Esercizio per casa

$$x^4 - 3x^3 - 3x^2 + 11x - 6 \geq 0$$

$p(x) = x^4 - 3x^3 - 3x^2 + 11x - 6$ $\qquad \{\pm 1, \pm 2, \pm 3, \pm 6\}$

$p(1) = 1 - 3 - 3 + 11 - 6 = 0$

$p(-1) = 1 + 3 - 3 - 11 - 6 \neq 0$

$p(2) = 16 - 24 - 12 + 22 - 6 \neq 0$

$p(-2) = 16 + 24 - 12 - 22 - 6 = 0$

$1, -2$ sono radici di $p(x)$ $\Rightarrow$ $(x-1)(x+2) = x^2 + x - 2$ divide $p(x)$

$$\begin{array}{rrrrr|l}
x^4 & -3x^3 & -3x^2 & +11x & -6 & x^2+x-2 \\ \cline{6-6}
x^4 & +x^3 & -2x^2 & & & x^2-4x+3 \\ \cline{1-5}
/ & -4x^3 & -x^2 & +11x & -6 & \\
 & -4x^3 & -4x^2 & +8x & & \\ \cline{2-5}
 & // & 3x^2 & +3x & -6 & \\
 & & 3x^2 & +3x & -6 & \\ \cline{3-5}
 & & & // & &
\end{array}$$

$p(x) = (x-1)(x+2)(x^2 - 4x + 3)$

$= (x-1)(x+2)(x-3)(x-1) = (x-1)^2(x+2)(x-3)$

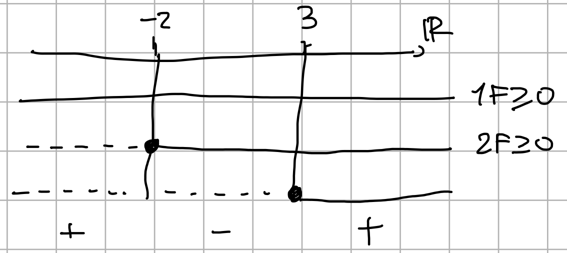
la disequazione di partenza è

$(x-1)^2(x+2)(x-3) \geq 0$

$1F \geq 0: \ (x-1)^2 \geq 0 \iff \forall x \in \mathbb{R}$

$2F \geq 0: \ x+2 \geq 0 \iff x \geq -2$

$3F \geq 0: \ x-3 \geq 0 \iff x \geq 3$

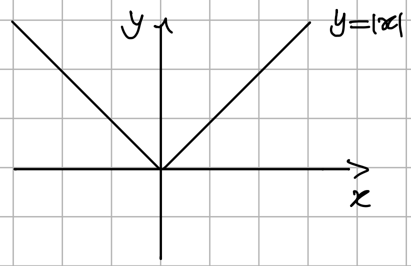


$x \leq -2 \ \cup \ x \geq 3$

Equazioni e disequazioni con valore assoluto

Ricordiamo che $\forall x \in \mathbb{R}$ $\quad |x| := \max\{x,-x\} = \begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } x \leq 0 \end{cases}$

Pertanto



$$|x| = K \iff \begin{cases} x = K \vee x = -K & \text{se } K \geq 0 \\ \nexists x \in \mathbb{R} & \text{se } K < 0 \end{cases}$$

$$|x| < K \iff \begin{cases} -K < x < K & \text{se } K > 0 \\ \nexists x \in \mathbb{R} & \text{se } K \leq 0 \end{cases}$$

$$|x| \leq K \iff \begin{cases} -K \leq x \leq K & \text{se } K \geq 0 \\ \nexists x \in \mathbb{R} & \text{se } K < 0 \end{cases}$$

$$|x| > K \iff \begin{cases} x < -K \vee x > K & \text{se } K \geq 0 \\ \forall x \in \mathbb{R} & \text{se } K < 0 \end{cases}$$

$$|x| \geq K \iff \begin{cases} x \leq -K \vee x \geq K & \text{se } K > 0 \\ \forall x \in \mathbb{R} & \text{se } K \leq 0 \end{cases}$$

Nota In quanto segue $p(x)$ e $q(x)$ sono polinomi rispetto alla variabile x, tuttavia tutto ciò che diremo sulle equazioni e sulle disequazioni con valore assoluto continuerà a valere anche per funzioni $f(x)$, $g(x)$ (per la definizione di funzione si veda la prossima lezione del Prof. Mancini)

$$|p(x)| = q(x) \iff \begin{cases} q(x) \geq 0 \\ p(x) = q(x) \vee p(x) = -q(x) \end{cases}$$

$$|p(x)| < q(x) \iff \begin{cases} q(x) > 0 \\ p(x) < q(x) \\ p(x) > -q(x) \end{cases}$$

$$|p(x)| \leq q(x) \iff \begin{cases} q(x) \geq 0 \\ p(x) \leq q(x) \\ p(x) \geq -q(x) \end{cases}$$

$$|p(x)| > q(x) \iff \begin{cases} q(x) < 0 \\ \exists p(x) \end{cases} \vee \begin{cases} q(x) \geq 0 \\ p(x) < -q(x) \vee p(x) > q(x) \end{cases}$$

$$|p(x)| \geq q(x) \iff \begin{cases} q(x) \leq 0 \\ \exists p(x) \end{cases} \vee \begin{cases} q(x) > 0 \\ p(x) \leq -q(x) \vee p(x) \geq q(x) \end{cases}$$

Esempi

1) $|x-2| < 3 \iff -3 < x-2 < 3 \iff \begin{cases} x-2 > -3 \\ x-2 < 3 \end{cases} \iff \begin{cases} x > -1 \\ x < 5 \end{cases}$

$\iff x \in (-1, 5)$

2) $|x^2-2x| \geq 1 \iff x^2-2x \leq -1 \ \vee \ x^2-2x \geq 1$

$\iff x^2-2x+1 \leq 0 \ \vee \ x^2-2x-1 \geq 0$ $\qquad x_{1/2} = \dfrac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

$\iff (x-1)^2 \leq 0 \ \vee \ x \leq 1-\sqrt{2} \ \vee \ x \geq 1+\sqrt{2}$

$\iff x = 1 \ \vee \ x \leq 1-\sqrt{2} \ \vee \ x \geq 1+\sqrt{2}$

$\iff x \in (-\infty, 1-\sqrt{2}] \cup \{1\} \cup [1+\sqrt{2}, +\infty)$

3) $|x+1| > -1 \quad \forall x \in \mathbb{R}$

4) $|x^3-x+2| \leq -5 \quad \nexists x \in \mathbb{R}$

5) $|x^2-2| < x+2$

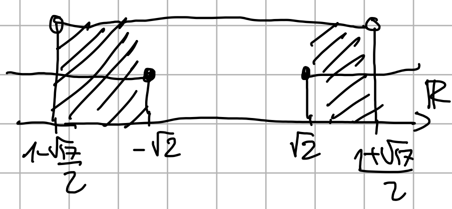
I metodo: $\begin{cases} x^2-2 \geq 0 \\ x^2-2 < x+2 \end{cases} \vee \begin{cases} x^2-2 \leq 0 \\ -x^2+2 < x+2 \end{cases}$

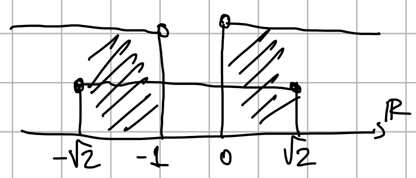
$\iff \begin{cases} x \leq -\sqrt{2} \vee x \geq \sqrt{2} \\ x^2-x-4 < 0 \end{cases} \vee \begin{cases} -\sqrt{2} \leq x \leq \sqrt{2} \\ x^2+x > 0 \end{cases}$

$\Delta = 1 - 4 \cdot (-4) = 17$

$x_{1/2} = \dfrac{1 \pm \sqrt{17}}{2}$

$\iff \begin{cases} x \leq -\sqrt{2} \vee x \geq \sqrt{2} \\ \dfrac{1-\sqrt{17}}{2} < x < \dfrac{1+\sqrt{17}}{2} \end{cases} \vee \begin{cases} -\sqrt{2} \leq x \leq \sqrt{2} \\ x < -1 \vee x > 0 \end{cases}$





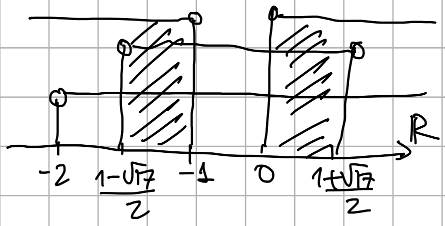
$\iff \dfrac{1-\sqrt{17}}{2} < x \leq -\sqrt{2} \ \vee \ \sqrt{2} \leq x < \dfrac{1+\sqrt{17}}{2} \ \vee \ -\sqrt{2} \leq x < -1 \ \vee \ 0 < x \leq \sqrt{2}$

$\iff \dfrac{1-\sqrt{17}}{2} < x < -1 \ \vee \ 0 < x < \dfrac{1+\sqrt{17}}{2}$

II metodo

$$|x^2-2| < x+2 \iff \begin{cases} x+2>0 \\ x^2-2 < x+2 \\ x^2-2 > -x-2 \end{cases} \iff \begin{cases} x > -2 \\ x^2-x-4<0 \\ x^2+x>0 \end{cases}$$

$$\iff \begin{cases} x>-2 \\ \frac{1-\sqrt{17}}{2} < x < \frac{1+\sqrt{17}}{2} \\ x<-1 \vee x>0 \end{cases}$$



$$\iff \frac{1-\sqrt{17}}{2} < x < -1 \quad \vee \quad 0 < x < \frac{1+\sqrt{17}}{2}$$

Equazioni e disequazioni irrazionali

Definizione Sia $n \in \mathbb{N}$, $n \geq 2$

- Se n è PARI $\forall a \geq 0$ si dice RADICE n-ESIMA di a e si denota $\sqrt[n]{a}$ l'unico $b \geq 0$ t.c. $a = b^n$
- Se n è DISPARI $\forall a \in \mathbb{R}$ si dice RADICE n-ESIMA di a e si denota $\sqrt[n]{a}$ l'unico $b \in \mathbb{R}$ t.c. $a = b^n$

Caso n - dispari

$$\sqrt[n]{p(x)} = q(x) \iff p(x) = (q(x))^n$$

$$\sqrt[n]{p(x)} \leq q(x) \iff p(x) \leq (q(x))^n$$

$$\sqrt[n]{p(x)} \geq q(x) \iff p(x) \geq (q(x))^n$$

Caso n pari

Se n è un numero pari possiamo definire la radice n-esima di un numero reale ≥ 0 e tale radice sarà a sua volta ≥ 0, pertanto

$$\sqrt[n]{p(x)} = q(x) \iff \begin{cases} p(x) \geq 0 & \text{[condizione di esistenza della radice]} \\ q(x) \geq 0 & \text{[condizione di accettabilità per il II membro]} \\ p(x) = (q(x))^n \end{cases}$$

$$\sqrt[n]{p(x)} < q(x) \iff \begin{cases} p(x) \geq 0 & \text{[c.e. radice]} \\ q(x) > 0 & \text{[c.a. per il II membro]} \\ p(x) < (q(x))^n \end{cases}$$

$$\sqrt[n]{p(x)} \leq q(x) \iff \begin{cases} p(x) \geq 0 \\ q(x) \geq 0 \\ p(x) \leq (q(x))^n \end{cases}$$

$$\sqrt[n]{p(x)} > q(x) \iff \begin{cases} q(x) < 0 \\ p(x) \geq 0 \end{cases} \vee \begin{cases} q(x) \geq 0 \\ \cancel{p(x) \geq 0} \\ p(x) > (q(x))^n \end{cases}$$

$$\sqrt[n]{p(x)} \geq q(x) \iff \begin{cases} q(x) \leq 0 \\ p(x) \geq 0 \end{cases} \vee \begin{cases} q(x) > 0 \\ p(x) \geq (q(x))^n \end{cases}$$

Esempi

1) $$\sqrt{x^2+x-6} = x-1 \iff \begin{cases} x^2+x-6 \geq 0 \\ x-1 \geq 0 \\ x^2+x-6 = (x-1)^2 \end{cases}$$

$$\iff \begin{cases} (x+3)(x-2) \geq 0 \\ x \geq 1 \\ \cancel{x^2}+x-6 = \cancel{x^2}-2x+1 \end{cases}$$

$$\iff \begin{cases} x \leq -3 \vee x \geq 2 \\ x \geq 1 \\ x = \frac{7}{3} \end{cases} \iff x = \frac{7}{3}$$

2) $$\sqrt[3]{x+2} = 2 \iff x+2 = 8 \iff x = 6$$

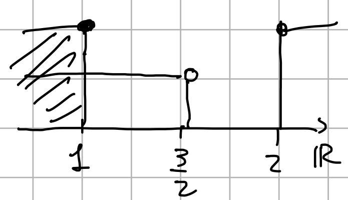
3) $$\sqrt{x^2-3x+2} > 2x-3 \iff \begin{cases} 2x-3 < 0 \\ x^2-3x+2 \geq 0 \end{cases} \vee \begin{cases} 2x-3 \geq 0 \\ x^2-3x+2 > (2x-3)^2 \end{cases}$$

$$\iff \begin{cases} x < \frac{3}{2} \\ (x-2)(x-1) \geq 0 \end{cases} \vee \begin{cases} x \geq \frac{3}{2} \\ x^2-3x+2 > 4x^2-12x+9 \end{cases}$$

$$\iff \begin{cases} x < \frac{3}{2} \\ x \leq 1 \vee x \geq 2 \end{cases} \vee \begin{cases} x \geq \frac{3}{2} \\ 3x^2-9x+7 < 0 \quad \nexists x \in \mathbb{R} \end{cases}$$

$$\Delta = 81-4\cdot 3\cdot 7 < 0$$

$$\iff \begin{cases} x < \frac{3}{2} \\ x \leq 1 \vee x \geq 2 \end{cases}$$



$x \leq 1$

4) $$\sqrt[3]{x^3+8} \geq x-2 \iff x^3+8 \geq (x-2)^3$$

$$\iff \cancel{x^3}+8 \geq \cancel{x^3}-6x^2+12x-8$$

$$(A+B)^3 = A^3+3A^2B+3AB^2+B^3$$

$$\iff 6x^2-12x+16 \geq 0 \iff 3x^2-6x+8 \geq 0 \quad \forall x \in \mathbb{R}$$

$$\Delta = 36-3\cdot 4\cdot 8 < 0$$