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Ricevimento su appuntamento da concordare tramite email

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Queste note e tutto il materiale prodotto durante le lezioni saranno disponibili sul canale TEAMS

LT - Matematica - 22-23

Disequazioni di 1° grado

Proprietà fondamentali (compatibilità di $\leq$ rispetto a somma e prodotto)

- $\forall a,b,c \in \mathbb{R} \quad : \quad a \leq b \iff a+c \leq b+c$
- $\forall a,b,c \in \mathbb{R} \quad : \quad a \leq b \ \wedge \ c>0 \implies ac \leq bc$
- $\forall a,b,c \in \mathbb{R} \quad : \quad a \leq b \ \wedge \ c<0 \implies ac \geq bc$

Es 1

$$-3x \leq -6 \implies \left(-\frac{1}{3}\right)\cdot(-3x) \geq \left(-\frac{1}{3}\right)\cdot(-6)$$
$$\implies x \geq 2$$

Es 2

$$-x-1>0 \implies -1>x \qquad S=(-\infty,-1)$$

Es 3

$$\frac{3}{2}x+\frac{7}{3}-x+\frac{5}{6} \geq -\frac{5}{8}x+\frac{11}{3} \implies \left(\frac{3}{2}-1+\frac{5}{8}\right)x \geq \frac{11}{3}-\frac{7}{3}-\frac{5}{6}$$

$$\implies \frac{12-8+5}{8}x \geq \frac{22-14-5}{6}$$

$$\implies \frac{9}{8}x \geq \frac{1}{2} \implies x \geq \frac{8}{9}\cdot\frac{1}{2}=\frac{4}{9} \qquad S=\left[\frac{4}{9},+\infty\right)$$

Disequazioni di 2° grado

Es 4 $x^2+3x+2<0$

$\Delta = 9-8=1$

le radici dell'equazione associata sono $x_{1/2} = \dfrac{-3\pm 1}{2} \begin{cases} -2 \\ -1 \end{cases}$

$\Leftrightarrow \quad -2<x<-1$

$S = (-2,-1)$

$]-2,-1[$

Es 5 $-x^2+7x-10\geq 0 \quad \Rightarrow \quad x^2-7x+10\leq 0$

$\Delta = 49-40=9>0$ le radici dell'equazione associata sono $x_{1/2} = \dfrac{7\pm 3}{2} \begin{cases} 2 \\ 5 \end{cases}$

$S = [2,5]$

Es 6 $x^2-9>0$ le radici dell'eq. associata sono $x=\pm 3$

$\Leftrightarrow \quad x<-3 \;\vee\; x>3$

$S = (-\infty,-3)\cup(3,+\infty)$

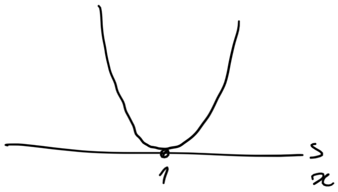
Es 7 $x^2-5\leq 0 \quad \Leftrightarrow \quad -\sqrt{5}\leq x\leq \sqrt{5}$

Es 8 $x^2+1\geq 0 \quad \forall x\in\mathbb{R}$

Es 9 $\quad x^2-2x+1>0 \quad (\geq 0,\ <0,\ \leq 0)$

$$x^2-2x+1 = (x-1)^2$$

$\iff (x-1)^2>0 \iff x\neq 1$



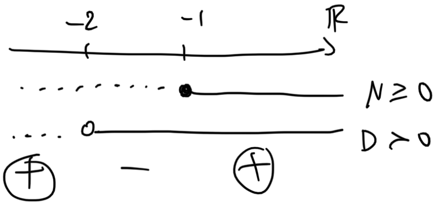
- $x^2-2x+1 \geq 0 \quad \forall x\in\mathbb{R}$
- $x^2-2x+1 < 0 \quad \nexists\, x\in\mathbb{R}$
- $x^2-2x+1 \leq 0 \iff x=1$

Disequazioni razionali (studio del segno)

Es 10 $\quad \dfrac{x^3+x^2}{x+2}\geq 0$

$N\geq 0$: $\quad x^3+x^2\geq 0 \iff x^2(x+1)\geq 0 \iff x+1\geq 0 \iff x\geq -1$

$D>0$: $\quad x+2>0 \iff x>-2$



$\dfrac{x^3+x^2}{x+2}\geq 0 \iff x<-2 \ \vee\ x\geq -1$

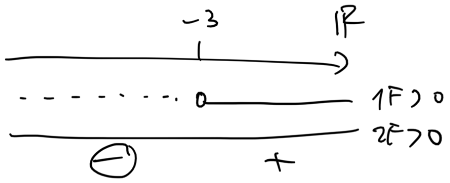
$S=(-\infty,-2)\cup[-1,+\infty)$

Es 11 $(x+3)(x^2+x+5) < 0$

$1F > 0 \;:\; x+3 > 0 \iff x > -3$

$2F > 0 \;:\; x^2+x+5 > 0 \iff \forall x \in \mathbb{R}$

$\Delta = 1 - 4\cdot 5 < 0$



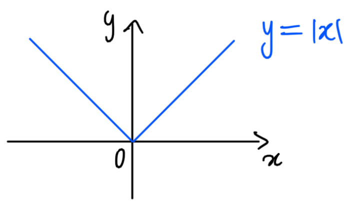
$S = (-\infty, -3)$

Equazioni e disequazioni con valore assoluto

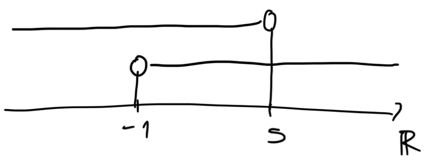
Ricordiamo che

- $|f(x)| < K \iff -K < f(x) < K$
- $|f(x)| \leq K \iff -K \leq f(x) \leq K$
- $|f(x)| > K \iff f(x) < -K \lor f(x) > K$
- $|f(x)| \geq K \iff f(x) \leq -K \lor f(x) \geq K$
- $|f(x)| = K \iff f(x) = K \lor f(x) = -K$ quando $K \geq 0$

Grafico di $y = |x|$



Es 12 $|x-2| < 3 \iff -3 < x-2 < 3 \iff \begin{cases} x-2 > -3 \\ x-2 < 3 \end{cases} \iff \begin{cases} x > -1 \\ x < 5 \end{cases}$

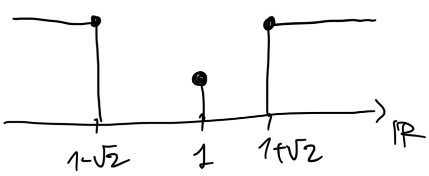


$\iff -1 < x < 5$

Es 13 $|x^2-2x| \geq 1 \iff x^2-2x \leq -1 \lor x^2-2x \geq 1$

$\iff x^2-2x+1 \leq 0 \lor x^2-2x-1 \geq 0$

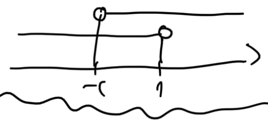
$\iff x = 1$ (vedi Es 9) $\lor\; x \leq 1-\sqrt{2} \lor x \geq 1+\sqrt{2}$



$S = (-\infty, 1-\sqrt{2}] \cup \{1\} \cup [1+\sqrt{2}, +\infty)$

Es 14 $\quad |x+1| > -2 \quad \forall x \in \mathbb{R}$

$|x+1| > -2 \iff x+1 < 2 \ \vee \ x+1 > -2$
$\iff x < 1 \ \vee \ x > -1$



Es 15 $\quad |x^3 - x + 2| \leq -5 \quad \nexists x \in \mathbb{R}$

Es 16 $\quad |x^2 - x| < x+2 \iff \begin{cases} x^2 - x \geq 0 \\ x^2 - x < x+2 \end{cases} \vee \begin{cases} x^2 - x < 0 \\ -x^2 + x < x+2 \end{cases}$

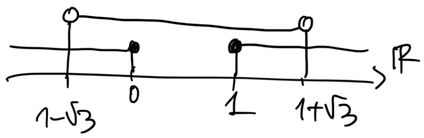
$$\iff \begin{cases} x \leq 0 \ \vee \ x \geq 1 \\ x^2 - 2x - 2 < 0 \end{cases} \vee \begin{cases} 0 < x < 1 \\ x^2 + 2 > 0 \quad \forall x \in \mathbb{R} \end{cases}$$

radici dell'eq. ass.

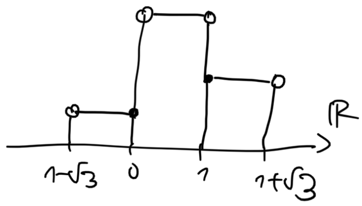
$\Delta = 12 = 3 \cdot 4$

$$x_{1/2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

$$\iff \begin{cases} x \leq 0 \ \vee \ x \geq 1 \\ 1-\sqrt{3} < x < 1+\sqrt{3} \end{cases} \vee \ 0 < x < 1 \qquad (A \cap \mathbb{R} = A)$$



$\iff 1-\sqrt{3} < x \leq 0 \ \vee \ 1 \leq x < 1+\sqrt{3} \ \vee \ 0 < x < 1$



$\iff 1-\sqrt{3} < x < 1+\sqrt{3}$

risoluzione alternativa

$|x^2 - x| < x+2 \iff -(x+2) < x^2 - x < x+2$

$\iff \begin{cases} x^2 - x > -x - 2 \\ x^2 - x < x+2 \end{cases} \iff \begin{cases} x^2 > -2 \quad \forall x \in \mathbb{R} \\ x^2 - 2x - 2 < 0 \end{cases}$

$\iff x^2 - 2x - 2 < 0 \iff 1-\sqrt{3} < x < 1+\sqrt{3}$

Equazioni e disequazioni irrazionali

Ricordiamo che

- Se $n \geq 2$ è un numero pari allora

$$\sqrt[n]{f(x)} = g(x) \iff \begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) = (g(x))^n \end{cases}$$

$$\underset{(\leq)}{\sqrt[n]{f(x)} < g(x)} \iff \begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) \underset{(\leq)}{<} (g(x))^n \end{cases}$$

$$\underset{(\geq)}{\sqrt[n]{f(x)} > g(x)} \iff \begin{cases} f(x) \geq 0 \\ g(x) < 0 \end{cases} \vee \begin{cases} \cancel{f(x) \geq 0} \\ g(x) \geq 0 \\ f(x) \underset{(\geq)}{>} (g(x))^n \end{cases}$$

- Se $n \geq 3$ è un numero dispari allora

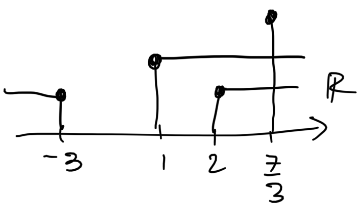
$$\sqrt[n]{f(x)} = g(x) \iff f(x) = (g(x))^n$$

$$\sqrt[n]{f(x)} \underset{(\leq)}{<} g(x) \iff f(x) \underset{(\leq)}{<} (g(x))^n$$

$$\sqrt[n]{f(x)} \underset{(\geq)}{>} g(x) \iff f(x) \underset{(\geq)}{>} (g(x))^n$$

Es 17 $\sqrt{x^2+x-6} = x-1 \iff \begin{cases} x^2+x-6 \geq 0 \to (x+3)(x-2) \\ x-1 \geq 0 \\ x^2+x-6 = (x-1)^2 = x^2-2x+1 \end{cases}$

$$\iff \begin{cases} x \leq -3 \vee x \geq 2 \\ x \geq 1 \\ x = \frac{7}{3} \end{cases}$$



$x = \frac{7}{3}$

Es 18 $\sqrt[3]{x+1} = 2 \iff x+1 = 2^3 \iff x = 7$

Es 19 $\sqrt{x^2-3x+2} > 2x-3 \iff \begin{cases} x^2-3x+2 \ge 0 \\ 2x-3<0 \end{cases} \vee \begin{cases} 2x-3 \ge 0 \\ x^2-3x+2 > (2x-3)^2 \end{cases}$

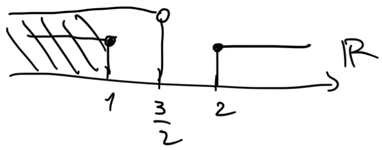
$(A+B)^2 = A^2+2AB+B^2$

$\iff \begin{cases} x \le 1 \ \vee \ x \ge 2 \\ x < \frac{3}{2} \end{cases} \vee \begin{cases} x \ge \frac{3}{2} \\ x^2-3x+2 > 4x^2-12x+9 \end{cases}$

$\iff \begin{cases} x \le 1 \ \vee \ x \ge 2 \\ x < \frac{3}{2} \end{cases} \vee \begin{cases} x \ge \frac{3}{2} \\ 3x^2-9x+7 < 0 \end{cases}$ $\nexists x \in \mathbb{R}$

$\Delta = 81 - 4 \cdot 3 \cdot 7 < 0$

$\iff \begin{cases} x \le 1 \ \vee \ x \ge 2 \\ x < \frac{3}{2} \end{cases}$

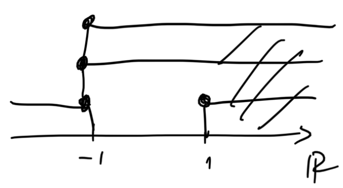


$\iff x \le 1$

$S = (-\infty, 1]$

Es 20 $\sqrt{2x+5} \geqslant x+2$

Es 21 $\sqrt{x^2-1} \le x+1 \iff \begin{cases} x^2-1 \ge 0 \\ x+1 \ge 0 \\ x^2-1 \le x^2+2x+1 \end{cases} \iff \begin{cases} x \le -1 \ \vee \ x \ge 1 \\ x \ge -1 \\ x \ge -1 \end{cases} \iff x \ge 1$



Es 22 $\sqrt[3]{x^3+8} \geq x-2 \iff x^3+8 \geq (x-2)^3 = x^3-6x^2+12x-8$

$(A+B)^3 = A^3+3A^2B+3AB^2+B^3$

$\iff 6x^2-12x+16>0$

$\iff 3x^2-6x+8>0 \quad \forall x \in \mathbb{R}$

$\Delta = 36-4\cdot3\cdot8<0$

$S=\mathbb{R}=(-\infty,+\infty)$

Equazioni e disequazioni esponenziali

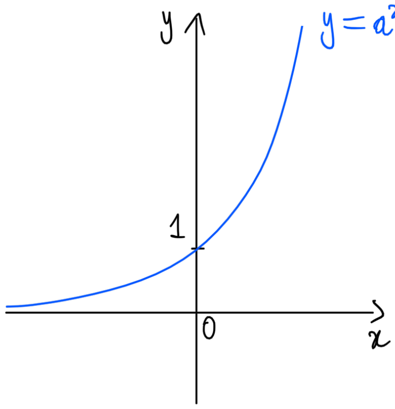
Proprietà delle potenze

- $\forall x,y \in \mathbb{R}, \forall a>0$: $a^x \cdot a^y = a^{x+y}$
- $\forall x,y \in \mathbb{R}, \forall a>0$: $\frac{a^x}{a^y} = a^{x-y}$
- $\forall x,y \in \mathbb{R}, \forall a>0$: $(a^x)^y = a^{xy}$
- $\forall x \in \mathbb{R}, \forall a,b>0$: $a^x \cdot b^x = (ab)^x$

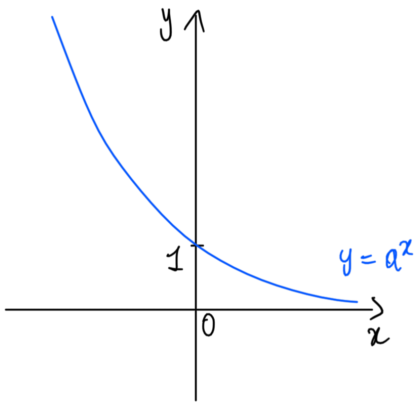
Ricordiamo inoltre che

- $\forall x \in \mathbb{R}, \forall a>0$: $a^{-x} = \frac{1}{a^x}$
- $\forall a>0$: $a^0 = 1$
- $\forall m \in \mathbb{N} \setminus \{0,1\}, \forall a>0$: $a^{\frac{1}{m}} = \sqrt[m]{a}$
- $\forall x \in \mathbb{R}, \forall a>0$: $a^x > 0$

Grafici delle funzioni esponenziali



CASO $a>1$



CASO $a \in (0,1)$

In particolare, dobbiamo considerare diversamente le disequazioni esponenziali a seconda che la base sia >1 oppure in $(0,1)$

Siano $a \in (0,1) \cup (1,+\infty)$ e $K>0$

- $a^{f(x)} = K \iff f(x) = \log_a K$
- Se $a>1$ allora : $a^{f(x)} \underset{(\leq)}{<} K \iff f(x) \underset{(\leq)}{<} \log_a K$

  $a^{f(x)} \underset{(\geq)}{>} K \iff f(x) \underset{(\geq)}{>} \log_a K$

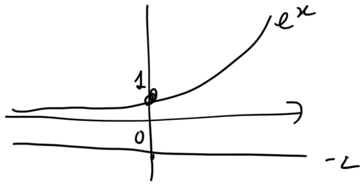
"il verso della disuguaglianza viene conservato"

• Se $a \in (0,1)$ allora: $a^{f(x)} < k$ $(\leq)$ $\iff$ $f(x) > \log_a k$ $(\geq)$

$a^{f(x)} > k$ $(\geq)$ $\iff$ $f(x) < \log_a k$ $(\leq)$

"il verso della disuguaglianza viene invertito"

Es 23 $\quad e^x \geq -2 \quad \forall x \in \mathbb{R}$



Es 24 $\quad 3^x = \dfrac{1}{\sqrt[5]{3}} = \dfrac{1}{3^{\frac{1}{5}}} = 3^{-\frac{1}{5}} \implies x = -\dfrac{1}{5}$

Es 25 $\quad \left(\dfrac{1}{2}\right)^x \geq \dfrac{8 \cdot \sqrt{2}}{\sqrt[3]{4}} = \dfrac{2^3 \cdot 2^{\frac{1}{2}}}{2^{\frac{2}{3}}} = 2^{3+\frac{1}{2}-\frac{2}{3}} = 2^{\frac{18+3-4}{6}} = 2^{\frac{17}{6}}$

$= \left(\dfrac{1}{2}\right)^{-\frac{17}{6}}$

$\iff \left(\dfrac{1}{2}\right)^x \geq \left(\dfrac{1}{2}\right)^{-\frac{17}{6}} \implies x \leq -\dfrac{17}{6}$

risoluzione alternativa

$\left(\dfrac{1}{2}\right)^x \geq 2^{\frac{17}{6}} \iff 2^{-x} \geq 2^{\frac{17}{6}} \iff -x \geq \dfrac{17}{6} \iff x \leq -\dfrac{17}{6}$

Es 26 $\quad 4^x - 5 \cdot 2^x + 4 > 0 \qquad 4^x = (2^2)^x = 2^{2x} = (2^x)^2$

$(2^x)^2 - 5 \cdot 2^x + 4 > 0 \qquad t = 2^x$

$t^2 - 5t + 4 > 0$

$\Delta = 25 - 16 = 9$ le radici dell'eq. ass. sono $t = \dfrac{5 \pm 3}{2} < \begin{matrix} 1 \\ 4 \end{matrix}$

$(2^x)^2 - 5 \cdot 2^x + 4 > 0 \iff t < 1 \ \vee\ t > 4 \iff 2^x < 1 \ \vee\ 2^x > 4$

$\iff 2^x < 2^0 \ \vee\ 2^x > 2^2 \iff x < 0 \ \vee\ x > 2$