MATEMATICA - LEZIONE 28 - 23/11/23 14.30 - 16.30

Esercizi sui limiti (formule di Taylor)

1) $\lim_{x\to 0} \frac{\sin(x+2x^2) - \cos x - x + 1}{\ln(1+x) - \sin x}$ $\left[\frac{0}{0}\right]$

denominatore

$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$ $x\to 0$

$\sin x = x - \frac{x^3}{6} + o(x^4)$ $x\to 0$

$\ln(1+x) - \sin x = \left(\cancel{x} - \frac{x^2}{2} + o(x^2)\right) - \left(\cancel{x} - \frac{x^3}{6} + o(x^4)\right)$

$= -\frac{x^2}{2} + o(x^2) + \frac{x^3}{6} + o(x^4) = -\frac{x^2}{2} + o(x^2) + o(x^2) + o(x^4)$

$= -\frac{x^2}{2} + o(x^2) + o(x^4) = -\frac{x^2}{2} + o(x^2)$ $x\to 0$

$\sin x = x + o(x^2)$ $x\to 0$

$\sin(x+2x^2) = x + 2x^2 + o\left((x+2x^2)^2\right)$

$= x + 2x^2 + o\left(x^2(1+4x+4x^2)\right)$

$= x + 2x^2 + o(x^2 + 4x^3 + 4x^4) = x + 2x^2 + o(x^2 + o(x^2))$

$= x + 2x^2 + o(x^2)$

$\cos x = 1 - \frac{x^2}{2} + o(x^3)$ $x\to 0$

$\sin(x+2x^2) - \cos x - x + 1 = \cancel{x} + 2x^2 + o(x^2) - \left(\cancel{1} - \frac{x^2}{2} + o(x^3)\right) - \cancel{x} + \cancel{1}$

$= \left(2 + \frac{1}{2}\right)x^2 + o(x^2) + o(x^3)$

$= \frac{5}{2}x^2 + o(x^2)$

$\lim_{x\to 0} \frac{\sin(x+2x^2) - \cos x - x + 1}{\ln(1+x) - \sin x} = \lim_{x\to 0} \frac{\frac{5}{2}x^2 + o(x^2)}{-\frac{x^2}{2} + o(x^2)} = \lim_{x\to 0} \frac{\frac{5}{2} + \frac{o(x^2)}{x^2}}{-\frac{1}{2} + \frac{o(x^2)}{x^2}} = \frac{\frac{5}{2}}{-\frac{1}{2}} = -5$

2) $\lim_{x\to 0^+} \frac{x\cos(2x) + e^{-x} - 1}{(\ln(1+\sqrt{x}) - \sqrt{x})^2}$ $\left[\frac{0}{0}\right]$

numeratore

$\cos x = 1 - \frac{x^2}{2} + o(x^3)$ per $x\to 0$

$\cos(2x) = 1 - \frac{(2x)^2}{2} + o((2x)^3) = 1 - 2x^2 + o(8x^3) = 1 - 2x^2 + o(x^3)$ per $x\to 0$

$x\cos(2x) = x\,(1 - 2x^2 + o(x^3)) = x - 2x^3 + x\cdot o(x^3) = x - 2x^3 + o(x^4)$ per $x\to 0$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$ per $x\to 0$

$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3)$ per $x\to 0$

$x\cos(2x) + e^{-x} - 1 = \cancel{x} - 2x^3 + o(x^4) + \cancel{1} - \cancel{x} + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3) - \cancel{1}$

$= \frac{x^2}{2} + \left(-2 - \frac{1}{6}\right)x^3 + o(x^4) + o(x^3) = \frac{x^2}{2} - \frac{13}{6}x^3 + o(x^3)$

denominatore

$\ln(1+x) = x - \frac{x^2}{2} + o(x^2) \quad x \to 0$

$\ln(1+\sqrt{x}) = \sqrt{x} - \frac{x}{2} + o(x) \quad x \to 0^+$

$(\ln(1+\sqrt{x}) - \sqrt{x})^2 = (-\frac{x}{2} + o(x))^2 = \frac{x^2}{4} + 2(-\frac{x}{2}) \cdot o(x) + (o(x))^2$

$= \frac{x^2}{4} + o(x^2) + o(x^2) = \frac{x^2}{4} + o(x^2)$

$$\lim_{x \to 0^+} \frac{x\cos(2x) + e^{-x} - 1}{(\ln(1+\sqrt{x}) - \sqrt{x})^2} = \lim_{x \to 0^+} \frac{\frac{x^2}{2} - \frac{13}{6}x^3 + o(x^3)}{\frac{x^2}{4} + o(x^2)} = \lim_{x \to 0^+} \frac{\frac{1}{2} - \frac{13}{6}x + \frac{o(x^3)}{x^2}}{\frac{1}{4} + \frac{o(x^2)}{x^2}} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$$

3) $$\lim_{x \to 0} \frac{x^2(\sqrt{1+2x} - e^x)}{\sin^2 x - x^2}$$

numeratore

$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2) \quad$ per $x \to 0$

$\sqrt{1+2x} = 1 + x - \frac{x^2}{2} + o(x^2) \quad$ per $x \to 0$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) \quad x \to 0$

$\sqrt{1+2x} - e^x = \cancel{1} + \cancel{x} - \frac{x^2}{2} + o(x^2) - (\cancel{1} + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3))$

$= -x^2 + o(x^2) - \frac{x^3}{6} + o(x^3) = -x^2 + o(x^2) \quad x \to 0$

$x^2(\sqrt{1+2x} - e^x) = x^2(-x^2 + o(x^2)) = -x^4 + o(x^4) \quad x \to 0$

denominatore

$\sin x = x - \frac{x^3}{6} + o(x^4) \quad x \to 0$

$\sin^2 x = (x - \frac{x^3}{6} + o(x^4))^2 = x^2 + \frac{x^6}{36} + (o(x^4))^2 + 2x(-\frac{x^3}{6}) + 2x \cdot o(x^4) + 2(-\frac{x^3}{6}) \cdot o(x^4)$

$(A+B+C)^2 = A^2 + B^2 + C^2 + 2AB + 2AC + 2BC$

$= x^2 + \frac{x^6}{36} + o(x^8) - \frac{x^4}{3} + o(x^5) + o(x^7)$

$= x^2 - \frac{x^4}{3} + \frac{x^6}{36} + o(x^5) = x^2 - \frac{x^4}{3} + o(x^5) \quad x \to 0$

$\sin^2 x - x^2 = -\frac{x^4}{3} + o(x^5) \quad x \to 0$

$$\lim_{x \to 0} \frac{x^2(\sqrt{1+2x} - e^x)}{\sin^2 x - x^2} = \lim_{x \to 0} \frac{-x^4 + o(x^4)}{-\frac{x^4}{3} + o(x^5)} = \frac{-1}{-\frac{1}{3}} = 3$$

4) $$\lim_{x \to 0} \frac{\cos^2 x - \sqrt{1-2x^2}}{x^3 - e^x \ln(1+x^3)} \quad \left[\frac{0}{0}\right]$$

numeratore

$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2) \quad x \to 0$

$\sqrt{1-2x^2} = 1 - x^2 - \frac{x^4}{2} + o(x^4) \quad x \to 0$

$\cos x = 1 - \frac{x^2}{2} + o(x^3) \quad x \to 0$

$\cos^2 x = (1 - \frac{x^2}{2} + o(x^3))^2 = 1 + \frac{x^4}{4} + o(x^6) - x^2 + o(x^3) + o(x^5)$

$= 1 - x^2 + \frac{x^4}{4} + o(x^3)$

$$\cos^2 x - \sqrt{1-2x^2} = \cancel{1} - \cancel{x^2} + \frac{x^4}{4} + o(x^3) - \left(\cancel{1} - \cancel{x^2} - \frac{x^4}{2} + o(x^4)\right)$$
$$= \frac{3}{4}x^4 + o(x^3) + o(x^4) = o(x^3) \quad \text{non va bene}$$

aggiungiamo un altro termine nello sviluppo del coseno

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) \qquad x \to 0$$
$$\cos^2 x = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)\right)^2$$
$$= 1 + \frac{x^4}{4} + \frac{x^8}{(24)^2} + o(x^{10}) - x^2 + \frac{x^4}{12} + o(x^5) - \frac{x^6}{24} + o(x^7) + o(x^9)$$
$$(A+B+C+D)^2 = A^2+B^2+C^2+D^2+2AB+2AC+2AD+2BC+2BD+2CD$$
$$= 1 - x^2 + \frac{x^4}{3} - \frac{x^6}{24} + \frac{x^8}{(24)^2} + o(x^5) = 1 - x^2 + \frac{x^4}{3} + o(x^5)$$

$$\cos^2 x - \sqrt{1-2x^2} = \cancel{1} - \cancel{x^2} + \frac{x^4}{3} + o(x^5) - \left(\cancel{1} - \cancel{x^2} - \frac{x^4}{2} + o(x^4)\right)$$
$$= \frac{5}{6}x^4 + o(x^4)$$

denominatore

$$e^x = 1 + x + o(x) \quad \text{per } x \to 0$$
$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2) \quad \text{per } x \to 0$$
$$\ln(1+x^3) = x^3 - \frac{x^6}{2} + o(x^6) \quad \text{per } x \to 0$$
$$e^x \ln(1+x^3) = (1+x+o(x))\left(x^3 - \frac{x^6}{2} + o(x^6)\right)$$
$$= x^3 - \frac{x^6}{2} + o(x^6) + x^4 - \frac{x^7}{2} + o(x^7) + o(x^4) + o(x^7) + o(x^7)$$
$$= x^3 + x^4 - \frac{x^6}{2} - \frac{x^7}{2} + o(x^4) = x^3 + x^4 + o(x^4) \quad \text{per } x \to 0$$

$$x^3 - e^x \ln(1+x^3) = \cancel{x^3} - \left(\cancel{x^3} + x^4 + o(x^4)\right) = -x^4 + o(x^4) \quad \text{per } x \to 0$$

$$\lim_{x \to 0} \frac{\cos^2 x - \sqrt{1-2x^2}}{x^3 - e^x \ln(1+x^3)} = \lim_{x \to 0} \frac{\frac{5}{6}x^4 + o(x^4)}{-x^4 + o(x^4)} = -\frac{5}{6}$$

5) $$\lim_{x \to 0} \frac{x \ln(1+2x) - \sin(2x^2)}{(1-e^x)^2 - x^2}$$

numeratore

$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2) \qquad x \to 0$$
$$\ln(1+2x) = 2x - 2x^2 + o(x^2) \qquad x \to 0$$
$$x \ln(1+2x) = 2x^2 - 2x^3 + o(x^3) \qquad x \to 0$$

$$\sin x = x - \frac{x^3}{6} + o(x^4) \quad \text{per } x \to 0$$
$$\sin(2x^2) = 2x^2 - \frac{4}{3}x^6 + o(x^8) \quad \text{per } x \to 0$$

$$x \ln(1+2x) - \sin(2x^2) = \cancel{2x^2} - 2x^3 + o(x^3) - \left(\cancel{2x^2} - \frac{4}{3}x^6 + o(x^8)\right)$$
$$= -2x^3 + \frac{4}{3}x^6 + o(x^3) + o(x^8) = -2x^3 + o(x^3) \qquad x \to 0$$

denominatore

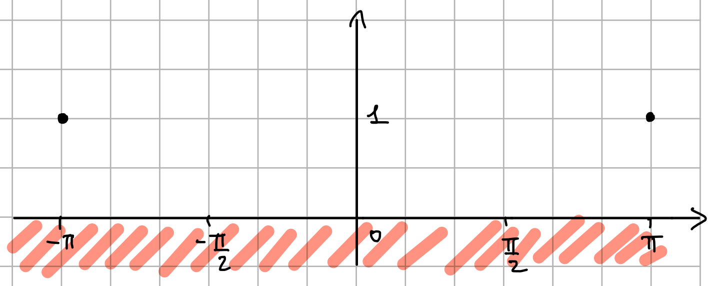
$$e^x = 1 + x + \frac{x^2}{2} + o(x^2) \qquad x \to 0$$
$$\Rightarrow 1 - e^x = -x - \frac{x^2}{2} + o(x^2) \qquad x \to 0$$
$$\Rightarrow (1-e^x)^2 = \left(-x - \frac{x^2}{2} + o(x^2)\right)^2 = \left(x + \frac{x^2}{2} + o(x^2)\right)^2 = x^2 + \frac{x^4}{4} + o(x^4) + x^3 + o(x^3) + o(x^4)$$
$$= x^2 + x^3 + \frac{x^4}{4} + o(x^3) = x^2 + x^3 + o(x^3) \quad \text{per } x \to 0$$

$(1-e^x)^2 - x^2 = x^3 + o(x^3) \qquad x \to 0$

$$\lim_{x\to 0} \frac{x \ln(1+2x) - \sin(2x^2)}{(1-e^x)^2 - x^2} = \lim_{x\to 0} \frac{-2x^3 + o(x^3)}{x^3 + o(x^3)} = -2$$

Studio di funzione

$f(x) = e^{\sin x} \qquad \text{dom} f = \mathbb{R}$



f è 2π – periodica

studiamo la funzione in $[-\pi, \pi]$

f non ha zeri

$\forall x \in \mathbb{R} : f(x) > 0$

$$\lim_{x\to -\pi} f(x) = f(-\pi) = e^{\sin(-\pi)} = 1$$

$$\lim_{x\to \pi} f(x) = f(\pi) = e^{\sin(\pi)} = 1$$

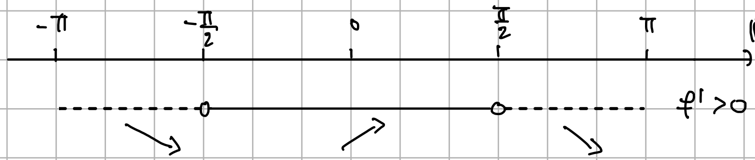
$f'(x) = e^{\sin x} \cdot \cos x$

punti stazionari

$f'(x) = 0 \iff e^{\sin x} \cdot \cos x = 0 \iff \cos x = 0 \iff x = \pm \frac{\pi}{2}$

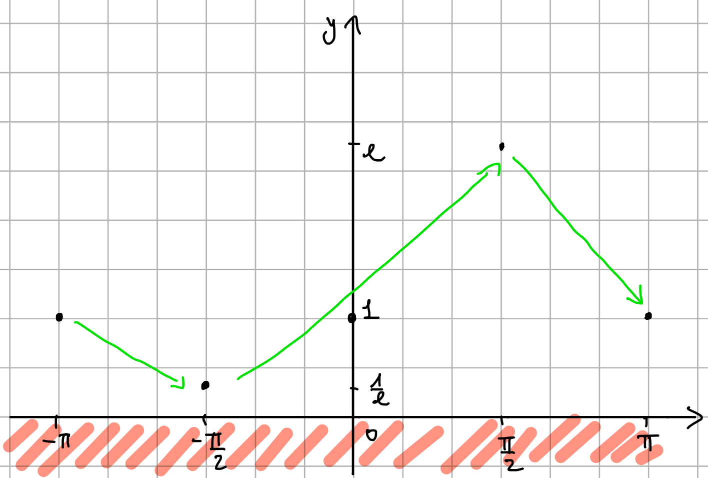
studio del segno di f'

$f'(x) > 0 \iff e^{\sin x} \cos x > 0 \iff \cos x > 0 \iff -\frac{\pi}{2} < x < \frac{\pi}{2}$



$x = -\frac{\pi}{2}$ è punto di minimo relativo $\qquad f(-\frac{\pi}{2}) = e^{\sin(-\frac{\pi}{2})} = \frac{1}{e} \simeq 0.37$

$x = \frac{\pi}{2}$ è punto di massimo relativo $\qquad f(\frac{\pi}{2}) = e^{\sin\frac{\pi}{2}} = e \simeq 2.72$



NB: Gli intervalli di monotonia sono uguali a quelli della funzione seno (poiché la funzione esponenziale è strettamente crescente)

$$f'(x) = e^{\sin x} \cdot \cos x$$

$$\begin{aligned} f''(x) &= e^{\sin x} \cos^2 x + e^{\sin x}(-\sin x) \\ &= e^{\sin x}(\cos^2 x - \sin x) \\ &= e^{\sin x}(1 - \sin^2 x - \sin x) = -e^{\sin x}(\sin^2 x + \sin x - 1) \end{aligned}$$

punti di flesso

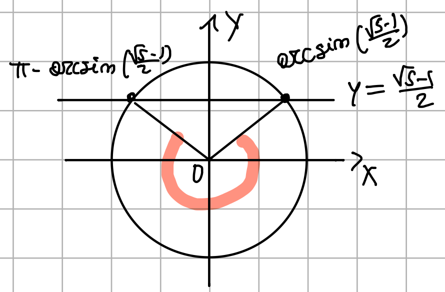
$$f''(x) = 0 \iff \sin^2 x + \sin x - 1 = 0 \iff \sin x = \frac{-1 \pm \sqrt{5}}{2}$$

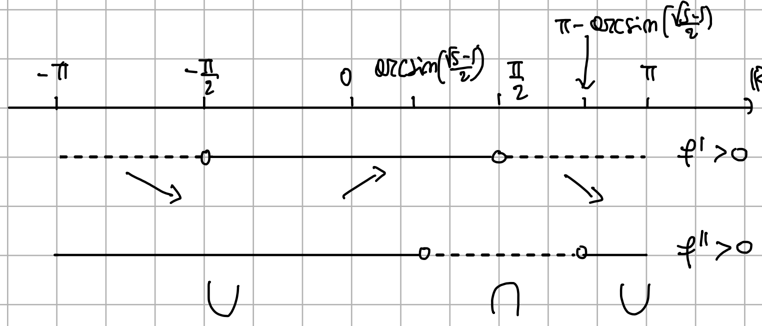
$$\iff \sin x = \frac{\sqrt{5}-1}{2} \iff x = \arcsin\left(\frac{\sqrt{5}-1}{2}\right) \vee x = \pi - \arcsin\left(\frac{\sqrt{5}-1}{2}\right)$$

$$f''(x) > 0 \iff \sin^2 x + \sin x - 1 < 0$$

$$\iff \frac{-1-\sqrt{5}}{2} < \sin x < \frac{-1+\sqrt{5}}{2} \iff \sin x < \frac{\sqrt{5}-1}{2}$$

$$\iff -\pi \le x < \arcsin\left(\frac{\sqrt{5}-1}{2}\right) \vee \pi - \arcsin\left(\frac{\sqrt{5}-1}{2}\right) < x \le \pi$$





$x = \arcsin\left(\frac{\sqrt{5}-1}{2}\right)$ punto di flesso (discendente)

$x = \pi - \arcsin\left(\frac{\sqrt{5}-1}{2}\right)$ punto di flesso (ascendente)

$$f\left(\arcsin\left(\frac{\sqrt{5}-1}{2}\right)\right) = e^{\frac{\sqrt{5}-1}{2}} \simeq 1.86$$

$$f\left(\pi - \arcsin\left(\frac{\sqrt{5}-1}{2}\right)\right) = e^{\frac{\sqrt{5}-1}{2}}$$