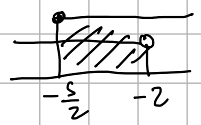
MATEMATICA - LEZIONE 7 - 12/10/23 14.30 - 16.30

Esempi

5) $\sqrt{2x+5} \geq x+2 \iff \begin{cases} x+2<0 \\ 2x+5 \geq 0 \end{cases} \vee \begin{cases} x+2 \geq 0 \\ \cancel{2x+5 \geq 0} \\ 2x+5 \geq (x+2)^2 \end{cases}$

$\iff \begin{cases} x<-2 \\ x \geq -\frac{5}{2} \end{cases} \vee \begin{cases} x \geq -2 \\ 2x+5 \geq x^2+4x+4 \end{cases}$

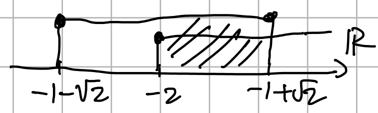
$\iff \begin{cases} x<-2 \\ x \geq -\frac{5}{2} \end{cases} \vee \begin{cases} x \geq -2 \\ x^2+2x-1 \leq 0 \end{cases}$



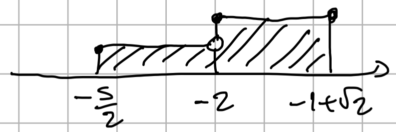
$\Delta = 4+4 = 8$

$x_{1/2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

$\iff -\frac{5}{2} \leq x < -2 \vee \begin{cases} x \geq -2 \\ -1-\sqrt{2} \leq x \leq -1+\sqrt{2} \end{cases}$



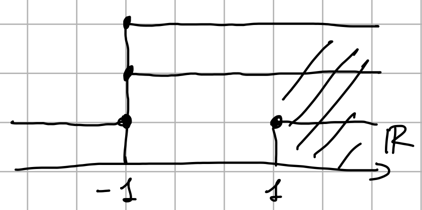
$\iff -\frac{5}{2} \leq x < -2 \vee -2 \leq x \leq -1+\sqrt{2}$



$\iff -\frac{5}{2} \leq x \leq -1+\sqrt{2}$

6) $\sqrt{x^2-1} \leq x+1 \iff \begin{cases} x^2-1 \geq 0 \\ x+1 \geq 0 \\ x^2-1 \leq (x+1)^2 \end{cases} \iff \begin{cases} x \leq -1 \vee x \geq 1 \\ x \geq -1 \\ \cancel{x^2}-1 \leq \cancel{x^2}+2x+1 \end{cases}$

$\iff \begin{cases} x \leq -1 \vee x \geq 1 \\ x \geq -1 \\ x \geq -1 \end{cases} \iff x = -1 \vee x \geq 1$



Proprietà dell'ESPONENZIALE

- Per $a>0$ possiamo definire $\forall x \in \mathbb{R}$ a^x

  se $a>1$ $\quad a^x := \sup\{a^q : q \in \mathbb{Q}, q<x\}$

  se $a \in (0,1)$ $\quad a^x := \inf\{a^q : q \in \mathbb{Q}, q<x\}$

In particolare $\forall x \in \mathbb{R}, \forall a>0 : a^x > 0$

- $\forall x,y \in \mathbb{R} \ \forall a>0 : a^{x+y} = a^x \cdot a^y$
- $\forall x,y \in \mathbb{R}, \forall a>0 : a^{x-y} = \frac{a^x}{a^y}$
- $\forall x \in \mathbb{R} \ \forall a>0 : a^{-x} = \frac{1}{a^x}$
- $\forall x,y \in \mathbb{R}, \forall a>0 : (a^x)^y = a^{x \cdot y}$
- $\forall x \in \mathbb{R}, \forall a,b>0 : (a \cdot b)^x = a^x \cdot b^x$
- $\forall x \in \mathbb{R}, \ \forall a,b>0 : \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
- se $a>1$ allora $\forall x_1, x_2 \in \mathbb{R} : x_1 < x_2 \iff a^{x_1} < a^{x_2}$
- se $a \in (0,1)$ allora $\forall x_1, x_2 \in \mathbb{R} : x_1 < x_2 \iff a^{x_1} > a^{x_2}$
- se $a>0$ $\forall x_1, x_2 \in \mathbb{R} : x_1 = x_2 \iff a^{x_1} = a^{x_2}$

Definizione Sia $a \in (0,1) \cup (1,+\infty)$ e $x>0$

Si dice LOGARITMO in base a di x l'unico numero reale y tale che $a^y = x$ e si denota con $\log_a x = y$

Proprietà dei logaritmi

Nel seguito a,b denotano basi dei logaritmi $(a,b \in (0,1) \cup (1,+\infty))$

- $\log_a 1 = 0$
- $\forall x \in \mathbb{R} : \log_a a^x = x$
- $\forall x>0 : a^{\log_a x} = x$
- $\forall x,y>0 : \log_a (x \cdot y) = \log_a x + \log_a y$
- $\forall x,y>0 : \log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
- $\forall x>0, \forall \alpha \in \mathbb{R} : \log_a x^\alpha = \alpha \log_a x$
- $\forall x>0 : \log_a x = \frac{\log_b x}{\log_b a}$

- se $a>1$ allora $\forall x>1 : \log_a x > 0$, $\forall x \in (0,1) : \log_a x < 0$
- se $a \in (0,1)$ allora $\forall x>1 : \log_a x < 0$, $\forall x \in (0,1) : \log_a x > 0$
- se $a>1$ allora $\forall x_1, x_2 > 0 : x_1 < x_2 \iff \log_a x_1 < \log_a x_2$
- se $a \in (0,1)$ allora $\forall x_1, x_2 > 0 : x_1 < x_2 \iff \log_a x_1 > \log_a x_2$
- se $a \in (0,1) \cup (1,+\infty)$ allora $\forall x_1, x_2 > 0 : x_1 = x_2 \iff \log_a x_1 = \log_a x_2$

Esempi

- $\log_4 2 = \frac{1}{2}$ $\qquad \log_4 2 = \frac{\log_2 2}{\log_2 4} = \frac{1}{2}$
- $\log_{\frac{1}{9}} \frac{1}{\sqrt{3}}$ $\qquad \log_{\frac{1}{9}} \frac{1}{\sqrt{3}} = \frac{\log_3 \frac{1}{\sqrt{3}}}{\log_3 \frac{1}{9}} = \frac{\log_3 3^{-\frac{1}{2}}}{\log_3 3^{-2}} = \frac{-\frac{1}{2}}{-2}$
  $= -\frac{1}{2} \cdot \left(-\frac{1}{2}\right) = \frac{1}{4}$
- $\log_{\sqrt{2}} 1 = 0$
- $\log_{100} \sqrt[11]{10} = \log_{100} (10)^{\frac{1}{11}} = \frac{1}{11} \log_{100} 10 = \frac{1}{11} \cdot \frac{1}{2} = \frac{1}{22}$
- $\log_{\sqrt[3]{3}} \sqrt[4]{27} = \log_{\sqrt[3]{3}} 3^{\frac{3}{4}} = \frac{3}{4} \log_{\sqrt[3]{3}} 3 = \frac{3}{4} \cdot 3 = \frac{9}{4}$

Equazioni e disequazioni esponenziali

Sia $a \in (0,1) \cup (1,+\infty)$ e $K \in \mathbb{R}$ allora

$$a^{p(x)} = K \iff \begin{cases} p(x) = \log_a K & \text{se } K > 0 \\ \nexists x \in \mathbb{R} & \text{se } K \leq 0 \end{cases}$$

$$a^{p(x)} \leq K \iff \begin{cases} p(x) \leq \log_a K & \text{se } K>0 \text{ e } a>1 \\ p(x) \geq \log_a K & \text{se } K>0 \text{ e } a \in (0,1) \\ \nexists x \in \mathbb{R} & \text{se } K \leq 0 \end{cases}$$

$$a^{p(x)} \geq K \iff \begin{cases} p(x) \geq \log_a K & \text{se } K>0 \text{ e } a>1 \\ p(x) \leq \log_a K & \text{se } K>0 \text{ e } a \in (0,1) \\ \forall x \in \mathbb{R} : \exists p(x) & \text{se } K \leq 0 \end{cases}$$

Esempi

1) $e^x \geq -2 \qquad \forall x \in \mathbb{R}$

2) $3^x = \frac{1}{\sqrt[5]{3}} = 3^{-\frac{1}{5}} \iff x = -\frac{1}{5}$

3) $\left(\frac{1}{2}\right)^x \geq \frac{8 \cdot \sqrt{2}}{\sqrt[3]{4}} = \frac{2^3 \cdot 2^{\frac{1}{2}}}{2^{2/3}} = \frac{2^{3+\frac{1}{2}}}{2^{2/3}} = \frac{2^{\frac{7}{2}}}{2^{2/3}}$

$= 2^{\frac{7}{2}-\frac{2}{3}} = 2^{\frac{17}{6}}$

$\iff \left(\frac{1}{2}\right)^x \geq 2^{\frac{17}{6}} \iff x \leq \log_{\frac{1}{2}} 2^{\frac{17}{6}} = -\frac{17}{6}$

4) $4^x - 5 \cdot 2^x + 4 > 0 \iff (2^x)^2 - 5 \cdot 2^x + 4 > 0$

$t = 2^x$

$\iff t^2 - 5t + 4 > 0$

$\iff (t-4)(t-1) > 0$

$\iff t < 1 \vee t > 4$

$\iff 2^x < 1 \quad \vee \quad 2^x > 4$

$\iff 2^x < 2^0 \quad \vee \quad 2^x > 2^2$

$\iff x < 0 \quad \vee \quad x > 2$

5) $3^x \geq 5 \iff x \geq \log_3 5$

6) $\left(\frac{1}{7}\right)^x < 6 \iff x > \log_{\frac{1}{7}} 6 = -\log_7 6$

$\Updownarrow$

$7^{-x} < 6 \iff -x < \log_7 6 \iff x > -\log_7 6$

Equazioni e disequazioni logaritmiche

Siano $a \in (0,1) \cup (1,+\infty)$ e $K \in \mathbb{R}$

$$\log_a P(x) = K \iff \begin{cases} P(x) > 0 & \text{[c.e. logaritmo]} \\ P(x) = a^K \end{cases}$$

$$\log_a P(x) \geq K \iff \begin{cases} P(x) > 0 & \text{[c.e. logaritmo]} \\ \begin{cases} P(x) \geq a^K & \text{se } a > 1 \\ P(x) \leq a^K & \text{se } a \in (0,1) \end{cases} \end{cases}$$

$$\log_a P(x) \leq K \iff \begin{cases} P(x) > 0 \\ \begin{cases} P(x) \leq a^K & \text{se } a > 1 \\ P(x) \geq a^K & \text{se } a \in (0,1) \end{cases} \end{cases}$$

Esempi

1) $\log_2(x+2) > 3 \iff \begin{cases} x+2 > 0 \\ x+2 > 2^3 \end{cases} \iff x > 6$

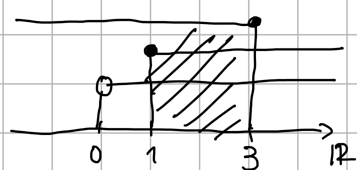
2) $(\log_3 x)^2 - \log_3 x \leq 0 \iff \log_3 x(\log_3 -1) \leq 0$

$\overset{\shortparallel}{\log_3^2 x}$

$t = \log_3 x$

$t(t-1) \leq 0 \iff 0 \leq t \leq 1 \iff 0 \leq \log_3 x \leq 1$

$\iff \begin{cases} \log_3 x \geq 0 \\ \log_3 x \leq 1 \end{cases} \iff \begin{cases} x > 0 \\ x \geq 3^0 = 1 \\ x \leq 3^1 = 3 \end{cases}$



$\iff 1 \leq x \leq 3$

3) $\log_2(x^2-5) = 2 + \log_2 x \iff \log_2(x^2-5) = \log_2 4 + \log_2 x$

$\iff \begin{cases} x^2-5 > 0 \\ x > 0 \\ \log_2(x^2-5) = \log_2(4x) \end{cases} \iff \begin{cases} x < -\sqrt{5} \ \vee \ x > \sqrt{5} \\ x > 0 \\ x^2-5 = 4x \end{cases}$

$$\Leftrightarrow \begin{cases} x<-\sqrt{5} \vee x>\sqrt{5} \\ x>0 \\ x^2-4x-5=0 \end{cases}$$
$(x-5)(x+1)$

$$\Leftrightarrow \begin{cases} x<-\sqrt{5} \vee x>\sqrt{5} \\ x>0 \\ x=5 \vee x=-1 \end{cases} \Leftrightarrow x=5$$

accettabile → non è accettabile

4) $\log_{\frac{1}{5}}\left(\frac{x+1}{x-1}\right) > \log_{\frac{1}{5}}\left(\frac{x}{x+1}\right)$

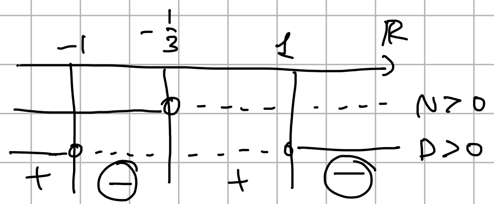
$$\Leftrightarrow \begin{cases} \frac{x+1}{x-1}>0 \\ \frac{x}{x+1}>0 \\ \frac{x+1}{x-1} < \frac{x}{x+1} \end{cases} \Leftrightarrow \begin{cases} x<-1 \vee x>1 \\ x<-1 \vee x>0 \\ \frac{x}{x+1} - \frac{x+1}{x-1} > 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x<-1 \vee x>1 \\ \frac{x(x-1)-(x+1)^2}{x^2-1} > 0 \end{cases} \Leftrightarrow \begin{cases} x<-1 \vee x>1 \\ \frac{-3x-1}{x^2-1} > 0 \end{cases}$$

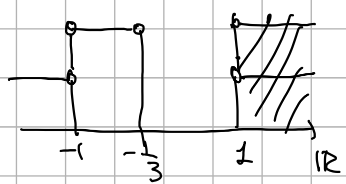
$\frac{-3x-1}{x^2-1} > 0 \Leftrightarrow \frac{3x+1}{x^2-1} < 0$ studio del segno

$N>0$: $x<-\frac{1}{3}$

$D>0$: $x<-1 \vee x>1$



$$\Leftrightarrow \begin{cases} x<-1 \vee x>1 \\ -1<x<-\frac{1}{3} \vee x>1 \end{cases} \Leftrightarrow x>1$$



Modelli d'esame

Aprile '22

a) $\frac{|x-1|-2}{4x-1} \geq 0$

b) $\frac{|\log x - 1|-2}{4\log x - 1} \geq 0$

ln, log "logaritmo in base e"
Log "logaritmo in base 10"

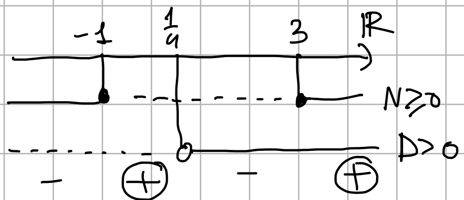
svolgimento: a) Studio del segno

$N \geq 0$: $|x-1|-2 \geq 0 \iff |x-1| \geq 2$

$\iff x-1 \leq -2 \quad \vee \quad x-1 \geq 2$

$\iff x \leq -1 \quad \vee \quad x \geq 3$

$D > 0$: $4x-1 > 0 \iff x > \frac{1}{4}$



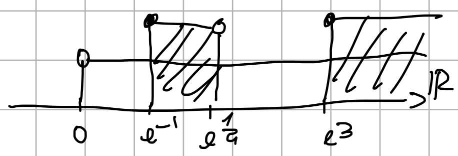
$$\frac{|x-1|-2}{4x-1} \geq 0 \iff -1 \leq x < \frac{1}{4} \quad \vee \quad x \geq 3$$

b) $$\frac{|\log x - 1| - 2}{4 \log x - 1} \geq 0 \qquad t = \log x$$

$$\frac{|t-1|-2}{4t-1} \geq 0 \iff -1 \leq t < \frac{1}{4} \quad \vee \quad t \geq 3$$

$$\iff -1 \leq \log x < \frac{1}{4} \quad \vee \quad \log x \geq 3$$

$$\iff \begin{cases} x > 0 \\ e^{-1} \leq x < e^{\frac{1}{4}} \quad \vee \quad x \geq e^3 \end{cases}$$



$$\iff e^{-1} \leq x < e^{\frac{1}{4}} \quad \vee \quad x \geq e^3$$

Luglio '22

a) $$\frac{3-|x-4|}{x^2-4} \geq 0$$

b) $$\frac{3-|e^x-4|}{e^{2x}-4} \geq 0$$

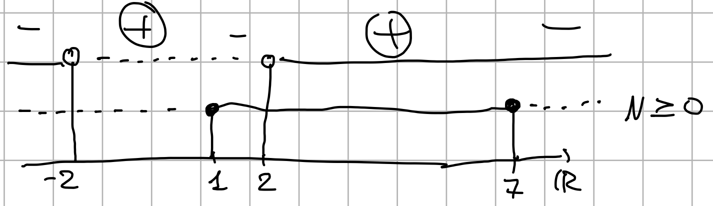
svolgimento: a) Studio del segno

$N \geq 0$: $3-|x-4| \geq 0 \iff |x-4| \leq 3$

$\iff -3 \leq x-4 \leq 3$

$\iff 1 \leq x \leq 7$

$D > 0 \,:\, x^2-4 > 0 \iff x<-2 \lor x>2$



$$\frac{3-|x-4|}{x^2-4} \geq 0 \iff -2 < x \leq 1 \lor 2 < x \leq 7$$

b) $\dfrac{3-|e^x-4|}{e^{2x}-4} \geq 0$ poniamo $t=e^x$

$$\frac{3-|t-4|}{t^2-4} \geq 0 \iff -2 < t \leq 1 \lor 2 < t \leq 7$$

$$\iff -2 < e^x \leq 1 \lor 2 < e^x \leq 7$$

$$\iff e^x \leq e^0 \lor e^{\ln 2} < e^x \leq e^{\ln 7}$$

$$\iff x \leq 0 \lor \ln 2 < x \leq \ln 7$$