LEZIONE 34

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ESERCIZIO 1

1) Calcolare $\int \frac{1}{x^2-2x}\,dx$

2) Calcolare $\int_{-\log 2}^{0} \frac{1}{e^{2x}-2}\,dx$

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1) $x^2 - 2x = x(x-2)$. Cerchiamo $A, B \in \mathbb{R}$ t.c.

$$\frac{1}{x^2-2x} = \frac{A}{x} + \frac{B}{x-2}$$

$$= \frac{A(x-2) + Bx}{x(x-2)} = \frac{(A+B)x - 2A}{x^2-2x}$$

$$\begin{cases} A + B = 0 \\ -2A = 1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{2} \\ B = \frac{1}{2} \end{cases}$$

Quindi: $\frac{1}{x^2-2x} = -\frac{1}{2}\frac{1}{x} + \frac{1}{2}\frac{1}{x-2}$ e

$$\int \frac{1}{x^2-2x}\,dx = -\frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{x-2}\,dx$$

$$= -\frac{1}{2}\log|x| + \frac{1}{2}\log|x-2| + C$$

2) $\int_{-\log 2}^{0} \frac{1}{e^{2x}-2}\,dx$

$t = e^x \qquad dt = e^x\,dx$

$$\int \frac{1}{(e^{2x}-2)} \frac{e^x}{e^x}\,dx = \int \frac{1}{(t^2-2)\,t}\,dt$$

Non è la sostituzione migliore

Scelta migliore:

$t = e^{2x}$ così:

$dt = e^{2x} \cdot 2\,dx = 2t\,dx \Rightarrow dx = \frac{1}{2t}\,dt$

$$\int \frac{1}{e^{2x}-2}\,dx = \int \frac{1}{t-2}\frac{dt}{2t} = \frac{1}{2}\int \frac{1}{t(t-2)}\,dt$$

$$= \frac{1}{2}\left[-\frac{1}{2}\log|t| + \frac{1}{2}\log|t-2|\right] + C$$

$$= -\frac{1}{4}\log|e^{2x}| + \frac{1}{4}\log|e^{2x}-2| + C$$

$$= \underbrace{-\frac{1}{2}x + \frac{1}{4}\log|e^{2x}-2|}_{F(x)} + C$$

$$F(0) = 0 + \frac{1}{4}\log|-1| = 0$$

$$F(-\log 2) = +\frac{1}{2}\log 2 + \frac{1}{4}\log|\underbrace{e^{-2\log 2}}_{\frac{1}{4}} - 2|$$

$$= \frac{1}{2}\log 2 + \frac{1}{4}\log\frac{7}{4}$$

$$= \cancel{\frac{1}{2}\log 2} + \frac{1}{4}\log 7 - \cancel{\frac{1}{4}\log 4} = \frac{1}{4}\log 7$$

$$\int_{-\log 2}^{0} \frac{1}{e^{2x}-2}\,dx = F(x)\Big|_{-\log 2}^{0} = 0 - \frac{1}{4}\log 7 = -\frac{1}{4}\log 7.$$

ESERCIZIO 2

Calcolare $\int x\log(2+3x)\,dx$

Integrazione per parti:

$$\int \underbrace{x}_{f'}\log(2+3x)\,dx = \frac{1}{2}x^2\log(2+3x) - \int \frac{1}{2}x^2\frac{1}{2+3x}\cdot 3\,dx$$

$$= \frac{1}{2}x^2\log(2+3x) - \frac{3}{2}\underbrace{\int\frac{x^2}{2+3x}}_{I}$$

Per calcolare I facciamo la divisione tra x^2 e $2+3x$

$$
\begin{array}{rrr|l}
x^2 & + 0\cdot x & + 0 & 3x+2 \\ \hline
x^2 & + \frac{2}{3}x & & \frac{x}{3} - \frac{2}{9} \\ \hline
// & -\frac{2}{3}x & + 0 & \\
& -\frac{2}{3}x & -\frac{4}{9} & \\ \hline
& // & \frac{4}{9} &
\end{array}
$$

$$x^2 = (3x+2)\left(\frac{x}{3} - \frac{2}{9}\right) + \frac{4}{9}$$

cioè $\displaystyle \frac{x^2}{3x+2} = \frac{x}{3} - \frac{2}{9} + \frac{\frac{4}{9}}{3x+2}$

$$\int \frac{x^2}{2+3x}\,dx = \int \frac{x}{3} - \frac{2}{9} + \frac{\frac{4}{9}}{3x+2}\,dx$$

$$= \frac{x^2}{6} - \frac{2}{9}x + \frac{4}{9}\log|3x+2|\cdot\frac{1}{3} + C$$

$$= \frac{x^2}{6} - \frac{2}{9}x + \frac{4}{27}\log|3x+2| + C$$

Conclusione:

$$\int x\log(2+3x)\,dx = \frac{1}{2}x^2\log(2+3x) - \frac{3}{2}\left[\frac{x^2}{6} - \frac{2}{9}x + \frac{4}{27}\log|3x+2|\right] + C$$

$$= \frac{1}{2}x^2\log(2+3x) - \frac{x^2}{4} + \frac{1}{3}x - \frac{2}{9}\underbrace{\log(3x+2)} + C$$

Qui non serve il $|\cdot|$ perché dall'inizio sappiamo che $2+3x>0$ altrimenti $\log(2+3x)$ non sarebbe ben definito

ESERCIZIO 3

Calcolare $\displaystyle \int \frac{1}{2e^x + e^{-x} + 1}\,dx$

$$\frac{1}{2e^x + \frac{1}{e^x} + 1} = \frac{1}{\frac{2e^{2x} + 1 + e^x}{e^x}} = \frac{e^x}{2e^{2x} + 1 + e^x}$$

$$\int \frac{e^x}{2e^{2x} + 1 + e^x}\,dx \quad dt$$

$t = e^x$

$dt = e^x\,dx$

$$= \int \frac{1}{2t^2+t+1}\, dt$$

$\Delta = 1 - 4\cdot 2 = -7 < 0.$ $\quad t_{1,2} = \frac{-1 \pm \sqrt{-7}}{4} = \frac{-1 \pm i\sqrt{7}}{4}$

Quindi: $2t^2+t+1 = 2\left(\left(t+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2\right)$ $\quad = -\frac{1}{4} \pm i\,\frac{\sqrt{7}}{4}$

$$\int \frac{1}{2t^2+t+1}\, dt = \int \frac{1}{2\left[\left(t+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2\right]}\, dt$$

$$= \frac{1}{2}\,\frac{4}{\sqrt{7}} \arctan\left(\frac{t+\frac{1}{4}}{\frac{\sqrt{7}}{4}}\right) + C$$

$$= \frac{2}{\sqrt{7}} \arctan\left(\frac{4t+1}{\sqrt{7}}\right) + C$$

$$= \frac{2}{\sqrt{7}} \arctan\left(\frac{2e^x+1}{\sqrt{7}}\right) + C.$$

ESERCIZIO:

Calcolare $\displaystyle\int_1^e \frac{\log x - 1}{\underline{x}\,(\log^2 x - 2\log x + 10)}\, \underline{dx}$

$t = \log x \qquad dt = \frac{1}{x}\, dx$ $\qquad x=1 \rightsquigarrow t = \log x = 0$; $\quad x = e \rightsquigarrow t = \log x = 1$

$$\int_1^e \frac{\log x - 1}{x\,(\log^2 x - 2\log x + 10)} = \int_0^1 \frac{t-1}{t^2-2t+10}\, dt$$

$\Delta = 4 - 40 < 0$

$$\frac{t-1}{t^2-2t+10} = \frac{A(2t-2)}{t^2-2t+10} + \frac{B}{t^2-2t+10}$$

$$= \frac{2At - 2A + B}{t^2-2t+10} \Longrightarrow \begin{cases} 2A = 1 \\ -2A + B = -1 \end{cases}$$

$$\Rightarrow \begin{cases} A = \frac{1}{2} \\ B = 0 \end{cases}.$$

$$\int_0^1 \frac{t-1}{t^2-2t+10}\,dt = \frac{1}{2}\int_0^1 \frac{2t-2}{t^2-2t+10}\,dt$$

$$= \frac{1}{2}\log(t^2-2t+10)\Big|_0^1 = \frac{1}{2}\log 9 - \frac{1}{2}\log 10$$

$$= \frac{1}{2}\log\frac{9}{10}$$

Metodo alternativo:

$$\int \frac{\log x - 1}{x(\log^2 x - 2\log x + 10)}\,dx = \int \frac{t-1}{t^2-2t+10}\,dt$$

$$= \frac{1}{2}\log(t^2-2t+10) + C$$

$$= \underbrace{\frac{1}{2}\log(\log^2 x - 2\log x + 10)}_{F(x)} + C$$

$$F(e) = \frac{1}{2}\log(1^2 - 2\cdot 1 + 10) = \frac{1}{2}\log 9$$

$$F(1) = \frac{1}{2}\log(0^2 - 2\cdot 0 + 10) = \frac{1}{2}\log 10$$

Risultato $\displaystyle\int_1^e \frac{\log x - 1}{x(\log^2 x - 2\log x + 10)} = F(e) - F(1) = \frac{1}{2}\log\frac{9}{10}$

Come si calcola $\displaystyle\int \frac{N(x)}{D(x)}\,dx$ se $\deg(D(x)) \geq 3$?

ESEMPIO :

$$\int \frac{1}{(2x+1)\underbrace{(x^2+x+1)}_{\Delta = 1-4<0}}\,dx$$

Cerchiamo $A, B, C \in \mathbb{R}$ t.c.

$$\frac{1}{(2x+1)(x^2+x+1)} = \frac{A}{2x+1} + \frac{B(2x+1)}{x^2+x+1} + \frac{C}{x^2+x+1}$$

$$= \frac{A(x^2+x+1)+B(2x+1)^2+C(2x+1)}{(2x+1)(x^2+x+1)}$$

$$= \frac{A(x^2+x+1)+B(4x^2+4x+1)+C(2x+1)}{(2x+1)(x^2+x+1)}$$

$$1 = x^2(A+4B) + x(A+4B+2C) + A+B+C .$$

$$\begin{cases} A+4B = 0 \\ A+4B+2C = 0 \\ A+B+C = 1 \end{cases} \Rightarrow \begin{cases} A = -4B \\ C = 0 \\ -4B+B+0 = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{4}{3} \\ C = 0 \\ B = -\frac{1}{3} \end{cases}$$

$$\frac{1}{(2x+1)(x^2+x+1)} = \frac{4}{3} \cdot \frac{1}{2x+1} - \frac{1}{3}\frac{2x+1}{x^2+x+1}$$

$$\int \frac{1}{(2x+1)(x^2+x+1)}\,dx = \frac{4}{3}\,\frac{1}{2}\log|2x+1| - \frac{1}{3}\log(x^2+x+1) + C$$

$$= \frac{2}{3}\log|2x+1| - \frac{1}{3}\log(x^2+x+1) + C$$

ESEMPIO 2

$$\int \frac{6x}{(x-2)^2(x^2+2)}\,dx$$

$$\frac{6x}{(x-2)^2(x^2+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C \cdot 2x}{x^2+2} + \frac{D}{x^2+2}$$

Facendo i conti si trova $A = -\frac{1}{3}$, $B = 2$, $C = \frac{1}{6}$, $D = -\frac{4}{3}$

$$\int \frac{6x}{(x-2)^2(x^2+2)}\,dx = -\frac{1}{3}\int\frac{1}{x-2}\,dx + 2\int\frac{1}{(x-2)^2}\,dx + \frac{1}{6}\int\frac{2x}{x^2+2}\,dx - \frac{4}{3}\int\frac{1}{x^2+2}\,dx$$

$$= -\frac{1}{3}\log|x-2| - \frac{2}{x-2} + \frac{1}{6}\log(x^2+2) - \frac{4}{3}\frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C$$

Alcune sostituzioni standard

1) $\int R\left(x, \sqrt{\frac{ax+b}{cx+d}}\right) dx$

Si utilizza la sostituzione $y = \sqrt{\frac{ax+b}{cx+d}}$.

ESEMPIO

$$\int \frac{1}{2-3\sqrt{x}}\, dx \qquad y = \sqrt{x}, \quad dy = \frac{1}{2\sqrt{x}}\, dx$$

$$\int \frac{1}{2-3\sqrt{x}} \cdot \frac{2\sqrt{x}}{2\sqrt{x}}\, dx = \int \frac{2y}{2-3y}\, dy$$

dy

$$= -\frac{2}{3}\int \frac{-3y}{2-3y}\, dy = -\frac{2}{3}\int \frac{2-3y-2}{2-3y}\, dy$$

$$= -\frac{2}{3}\int 1 - \frac{2}{2-3y}\, dy = -\frac{2}{3}\int 1 + \frac{2}{3y-2}\, dy$$

$$= -\frac{2}{3}y - \frac{4}{3}\log|3y-2| \cdot \frac{1}{3} + C.$$

$$= -\frac{2}{3}\sqrt{x} - \frac{4}{9}\log|3\sqrt{x}-2| + C.$$

ESEMPIO

$$\int \frac{1}{(x+6)\sqrt{x+2}}\, dx \qquad y = \sqrt{x+2}, \quad dy = \frac{1}{2\sqrt{x+2}}\, dx$$

Per completare la sostituzione:

$$y^2 = x+2 \Rightarrow x = y^2 - 2$$
$$\Rightarrow x+6 = y^2+4.$$

Allora:

$$\int \frac{1}{(x+6)\sqrt{x+2}}\,dx = \int \frac{2}{(x+6)\,2\sqrt{x+2}}\,dx = \int \frac{2}{y^2+4}\,dy$$

$$= 2\int \frac{1}{y^2+4}\,dy$$

$$= 2\cdot\frac{1}{2}\arctan\left(\frac{y}{2}\right) + c$$

$$= \arctan\left(\frac{y}{2}\right) + c = \arctan\left(\frac{\sqrt{x+2}}{2}\right) + c\,.$$

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2) Sostituzioni con formule parametriche di seno e coseno.
Ricordiamo che se $t = \tan\frac{x}{2}$, allora:

$$\sin x = \frac{2t}{1+t^2}\,,\quad \cos x = \frac{1-t^2}{1+t^2}\,,\quad \tan x = \frac{2t}{1+t^2}\,.$$

Si usano per integrali in cui compaiono seno e coseno.

$$t = \tan\frac{x}{2} \Rightarrow dt = \left(1+\tan^2\frac{x}{2}\right)\cdot\frac{1}{2}\,dx = \frac{(1+t^2)}{2}\,dx$$

cioè $dx = \frac{2}{1+t^2}\,dt$.

ESEMPIO

$$\int \frac{1}{\sin x}\,dx \qquad t = \tan\frac{x}{2}\,,\quad \sin x = \frac{2t}{1+t^2}\,,\quad dx = \frac{2}{1+t^2}\,dt$$

Allora

$$\int \frac{1}{\sin x}\,dx = \int \frac{1}{\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt$$

$$= \int \frac{\cancel{1+t^2}}{\cancel{2}t}\cdot\frac{\cancel{2}}{\cancel{1+t^2}}\,dt$$

$$= \int \frac{1}{t}\,dt = \log|t| + c$$

$$= \log\left|\tan\frac{x}{2}\right| + c\,.$$

Attenzione. Non sempre questa è la sostituzione migliore:

3) Per integrali del tipo $\int R(\cos x)\sin x\,dx$ conviene la sostituzione $y = \cos x$.

Per integrali del tipo $\int R(\sin x)\cos x\,dx$ conviene $y = \sin x$.

ESEMPIO

$$\int \frac{\sin x}{1+4\cos x}\,dx \qquad \text{Meglio} \quad t = \cos x$$
$$dt = -\sin x\,dx$$

$$= -\int \frac{1}{1+4t}\,dt = -\frac{1}{4}\log|1+4t| + c$$
$$= -\frac{1}{4}\log|1+4\cos x| + C$$

In questo caso, utilizzare la sostituzione con $\tan\frac{x}{2}$ sarebbe molto più complicato.

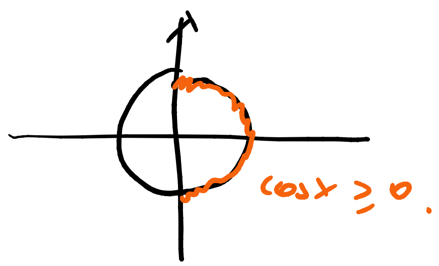
4) $\int R(x, \sqrt{a^2-x^2})\,dx$.

Sostituzione: $x = a\sin t$

$$\int_{-1}^{1}\sqrt{1-x^2}\,dx \qquad x = \sin t \qquad dx = \cos t\,dt.$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\sin^2 t}\,\cos t\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos^2 t}\,\cos t\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|\cos t|\cos t\,dt$$



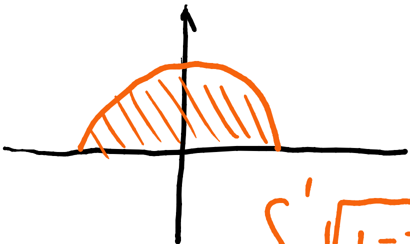
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt.$$

$$\cos 2t = 2\cos^2 t - 1$$
$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\, dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos(2t)\, dt = \frac{1}{2}t + \frac{1}{4}\sin(2t) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{4}\cdot 0 - \left(-\frac{1}{2}\frac{\pi}{2} + 0\right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

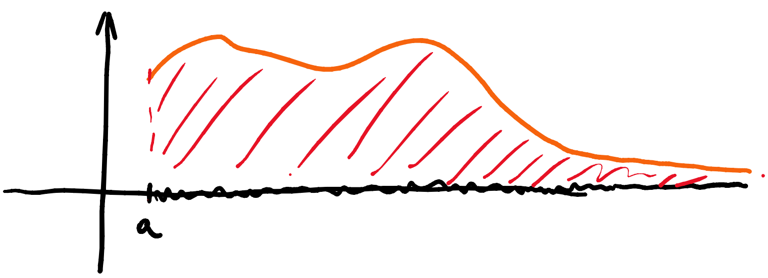


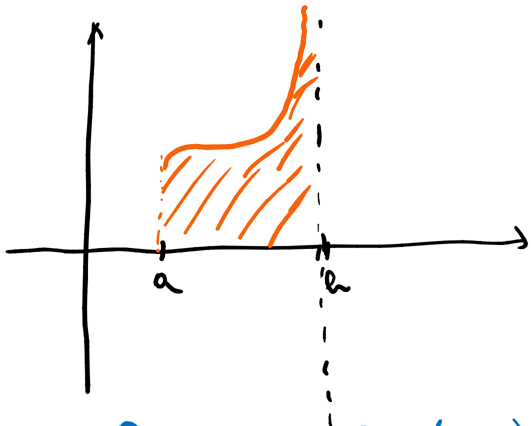
$x^2 + y^2 = 1$

$y = \pm\sqrt{1-x^2}$

$\int_{-1}^{1} \sqrt{1-x^2}\, dx$ è l'area del semicerchio di raggio 1.

Come si definiscono gli integrali se il dominio è un intervallo non limitato o se la funzione non è limitata?





DEF: Sia $f : [a, b) \longrightarrow \mathbb{R}$ con $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$ $b > a$. Si dice che f è INTEGRABILE IN SENSO GENERALIZZATO in $[a, b)$ se:

1) f è integrabile in $[a, c]$ $\forall\, c \in (a, b)$.

2) $\exists \lim_{c \to b^-} \int_a^c f(x)\, dx$. Tale limite si indica con il simbolo $\int_a^b f(x)\, dx$.

ESEMPI

$$\int_0^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{c\to+\infty} \int_0^c \frac{1}{1+x^2}\,dx = \lim_{c\to+\infty} \arctan x \Big|_0^c$$

$$= \lim_{c\to+\infty} \arctan c - 0$$

$$= \frac{\pi}{2}$$

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{c\to+\infty} \int_1^c \frac{1}{x}\,dx = \lim_{c\to+\infty} \log c - \underbrace{\log 1}_{=0} = +\infty .$$

$$\int_0^{+\infty} \sin x\,dx = \lim_{c\to+\infty} \int_0^c \sin x\,dx = \lim_{x\to+\infty} -\cos c + 1 \quad \nexists$$
