$= \sqrt{2}\left(\cos\left(\frac{5}{6}\pi + k\pi\right) + i\sin\left(\frac{5}{6}\pi + k\pi\right)\right) \quad k = 0, 1$$

$$w_0 = \sqrt{2}\left(\cos\left(\frac{5}{6}\pi\right) + i\sin\left(\frac{5}{6}\pi\right)\right)$$

$$w_0 = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right) = -\frac{\sqrt{6}}{2} + i\,\frac{\sqrt{2}}{2}$$

$$w_1 = \sqrt{2}\left(\cos\left(\frac{11}{6}\pi\right) + i\,\sin\left(\frac{11}{6}\pi\right)\right)$$
$$= \sqrt{2}\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right)$$
$$= \frac{\sqrt{6}}{2} - i\,\frac{\sqrt{2}}{2}$$

Osserviamo che $w_1 = -w_0$



Esercizio A)

Si consideri il numero complesso $w = \dfrac{1+9i}{4-5i}$

a) Determinare la forma algebrica di w

b) Determinare la forma trigonometrica e la forma esponenziale di w

c) Calcolare w^3, w^4, w^6

d) Calcolare le radici cubiche, seste e decime di w

Svolgimento:

a) $$w = \frac{1+9i}{4-5i}\cdot\frac{4+5i}{4+5i} = \frac{(1+9i)(4+5i)}{4^2-(5i)^2} = \frac{4+5i+36i+45i^2}{4^2+5^2}$$

$$= \frac{4-45+41i}{41} = \frac{-41+41i}{41} = -1+i$$ forma algebrica

b) $$|w| = \sqrt{(-1)^2+1^2} = \sqrt{2}$$

Determiniamo un argomento di w

$$\begin{cases} \cos\theta = \dfrac{-1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2} \\ \sin\theta = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2} \end{cases} \Rightarrow \theta = \frac{3}{4}\pi$$

$$w = \sqrt{2}\left(\cos\left(\frac{3}{4}\pi\right) + i\,\sin\left(\frac{3}{4}\pi\right)\right)$$ forma trigonometrica

$$= \sqrt{2}\,e^{i\frac{3}{4}\pi}$$ forma esponenziale

c) $$w^3 = (\sqrt{2})^3\left(\cos\left(\frac{9}{4}\pi\right) + i\sin\left(\frac{9}{4}\pi\right)\right)$$

$$= 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right) = 2\sqrt{2}\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) = 2(1+i)$$

$$w^4 = (\sqrt{2})^4 \left(\cos(3\pi) + i \sin(3\pi) \right)$$
$$= 4 \left(\cos\pi + i \sin\pi \right) = 4(-1 + i\cdot 0) = -4$$

$$w^6 = (\sqrt{2})^6 \left(\cos\left(\cancel{6}^{3}\cdot \frac{3}{\cancel{4}_{2}}\pi\right) + i \sin\left(\frac{9}{2}\pi\right) \right)$$
$$= 8 \left(\cos\left(4\pi + \frac{\pi}{2}\right) + i \sin\left(4\pi + \frac{\pi}{2}\right) \right)$$

$$\boxed{9 = 4\cdot 2 + 1 \qquad \frac{9}{2} = \frac{4\cdot 2+1}{2} = 4 + \frac{1}{2}}$$

$$= 8\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = 8(0 + i) = 8i$$

d) Denotiamo con z_0, z_1, z_2 le radice cubiche di w

$$z_k = \sqrt[3]{\sqrt{2}} \left(\cos\left(\frac{\frac{3}{4}\pi + 2k\pi}{3}\right) + i \sin\left(\frac{\frac{3}{4}\pi + 2k\pi}{3}\right) \right) \qquad k = 0,1,2$$
$$= \sqrt[6]{2} \left(\cos\left(\frac{\pi}{4} + \frac{2}{3}k\pi\right) + i \sin\left(\frac{\pi}{4} + \frac{2}{3}k\pi\right) \right)$$

$$z_0 = \sqrt[6]{2} \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right) = \sqrt[6]{2} \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right)$$
$$= \frac{1}{\sqrt[3]{2}} + i \frac{1}{\sqrt[3]{2}}$$

poiché $\dfrac{\sqrt[6]{2}\cdot\sqrt{2}}{2} = \dfrac{2^{\frac{1}{6}}\cdot 2^{\frac{1}{2}}}{2} = 2^{\frac{1}{6}+\frac{1}{2}-1} = 2^{\frac{1+3-6}{6}} = 2^{-\frac{2}{6}} = 2^{-\frac{1}{3}} = \dfrac{1}{\sqrt[3]{2}}$

$$z_1 = \sqrt[6]{2} \left(\cos\left(\frac{\pi}{4} + \frac{2}{3}\pi\right) + i \sin\left(\frac{\pi}{4} + \frac{2}{3}\pi\right) \right)$$
$$= \sqrt[6]{2} \left(\cos\left(\frac{11}{12}\pi\right) + i \sin\left(\frac{11}{12}\pi\right) \right) \rightarrow$$

$\cos\left(\frac{11}{12}\pi\right)$, $\sin\left(\frac{11}{12}\pi\right)$ si possono calcolare con le formule di addizione

$$z_2 = \sqrt[6]{2} \left(\cos\left(\frac{\pi}{4} + \frac{4}{3}\pi\right) + i \sin\left(\frac{\pi}{4} + \frac{4}{3}\pi\right) \right)$$
$$= \sqrt[6]{2} \left(\cos\left(\frac{19}{12}\pi\right) + i \sin\left(\frac{19}{12}\pi\right) \right)$$

Denotiamo con $z_0', z_1', \ldots, z_5'$ le radice seste di w

$$z_k' = \sqrt[6]{\sqrt{2}} \left(\cos\left(\frac{\frac{3}{4}\pi + 2k\pi}{6}\right) + i \sin\left(\frac{\frac{3}{4}\pi + 2k\pi}{6}\right) \right) \qquad k = 0,1,2,3,4,5$$
$$= \sqrt[12]{2} \left(\cos\left(\frac{\pi}{8} + k\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{8} + k\frac{\pi}{3}\right) \right)$$

Denotiamo con $z_0'', z_1'', \dots, z_9''$ le radici decime di w

$$z_k'' = \sqrt[20]{2}\left(\cos\left(\frac{\frac{3}{4}\pi + 2k\pi}{10}\right) + i\sin\left(\frac{\frac{3}{4}\pi + 2k\pi}{10}\right)\right)$$

$$= \sqrt[20]{2}\left(\cos\left(\frac{3}{40}\pi + k\frac{\pi}{5}\right) + i\sin\left(\frac{3}{40}\pi + k\frac{\pi}{5}\right)\right) \quad k = 0, 1, \dots, 9$$

Esercizio (I appello 23/24)

Si consideri il numero complesso $w = \frac{1+2i}{3+i}$

1. Determinare la formula algebrica di w
2. Calcolare w^9

Svolgimento:

1. $$w = \frac{1+2i}{3+i} \cdot \frac{3-i}{3-i} = \frac{3-i+6i+2}{3^2+1} = \frac{5+5i}{10} = \frac{1}{2} + \frac{i}{2}$$

2. Determiniamo la forma trigonometrica di w

$$|w| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\begin{cases} \cos\theta = \frac{\operatorname{Re} w}{|w|} = \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} = \frac{1}{2} \cdot \sqrt{2} = \frac{\sqrt{2}}{2} \\ \sin\theta = \frac{\operatorname{Im} w}{|w|} = \frac{\sqrt{2}}{2} \end{cases}$$

$$\Rightarrow \theta = \frac{\pi}{4} \qquad w = \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$w^9 = \left(\frac{1}{\sqrt{2}}\right)^9 \cdot \left(\cos\left(\frac{9}{4}\pi\right) + i\sin\left(\frac{9}{4}\pi\right)\right)$$

$$= 2^{-\frac{1}{2}\cdot 9}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$

$$= 2^{-4-\frac{1}{2}}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 2^{-4} \cdot \frac{1}{\cancel{\sqrt{2}}}\left(\frac{\cancel{\sqrt{2}}}{2} + i\frac{\cancel{\sqrt{2}}}{2}\right)$$

$$= 2^{-4}\left(\frac{1}{2} + i\frac{1}{2}\right) = 2^{-5}(1+i) = \frac{1}{32}(1+i)$$

Esercizio: (II appello 23/24)

Si consideri il numero complesso $w = \frac{5i-3}{1+4i}$

1. Determinare la forma algebrica di w
2. Calcolare w^7

svolgimenti:

1. $$w = \frac{5i-3}{1+4i} \cdot \frac{1-4i}{1-4i} = \frac{5i+20-3+12i}{1^2+4^2} = \frac{17+17i}{17} = 1+i$$

2. Calcoliamo la forma trigonometrica

$$|w| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\begin{cases} \cos\theta = \frac{1}{\sqrt{2}} \\ \sin\theta = \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \theta = \frac{\pi}{4}$$

$$w = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$w^7 = (\sqrt{2})^7\left(\cos\left(\tfrac{7}{4}\pi\right) + i\sin\left(\tfrac{7}{4}\pi\right)\right)$$

$$= 8\sqrt{2}\left[\cos\left(2\pi - \tfrac{\pi}{4}\right) + i\sin\left(2\pi - \tfrac{\pi}{4}\right)\right]$$

$$= 8\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = 8\sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= 8(1-i)$$

Foglio di esercizi del Prof. Mancini

Esercizio 1.5

Determinare parte reale, parte immaginaria, coniugato, modulo, argomento del numero complesso

$$z = \frac{(1-i)(1+i)^5}{1+\sqrt{3}i}$$

determiniamo modulo e argomento dei numeri

$z_1 = 1-i$, $z_2 = 1+i$, $z_3 = 1+\sqrt{3}i$

$|z_1| = \sqrt{1^2+(-1)^2} = \sqrt{2}$ $\qquad$ θ_1 argomento di z_1

$|z_2| = \sqrt{1^2+1^2} = \sqrt{2}$ $\qquad$ θ_2 argomento di z_2

$|z_3| = \sqrt{1^2+(\sqrt{3})^2} = 2$ $\qquad$ θ_3 argomento di z_3

determiniamo gli argomenti di z_1, z_2, z_3

$$\begin{cases} \cos\theta_1 = \frac{\operatorname{Re} z_1}{|z_1|} = \frac{\sqrt{2}}{2} \\ \sin\theta_1 = \frac{\operatorname{Im} z_1}{|z_1|} = -\frac{\sqrt{2}}{2} \end{cases} \Rightarrow \theta_1 = \frac{7}{4}\pi$$

$$\begin{cases} \cos\theta_2 = \frac{\sqrt{2}}{2} \\ \sin\theta_2 = \frac{\sqrt{2}}{2} \end{cases} \Rightarrow \quad \theta_2 = \frac{\pi}{4}$$

$$\begin{cases} \cos\theta_3 = \frac{1}{2} \\ \sin\theta_3 = \frac{\sqrt{3}}{2} \end{cases} \Rightarrow \quad \theta_3 = \frac{\pi}{3}$$

abbiamo dunque ottenuto le forme trigonometriche

$$z_1 = \sqrt{2}\left(\cos\left(\tfrac{7}{4}\pi\right) + i\sin\left(\tfrac{7}{4}\pi\right)\right)$$

$$z_2 = \sqrt{2}\left(\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right)$$

$$z_3 = 2\left(\cos\left(\tfrac{\pi}{3}\right) + i\sin\left(\tfrac{\pi}{3}\right)\right)$$

$$z = \frac{z_1 \cdot z_2^5}{z_3}$$

$z_2^5 = 4\sqrt{2}\left(\cos\left(\tfrac{5}{4}\pi\right) + i\sin\left(\tfrac{5}{4}\pi\right)\right)$ formula di De Moivre

$$z_1 \cdot z_2^5 = \sqrt{2}\cdot 4\sqrt{2}\left(\cos\left(\tfrac{7}{4}\pi + \tfrac{5}{4}\pi\right) + i\sin\left(\tfrac{7}{4}\pi + \tfrac{5}{4}\pi\right)\right)$$

$$= 8\left(\cos(3\pi) + i\sin(3\pi)\right)$$

$$= 8\left(\cos\pi + i\sin\pi\right)$$

facendo il prodotto di numeri complessi in forma trigonometrica gli argomenti si sommano, nel il quoziente si fa la differenza

$$z = \frac{z_1 \cdot z_2^5}{z_3} = \frac{8(\cos\pi + i\sin\pi)}{2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$$

$$= 4\left(\cos\left(\pi - \tfrac{\pi}{3}\right) + i\sin\left(\pi - \tfrac{\pi}{3}\right)\right)$$

$$= 4\left(\cos\left(\tfrac{2}{3}\pi\right) + i\sin\left(\tfrac{2}{3}\pi\right)\right) = 4\left(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right)$$

$$= -2 + i2\sqrt{3}$$

$$z = \underbrace{-2 + i2\sqrt{3}}_{\text{forma algebrica}} = \underbrace{4\left(\cos\left(\tfrac{2}{3}\pi\right) + i\sin\left(\tfrac{2}{3}\pi\right)\right)}_{\text{forma trigonometrica}} = \underbrace{4e^{i\frac{2}{3}\pi}}_{\text{forma esponenziale}}$$

$\mathrm{Re}(z) = -2$, $\mathrm{Im}(z) = 2\sqrt{3}$, $|z| = 4$, $\frac{2}{3}\pi$ argomento

$$\bar{z} = -2 - i2\sqrt{3}$$

Esercizio 4 Si consideri il numero complesso $z = \frac{4+2i}{i-3}$

1. Scrivere la forma trigonometrica di z
2. Si scrivano in forma esponenziale le radici cubiche di z

Svolgimento:

1. $$z = \frac{4+2i}{i-3} \cdot \frac{-i-3}{-i-3} = \frac{-4i-12+2-6i}{1+9} = \frac{-10-10i}{10} = -1-i$$

$$|z| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$$

$$\begin{cases} \cos\theta = \frac{\operatorname{Re} z}{|z|} = -\frac{\sqrt{2}}{2} \\ \sin\theta = \frac{\operatorname{Im} z}{|z|} = -\frac{\sqrt{2}}{2} \end{cases} \Rightarrow \quad \theta = \frac{5}{4}\pi$$

$$z = \sqrt{2}\left(\cos\left(\frac{5}{4}\pi\right) + i\sin\left(\frac{5}{4}\pi\right)\right)$$

2. Siano w_0, w_1, w_2 le radici cubiche di z

$$w_k = \sqrt[6]{2}\left(\cos\left(\frac{\frac{5}{4}\pi+2k\pi}{3}\right) + i\sin\left(\frac{\frac{5}{4}\pi+2k\pi}{3}\right)\right)$$

$$= \sqrt[6]{2}\left(\cos\left(\frac{5}{12}\pi+\frac{2}{3}k\pi\right) + i\sin\left(\frac{5}{12}\pi+\frac{2}{3}k\pi\right)\right)$$

$$= \sqrt[6]{2}\, e^{i\left(\frac{5}{12}\pi+\frac{2}{3}k\pi\right)} \qquad k=0,1,2$$

Esercizio 5 Si consideri il numero complesso $z = \frac{8i-1}{2i+3}$

1. Determinare la forma algebrica
2. Si scrivano in forma esponenziale le radici quinte di z

Svolgimento:

1. $$z = \frac{8i-1}{2i+3} \cdot \frac{-2i+3}{-2i+3} = \frac{16+24i+2i-3}{4+9} = \frac{13+26i}{13} = 1+2i$$

2. $$|z| = \sqrt{1^2+2^2} = \sqrt{5}$$

$$\begin{cases} \cos\theta = \frac{1}{\sqrt{5}} \\ \sin\theta = \frac{2}{\sqrt{5}} \end{cases} \Rightarrow \quad \tan\theta = \frac{\frac{2}{\sqrt{5}}}{\frac{1}{\sqrt{5}}} = 2 \quad \Rightarrow \theta = \arctan 2$$

(siamo nel I quadrante)

la forma trigonometrica di z è

$$z = \sqrt{5}\, e^{i \arctan 2}$$

siano $w_0, \dots, w_4$ le radici quinte di z, allora

$$w_k = \sqrt[10]{5}\, e^{i\left(\frac{\arctan 2}{5} + \frac{2}{5}k\pi\right)} \qquad k = 0,1,2,3,4$$