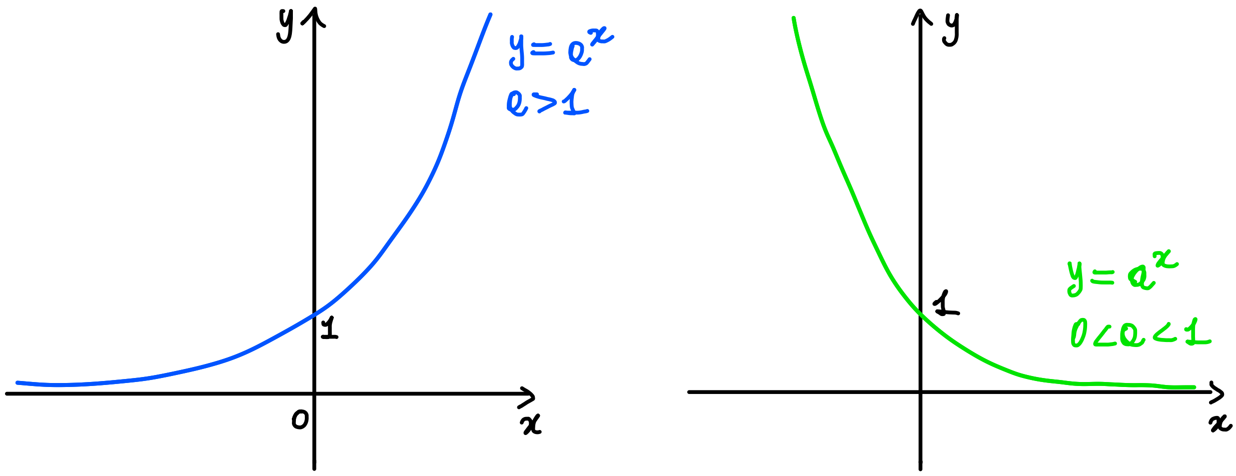
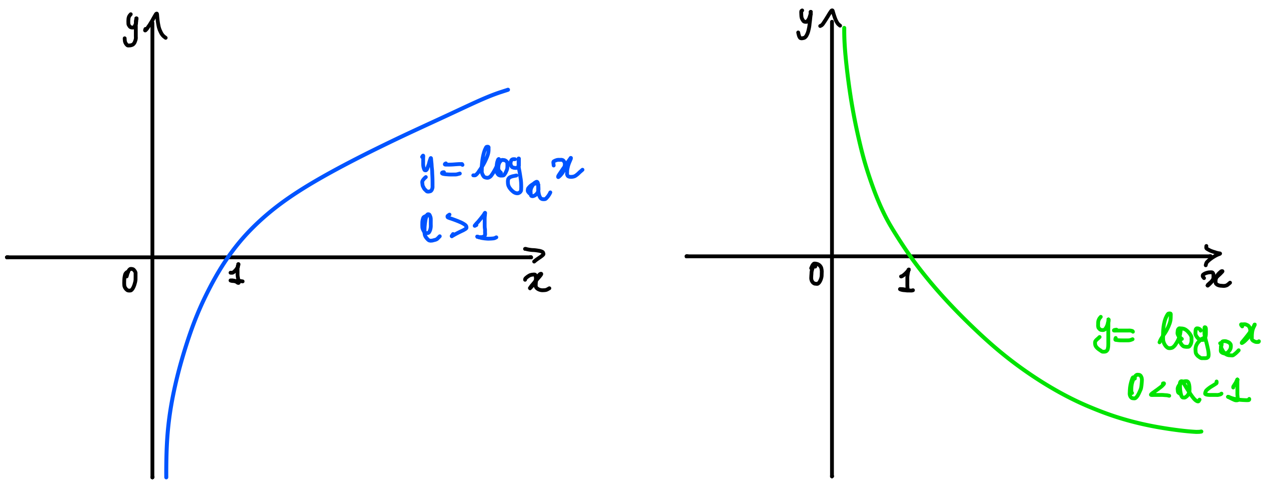
MATEMATICA – LEZIONE 10 – 17/10/24 14.00 - 16.00 Aula 4

Grafici delle funzioni esponenziali



Grafici delle funzioni logaritmiche



Equazioni e disequazioni esponenziali

Siano $a \in (0,1) \cup (1,+\infty)$, $K \in \mathbb{R}$

$$a^{f(x)} = K \iff \begin{cases} f(x) = \log_a K & \text{se } K > 0 \\ \nexists\, x \in \mathbb{R} & \text{se } K \leq 0 \end{cases}$$

$$a^{f(x)} > K \iff \begin{cases} \begin{cases} f(x) > \log_e K & \text{se } K > 0 \text{ e } a > 1 \\ f(x) < \log_a K & \text{se } K > 0 \text{ e } a \in (0,1) \end{cases} \\ \exists\, f(x) & \text{se } K \leq 0 \end{cases}$$

$$a^{f(x)} < K \iff \begin{cases} \begin{cases} f(x) < \log_a K & \text{se } K > 0 \text{ e } a > 1 \\ f(x) > \log_e K & \text{se } K > 0 \text{ e } a \in (0,1) \end{cases} \\ \nexists\, x \in \mathbb{R} & \text{se } K \leq 0 \end{cases}$$

Equazione e disequazioni esponenziali

1) $e^x \geq -2$ $\qquad$ e "numero di Nepero"

$\forall x \in \mathbb{R}$

2) $3^x = \dfrac{1}{\sqrt[5]{3}}$ $\Rightarrow$ $3^x = (\sqrt[5]{3})^{-1}$

$\Rightarrow$ $3^x = (3^{\frac{1}{5}})^{-1} = 3^{-\frac{1}{5}}$

$\Rightarrow$ $x = -\frac{1}{5}$

3) $\left(\frac{1}{2}\right)^x \geq \dfrac{8 \cdot \sqrt{2}}{\sqrt[3]{4}}$ $\Rightarrow$ $2^{-x} \geq \dfrac{2^3 \cdot 2^{\frac{1}{2}}}{2^{\frac{2}{3}}} = \dfrac{2^{3+\frac{1}{2}}}{2^{2/3}} = \dfrac{2^{\frac{7}{2}}}{2^{\frac{2}{3}}}$

$\Rightarrow$ $2^{-x} \geq 2^{\frac{7}{2}-\frac{2}{3}} = 2^{\frac{17}{6}}$

$\Rightarrow$ $-x \geq \frac{17}{6}$ $\Rightarrow$ $x \leq -\frac{17}{6}$

4) $4^x - 5 \cdot 2^x + 4 > 0$ $\Rightarrow$ $2^{2x} - 5 \cdot 2^x + 4 > 0$

$\Rightarrow$ $(2^x)^2 - 5 \cdot 2^x + 4 > 0$

$t = 2^x$ $\qquad$ $t^2 - 5t + 4 > 0$

Studiamo l'equazione associata $t^2 - 5t + 4 = 0$

$\Delta = (-5)^2 - 4 \cdot 1 \cdot 4 = 9$ $\qquad$ $t_{1/2} = \dfrac{5 \pm 3}{2}$ $\begin{cases} 1 \\ 4 \end{cases}$

$t^2 - 5t + 4 > 0$ $\Leftrightarrow$ $t < 1 \ \vee \ t > 4$

$\Leftrightarrow$ $2^x < 1 \ \vee \ 2^x > 4$

$\Leftrightarrow$ $2^x < 2^0 \ \vee \ 2^x > 2^2$

$\Leftrightarrow$ $x < 0 \ \vee \ x > 2$

5) $3^x \geq 5$ $\Rightarrow$ $x \geq \log_3 5$

6) $\left(\frac{1}{7}\right)^x < 6$

formula del cambiamento di base

I metodo $\quad \left(\frac{1}{7}\right)^x < 6$ $\Rightarrow$ $x > \log_{\frac{1}{7}} 6 = \dfrac{\log_7 6}{\log_7 \frac{1}{7}} = \dfrac{\log_7 6}{-1} = -\log_7 6$

II metodo $\quad 7^{-x} < 6$ $\Rightarrow$ $-x < \log_7 6$

$\Rightarrow$ $x > -\log_7 6$

7) $3^{3(x+2)} = 9^{\frac{1}{x}+1}$ $\Leftrightarrow$ $3^{3x+6} = 3^{\frac{2}{x}+2}$ $\Leftrightarrow$ $3x+6 = \frac{2}{x}+2$

C.E. $x \neq 0$

$$3x^2+6x=2+2x \iff 3x^2+4x-2=0$$

$$\Delta = 4^2-4\cdot 3\cdot(-2) = 16+24 = 40 = 2^2\cdot 10$$

$$\iff x_{1/2} = \frac{-4\pm 2\sqrt{10}}{6} = \frac{-2\pm\sqrt{10}}{3}$$ soluzioni accettabili perché sono $\neq 0$

8) $$\left(3^{x+1}\right)^{x-2}\cdot 9^{3x+2} \geq 81^{x+2} \iff$$

$$3^{(x+1)(x-2)}\cdot 3^{6x+4} \geq 3^{4x+8} \iff$$

$$3^{(x+1)(x-2)+6x+4} \geq 3^{4x+8} \iff$$

$$(x+1)(x-2)+6x+4 \geq 4x+8 \iff$$

$$x^2-x-2+6x+4 \geq 4x+8 \iff x^2+x-6\geq 0$$

$$\iff (x+3)(x-2)\geq 0$$

$$\iff x\leq -3 \lor x\geq 2$$

Esercizi sui logaritmi

- $\log_4 2 = \log_4\sqrt{4} = \log_4 4^{\frac{1}{2}} = \frac{1}{2}$
- $\log_{\frac{1}{9}}\frac{1}{\sqrt{3}} = \dfrac{\log_3\frac{1}{\sqrt{3}}}{\log_3\frac{1}{9}} = \dfrac{\log_3 3^{-\frac{1}{2}}}{\log_3 3^{-2}} = \dfrac{-\frac{1}{2}}{-2} = \frac{1}{4}$
- $\log_{100}\sqrt[11]{10} = \log_{100} 10^{\frac{1}{11}} = \log_{100}(100)^{\frac{1}{22}} = \frac{1}{22}$
- $\log_{\sqrt{2}} 1 = 0$
- $\log_5 5 = 1$
- $\log_3\frac{1}{3} = -1$
- $\log_{\frac{\sqrt{3}}{9}} 27 = \dfrac{\log_3 27}{\log_3\frac{\sqrt{3}}{9}} = \dfrac{3}{\log_3\sqrt{3}-\log_3 9} = \dfrac{3}{\frac{1}{2}-2} = \dfrac{3}{-\frac{3}{2}}$

$$= 3\left(-\frac{2}{3}\right) = -2$$

Equazioni e disequazioni logaritmiche

Siano $a\in(0,1)\cup(1,+\infty)$ e $K\in\mathbb{R}$

$$\log_a f(x) = K \iff \begin{cases} f(x)>0 \\ f(x)=a^K\end{cases}$$

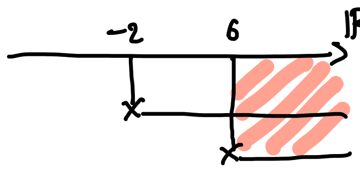
$$\log_a f(x) > K \iff \begin{cases} f(x) > 0 \\ \begin{cases} f(x) > a^k & \text{se } a > 1 \\ f(x) < a^k & \text{se } a \in (0,1) \end{cases} \end{cases}$$

$$\log_a f(x) < K \iff \begin{cases} f(x) > 0 \\ \begin{cases} f(x) < a^k & \text{se } a > 1 \\ f(x) > a^k & \text{se } a \in (0,1) \end{cases} \end{cases}$$

Equazioni e disequazioni logaritmiche

1) $\log_2 (x+2) > 3 \iff \begin{cases} x+2 > 0 \\ x+2 > 8 = 2^3 \end{cases}$

$$\iff \begin{cases} x > -2 \\ x > 6 \end{cases}$$



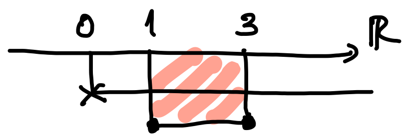
$x > 6$

2) $(\log_3 x)^2 - \log_3 x \leq 0 \iff \log_3 x \, (\log_3 x - 1) \leq 0$

$t = \log_3 x$

$t(t-1) \leq 0 \iff 0 \leq t \leq 1 \iff \log_3 1 = 0 \leq \log_3 x \leq 1 = \log_3 3$

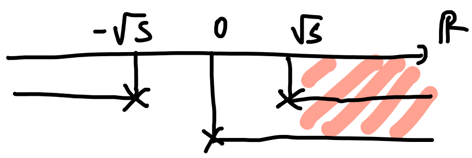
$$\iff \begin{cases} x > 0 \\ 1 \leq x \leq 3 \end{cases}$$



$1 \leq x \leq 3$

3) $\log_2 (x^2 - 5) = 2 + \log_2 x$

C.E. $\begin{cases} x^2 - 5 > 0 \\ x > 0 \end{cases} \iff \begin{cases} x < -\sqrt{5} \vee x > \sqrt{5} \\ x > 0 \end{cases} \iff x > \sqrt{5}$



$$\log_2(x^2-5) = \log_2 4 + \log_2 x$$

$$\log_2(x^2-5) = \log_2(4x) \quad \Rightarrow \quad x^2-6 = 4x$$

$$\Rightarrow \quad x^2-4x-5=0$$

$$\Rightarrow \quad (x-5)(x+1)=0$$

$$\Rightarrow \quad x=5 \lor x=-1$$

accettabile ↳ non è accettabile

C.E. $x > \sqrt{5}$

$x=5$ è l'unica soluzione

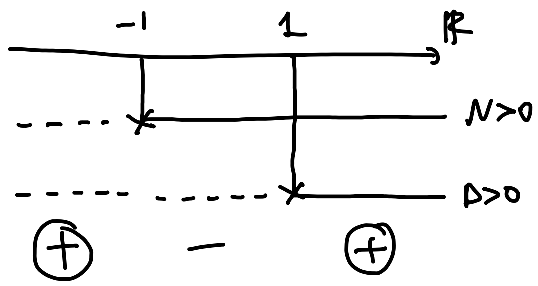
4) $\log_{\frac{1}{5}}\left(\frac{x+1}{x-1}\right) > \log_{\frac{1}{5}}\left(\frac{x}{x+1}\right)$

C.E. $\begin{cases} \frac{x+1}{x-1} > 0 & (*) \\ \frac{x}{x+1} > 0 & (**) \end{cases}$

$(*)$ $\frac{x+1}{x-1} > 0$ studio del segno

$N>0$: $x>-1$

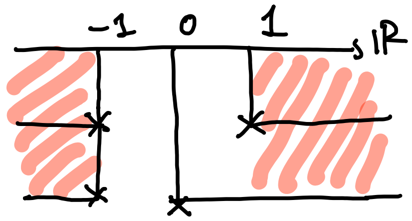
$D>0$: $x>1$



$(*)$ $x<-1 \lor x>1$

$(**)$ $\Leftrightarrow$ $x<-1 \lor x>0$ (facendo lo studio del segno)

C.E. $\begin{cases} x<-1 \lor x>1 \\ x<-1 \lor x>0 \end{cases}$



C.E. $x<-1 \lor x>1$

$$\log_{\frac{1}{5}}\left(\frac{x+1}{x-1}\right) > \log_{\frac{1}{5}}\left(\frac{x}{x+1}\right) \Rightarrow \frac{x+1}{x-1} < \frac{x}{x+1}$$

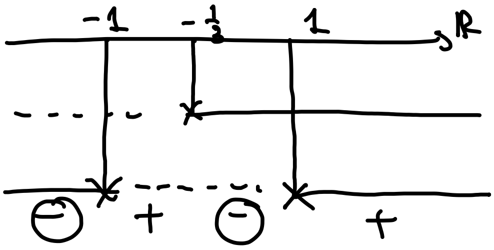
$$\Rightarrow \frac{x+1}{x-1} - \frac{x}{x+1} < 0 \Rightarrow \frac{(x+1)^2 - x(x-1)}{(x-1)(x+1)} < 0$$

$$\Rightarrow \quad \frac{\cancel{x^2}+2x+1-\cancel{x^2}+x}{x^2-1}<0 \quad \Rightarrow \quad \frac{3x+1}{x^2-1}<0$$

Studio del segno

$N>0 \; : \quad x>-\frac{1}{3}$

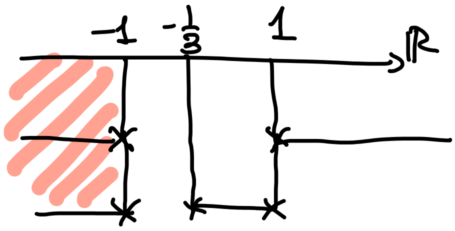
$D>0 \; : \quad x<-1 \vee x>1$



$$x<-1 \quad \vee \quad -\frac{1}{3}<x<1$$

Applichiamo le condizioni di esistenza

$$\begin{cases} x<-1 \vee x>1 \\ x<-1 \vee -\frac{1}{3}<x<1 \end{cases}$$



L'insieme delle soluzioni è $x<-1$

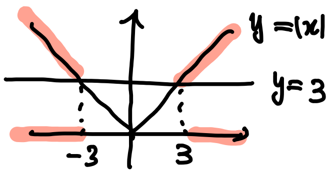
Esercizi modello esame

III appello 2023/24

a) $\dfrac{|x-4|-3}{1-2x} \geq 0$

b) $\dfrac{|\log x-4|-3}{1-2\log x} \geq 0$

Svolgimento: a) Studio del segno

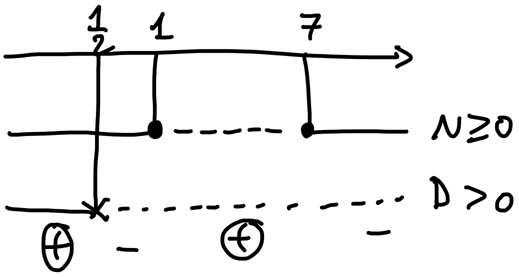


$N \geq 0 \; : \quad |x-4|-3 \geq 0 \iff |x-4| \geq 3$

$\iff x-4 \leq -3 \quad \vee \quad x-4 \geq 3$

$\iff x \leq 1 \quad \vee \quad x \geq 7$

$D>0 \; : \quad 1-2x>0 \iff x<\frac{1}{2}$



$$x<\frac{1}{2} \quad \vee \quad 1 \leq x \leq 7$$

b) $\dfrac{|\log x - 4| - 3}{1-2\log x} \geq 0$ poniamo $t = \log x$

$\dfrac{|t-4|-3}{1-2t} \geq 0 \iff t < \frac{1}{2} \vee 1 \leq t \leq 7$

$\iff \log x < \frac{1}{2} \vee 1 \leq \log x \leq 7$

$\iff \begin{cases} x > 0 \\ \log x < \log e^{\frac{1}{2}} \quad \vee \quad \log e \leq \log x \leq \log e^{7} \end{cases}$

$\iff \begin{cases} x > 0 \\ x < e^{\frac{1}{2}} \quad \vee \quad e \leq x \leq e^{7} \end{cases}$

$\iff 0 < x < e^{\frac{1}{2}} \vee e \leq x \leq e^{7}$

Esercizio per casa:

II appello 2022/23

a) $\dfrac{|1-4x|-3}{4-x} < 0$

b) $\dfrac{|1-4e^{2x}|-3}{4-e^{2x}} < 0$