LEZIONE 29

mercoledì 30 novembre 2022 09:34

Integrali:

- $\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx$

  (integrazione per parti)

- $\int f(y)\, dy \overset{y=\varphi(x)}{=} \int f(\varphi(x))\varphi'(x)\, dx$ " $dy = \varphi'(x)\, dx$ " " $\frac{dy}{dx} = \varphi'(x)$ "

  (Integrazione per sostituzione)

ESEMPI

- $\int \frac{x}{\sqrt{1-x^2}}\, dx$ $\quad -\frac{1}{2} dy$

$y = 1 - x^2$
$dy = -2x\, dx$
$x \cdot dx = -\frac{1}{2} dy$

$$= \int \frac{1}{\sqrt{y}} \left(-\frac{1}{2}\right) dy = -\frac{1}{2} \int \frac{1}{\sqrt{y}}\, dy = -\frac{1}{2} \int y^{-\frac{1}{2}}\, dy$$

$$= -\frac{1}{2} \frac{y^{1-\frac{1}{2}}}{1-\frac{1}{2}} + C = -\frac{1}{2} \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + C = -y^{\frac{1}{2}} + C$$

$$= -(1-x^2)^{\frac{1}{2}} + C = -\sqrt{1-x^2} + C.$$

ESEMPIO

$\int \frac{x}{\sqrt{1-x^4}}\, dx$

$y = x^2$
$dy = 2x\, dx$
$x\, dx = \frac{1}{2} dy$

$$= \int \frac{1}{\sqrt{1-y^2}} \frac{1}{2} dy = \frac{1}{2} \arcsin(y) + C$$

$$= \frac{1}{2} \arcsin(x^2) + C.$$

Integrali di funzioni razionali

Funzioni razionali: $\frac{p(x)}{q(x)}$ dove p e q sono polinomi.

Vogliamo calcolare $\int \frac{p(x)}{q(x)}\,dx$.

⌈Ricordiamo che dato un polinomio: $a_0 + a_1 x + \dots + a_n x^n$
se $a_n \neq 0$ si dice che $\deg(a_0 + a_1 x + \dots + a_n x^n) = n$.⌋

• **Caso $\deg(q(x)) = 1$.** (cioè $q(x) = ax + b$)

Ci sono due possibilità:

1) $p(x)$ è costante:

In questo caso:

$$\int \frac{c}{ax+b}\,dx = c\int \frac{1}{ax+b}\,dx = \frac{c}{a}\log|ax+b| + C_2.$$

Si può dimostrare per sostituzione

$$\int \frac{1}{ax+b}\,dx \qquad y = ax+b, \quad dy = a\,dx, \quad dx = \frac{1}{a}\,dy$$

$$= \int \frac{1}{y}\frac{1}{a}\,dy = \frac{1}{a}\int \frac{1}{y}\,dy = \frac{1}{a}\log|y| + c_2$$

$$= \frac{1}{a}\log|ax+b| + c_2$$

2) Se $\deg(p(x)) \geq 1$

Si fa la divisione tra $p(x)$ + $q(x)$ e ci si riconduce al caso 1).

ESEMPI

• $\int \frac{1}{3x-1}\,dx = \frac{1}{3}\log|3x-1| + C.$

- $\displaystyle\int \frac{1}{1-x}\,dx = -\log|1-x| + C$
- $\displaystyle\int \frac{\sqrt{3}}{1+\pi x}\,dx = \sqrt{3}\int \frac{1}{1+\pi x}\,dx = \frac{\sqrt{3}}{\pi}\log|1+\pi x| + C.$
- $\displaystyle\int \frac{x^2}{2x-1}\,dx$

$$
\begin{array}{rrr|l}
x^2 & +\,0\cdot x & +\,0 & 2x-1 \\
\cline{4-4}
x^2 & -\,\frac{x}{2} & & \frac{x}{2}+\frac{1}{4} \\
\hline
// & \frac{x}{2} & +\,0 & \\
 & \frac{x}{2} & -\,\frac{1}{4} & \\
\cline{2-3}
 & // & \frac{1}{4} &
\end{array}
$$

Quindi:

$$x^2 = (2x-1)\left(\frac{x}{2}+\frac{1}{4}\right) + \frac{1}{4}$$

$$\frac{x^2}{2x-1} = \frac{x}{2} + \frac{1}{4} + \frac{\frac{1}{4}}{2x-1}$$

$$\int \frac{x^2}{2x-1}\,dx = \int \frac{x}{2} + \frac{1}{4} + \frac{\frac{1}{4}}{2x-1}\,dx =$$

$$= \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{4}\int \frac{1}{2x-1}\,dx$$

$$= \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{4}\cdot\frac{1}{2}\log|2x-1| + C$$

$$= \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{8}\log|2x-1| + C.$$

ESEMPIO

$$\int \frac{4x^3-2x+1}{2-3x}\,dx = -\int \frac{4x^3-2x+1}{3x-2}\,dx$$

Divisione tra polinomi:

$$
\begin{array}{l|l}
4x^3 + 0\cdot x^2 - 2x + 1 & 3x - 2 \\
4x^3 - \frac{8}{3}x^2 & \hline \frac{4}{3}x^2 + \frac{8}{9}x - \frac{2}{27} \\
\hline
// \quad \frac{8}{3}x^2 - 2x + 1 & \\
\frac{8}{3}x^2 - \frac{16}{9}x & \\
\hline
// \quad -\frac{2}{9}x + 1 & \\
-\frac{2}{9}x + \frac{4}{27} & \\
\hline
// \quad \frac{23}{27} &
\end{array}
$$

$$4x^3 - 2x + 1 = (3x-2)\left(\frac{4}{3}x^2 + \frac{8}{9}x - \frac{2}{27}\right) + \frac{23}{27}$$

$$\frac{4x^3 - 2x + 1}{3x-2} = \frac{4}{3}x^2 + \frac{8}{9}x - \frac{2}{27} + \frac{\frac{23}{27}}{3x-2}$$

$$\int \frac{4x^3 - 2x + 1}{3x-2}\,dx = \int \frac{4}{3}x^2 + \frac{8}{9}x - \frac{2}{27} + \frac{\frac{23}{27}}{3x-2}\,dx$$

$$= \frac{4}{3}\cdot\frac{1}{3}x^3 + \frac{8}{9}\cdot\frac{1}{2}x^2 - \frac{2}{27}x + \frac{23}{27}\,\frac{1}{3}\log|3x-2| + C$$

$$= \frac{4}{9}x^3 + \frac{4}{9}x^2 - \frac{2}{27}x + \frac{23}{81}\log|3x-2| + C$$

Quindi:

$$\int \frac{4x^3 - 2x + 1}{2-3x}\,dx = -\frac{4}{9}x^3 - \frac{4}{9}x^2 + \frac{2}{27}x - \frac{23}{81}\log|3x-2| + C$$

- A volte la divisione si può fare anche in maniera più rapida aggiungendo e togliendo termini al numeratore;

ESEMPIO

$$\int \frac{x^2}{x-1}\,dx$$

$$\frac{x^2}{x-1} = \frac{x^2-1+1}{x-1} = \frac{x^2-1}{x-1} + \frac{1}{x-1} = x+1 + \frac{1}{x-1}$$

$$\int \frac{x^2}{x-1}\,dx = \int x+1 + \frac{1}{x-1}\,dx = \frac{1}{2}x^2 + x + \log|x-1| + c$$

ESEMPIO:

$$\int \frac{2x+5}{3x-1}\,dx$$

$$\frac{2x+5}{3x-1} = \frac{2}{3}\cdot\frac{3}{2}\,\frac{2x+5}{3x-1} = \frac{2}{3}\,\frac{3x+\frac{15}{2}}{3x-1}$$

$$= \frac{2}{3}\cdot\frac{3x-1+1+\frac{15}{2}}{3x-1}$$

$$= \frac{2}{3}\left(1+\frac{\frac{17}{2}}{3x-1}\right) = \frac{2}{3} + \frac{17}{3}\,\frac{1}{3x-1}$$

Quindi $\displaystyle\int \frac{2x+5}{3x-1}\,dx = \int \frac{2}{3} + \frac{17}{3}\,\frac{1}{3x-1}\,dx$

$$= \frac{2}{3}x + \frac{17}{3}\,\frac{1}{3}\log|3x-1| + c$$

$$= \frac{2}{3}x + \frac{17}{9}\log|3x-1| + c.$$

Caso $q(x)$ di grado 2 $\quad \left(\int \frac{p(x)}{q(x)}\,dx\right)$

1) Se $\deg(p(x)) \geq 2$ possiamo fare la divisione tra p e q e ricondurci al caso in cui il numeratore ha grado al più 1. (0 o 1).

2) Se $p(x)$ ha grado 0 o 1:
Si cerca di scomporre $q(x) = ax^2 + bx + c$.
Ci sono tre possibilità:

i) $\Delta > 0$. In questo caso $q(x)$ ha due radici x_1, x_2 e si può scomporre come $q(x) = a(x-x_1)(x-x_2)$

Si possono cercare $A, B \in \mathbb{R}$ tali che

$$\frac{p(x)}{q(x)} = \frac{A}{x-x_1} + \frac{B}{x-x_2}$$

Allora

$$\int \frac{p(x)}{q(x)}\,dx = A \log|x-x_1| + B \log|x-x_2| + C.$$

ii) $\Delta = 0$. In questo caso $q(x)$ ha una sola radice $x_1 \in \mathbb{R}$ e $q(x) = a(x-x_1)^2$.

In questo caso si cercano $A, B \in \mathbb{R}$ tali che

$$\frac{p(x)}{q(x)} = \frac{A}{x-x_1} + \frac{B}{(x-x_1)^2}$$

$$\int \frac{A}{x-x_1}\,dx = A \log|x-x_1| + C.$$

$$\int \frac{B}{(x-x_1)^2}\,dx = B \int \frac{1}{(x-x_1)^2}\,dx = B\,\frac{(x-x_1)^{-1}}{-1} + C$$

$$= -\frac{B}{x-x_1} + C.$$

iii) $\Delta < 0$. In questo caso:

$$q(x) = a\left(\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right)$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2 + \frac{|\Delta|}{4a^2}\right)$$

$$= \frac{|\Delta|}{4a}\left(\frac{4a^2}{|\Delta|}\left(x+\frac{b}{2a}\right)^2 + 1\right)$$

$$= \frac{|\Delta|}{4a}\left(\left(\frac{2ax+b}{\sqrt{|\Delta|}}\right)^2 + 1\right)$$

Si cercano due costanti A, B tali che

$$\frac{p(x)}{q(x)} = \frac{A\, q'(x)}{q(x)} + \frac{B}{q(x)}$$

e $\displaystyle\int \frac{A\, q'(x)}{q(x)}\, dx = A \int \frac{q'(x)}{q(x)}\, dx = A \log|q(x)| + C$

e $$\int \frac{B}{q(x)}\, dx = B \int \frac{1}{\frac{|\Delta|}{4a}\left(\left(\frac{2ax+b}{\sqrt{\Delta}}\right)^2 + 1\right)}\, dx$$

$$= \frac{4aB}{|\Delta|} \int \frac{1}{\left(\frac{2ax+b}{\sqrt{\Delta}}\right)^2 + 1}\, dx \quad (*)$$

con la sostituzione

$y = \frac{2ax+b}{\sqrt{|\Delta|}}$, $dy = \frac{2a}{\sqrt{|\Delta|}}\, dx$ e

$$(*) = \frac{4aB}{|\Delta|} \int \frac{1}{y^2+1}\, \frac{\sqrt{\Delta}}{2a}\, dx$$

$$= \frac{2aB}{\sqrt{|\Delta|}} \int \frac{1}{y^2+1}\, dy = \frac{2aB}{\sqrt{\Delta}} \arctan y + C$$

$$= \frac{2aB}{\sqrt{|\Delta|}} \arctan\left(\frac{2ax+b}{\sqrt{|\Delta|}}\right) + C.$$

ESEMPIO 1

- $\displaystyle\int \frac{1}{x^2-1}\, dx$

$$x^2-1 = (x-1)(x+1)$$

Cerchiamo A, B tali che:

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$= \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}$$

$$= \frac{Ax + A + Bx - B}{x^2-1}$$
$$= \frac{(A+B)x + A - B}{x^2-1}$$

Vogliamo che: $\begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow \begin{cases} B=-A \\ 2A=1 \end{cases} \Rightarrow \begin{cases} B=-\frac{1}{2} \\ A=\frac{1}{2} \end{cases}$

A questo punto:
$$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1} = \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}$$

Allora $\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$
$$= \frac{1}{2}\log|x-1| - \frac{1}{2}\log|x+1| + c.$$

ESEMPIO 2

$$\int \frac{x-1}{x^2-3x-4}\,dx$$

$q(x) = x^2 - 3x - 4 \qquad \Delta = 9 + 16 = 25 > 0$

$x_{1,2} = \frac{3 \pm 5}{2} = \left\langle \begin{matrix} 4 \\ -1 \end{matrix} \right.$

Allora $x^2 - 3x - 4 = (x-4)(x+1)$

Cerchiamo $A, B \in \mathbb{R}$ tali che
$$\frac{x-1}{x^2-3x-4} = \frac{A}{x-4} + \frac{B}{x+1}$$
$$= \frac{A(x+1) + B(x-4)}{(x-4)(x+1)}$$
$$= \frac{(A+B)x + A - 4B}{x^2-3x-4}$$

$\begin{cases} A+B=1 \\ A-4B=-1 \end{cases} \Rightarrow \begin{cases} 5B-1=1 \\ A=4B-1 \end{cases} \Rightarrow \begin{cases} B=\frac{2}{5} \\ A=\frac{3}{5} \end{cases}$

Quindi:

$$\frac{x-1}{x^2-3x-4} = \frac{\frac{3}{5}}{x-4} + \frac{\frac{2}{5}}{x+1}$$

Allora:

$$\int \frac{x-1}{x^2-3x-4}\,dx = \frac{3}{5}\int \frac{1}{x-4}\,dx + \frac{2}{5}\int \frac{1}{x+1}\,dx$$

$$= \frac{3}{5}\log|x-4| + \frac{2}{5}\log|x+1| + C.$$

Attenzione: Se $q(x) = ax^2+bx+c$ con $a \neq 1$ e $\Delta > 0$.

Allora $q(x) = a(x-x_1)(x-x_2)$

Se $\frac{p(x)}{q(x)} = \frac{A}{x-x_1} + \frac{B}{x-x_2} = \frac{A(x-x_2)+B(x-x_1)}{(x-x_1)(x-x_2)} = \frac{(A+B)x - Ax_2 - Bx_2}{(x-x_1)(x-x_2)}$

$$\frac{p(x)}{a(x-x_1)(x-x_2)} = \frac{(A+B)x - Ax_2 - Bx_1}{(x-x_1)(x-x_2)}$$

I denominatori non sono uguali! Si deve moltiplicare per a

$$\frac{p(x)}{(x-x_1)(x-x_2)} = \frac{a(A+B)x - aAx_2 - Bx_1}{(x-x_1)(x-x_2)}$$

In alternativa si possono cercare A e B tali che

$$\frac{p(x)}{q(x)} = \frac{A}{a(x-x_1)} + \frac{B}{x-x_2}$$

ESEMPIO 3

$$\int \frac{12-5x}{3x^2+x-10}\,dx$$

$q(x) = 3x^2+x-10 \qquad \Delta = 1+120 = 121 > 0$

$$x_{1,2} = \frac{-1 \pm 11}{6} \begin{cases} -2 \\ \frac{10}{6} = \frac{5}{3} \end{cases}$$

$$q(x) = 3(x+2)\left(x-\frac{5}{3}\right) = (x+2)(3x-5)$$

Cerco A e B tali che

$$\frac{12-5x}{3x^2+x-10} = \frac{A}{x+2} + \frac{B}{3x-5}$$

$$= \frac{A(3x-5)+B(x+2)}{(x+2)(3x-5)}$$

$$= \frac{(3A+B)x - 5A + 2B}{3x^2+x-10}$$

$$\begin{cases} 3A+B=-5 \\ -5A+2B=12 \end{cases} \quad \begin{cases} B=-5-3A \\ -5A-10-6A=12 \end{cases} \iff \begin{cases} B=-5-3A \\ -11A=22 \end{cases}$$

$$\iff \begin{cases} B=1 \\ A=-2 \end{cases}$$

Allora

$$\int \frac{12-5x}{3x^2+x-10}\,dx = \int \frac{-2}{x+2} + \frac{1}{3x-5}\,dx$$

$$= -2\log|x+2| + \frac{1}{3}\log|3x-5| + C.$$

ESEMPIO 4

$$\int \frac{x}{x^2+4x+4}\,dx$$

$q(x) = x^2+4x+4 \qquad \Delta = 16-4\cdot 4 = 0.$

$x_1 = -2$

$q(x) = (x+2)^2$

Cerchiamo $A, B \in \mathbb{R}$ tali che:

$$\frac{x}{x^2+4x+4} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2} = \frac{Ax+2A+B}{(x+2)^2}$$

$$\begin{cases} A = 1 \\ 2A + B = 0 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -2A = -2 \end{cases}$$

Quindi:

$$\int \frac{x}{x^2+4x+4}\,dx = \int \frac{1}{x+2} + \frac{-2}{(x+2)^2}\,dx$$

$$= \log|x+2| - 2\int \frac{1}{(x+2)^2}\,dx$$

$$= \log|x+2| + \frac{2}{x+2} + C$$

$$\boxed{\int \frac{1}{(x-x_1)^2}\,dx = -\frac{1}{x-x_1} + C}$$

ESEMPIO 5

$$\int \frac{5x^3-9x^2}{x^2-2x+1}\,dx$$

Dividiamo $5x^3-9x^2$ per x^2-2x+1

$$\begin{array}{cccc|l} 5x^3 & -9x^2 & +0x & +0 & x^2-2x+1 \\ \hline 5x^3 & -10x^2 & +5x & & 5x+1 \\ \hline // & x^2 & -5x & +0 & \\ & x^2 & -2x & +1 & \\ \hline & // & -3x & -1 & \end{array}$$

$$5x^3-9x^2 = (x^2-2x+1)(5x+1) - 3x - 1 = -(3x+1)$$

$$\frac{5x^3-9x^2}{x^2-2x+1} = 5x+1 - \frac{3x+1}{x^2-2x+1}$$

Quindi $$\int \frac{5x^3-9x^2}{x^2-2x+1}\,dx = \frac{5}{2}x^2 + x - \underbrace{\int \frac{3x+1}{x^2-2x+1}\,dx}_{(*)}$$

$$x^2-2x+1 = (x-1)^2$$

$$\frac{3x+1}{x^2-2x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2} = \frac{Ax-A+B}{(x-1)^2}$$

$$\begin{cases} A = 3 \\ -A+B = 1 \end{cases} \Rightarrow \begin{cases} A = 3 \\ B = 1 + A = 4 \end{cases}$$

Quindi:

$$\int \frac{3x+1}{x^2-2x+1}\,dx = \int \frac{3}{x-1} + \frac{4}{(x-1)^2}\,dx$$

$$= 3\log|x-1| - \frac{4}{x-1} + C. \quad (*)$$

$$\int \frac{5x^3-9x^2}{x^2-2x+1}\,dx = \frac{5}{2}x^2 + x - 3\log|x-1| + \frac{4}{x-1} - C$$

Note

Se p è costante e q è un quadrato:

$$\int \frac{c}{a(x-x_0)^2}\,dx = \frac{c}{a}\int \frac{1}{(x-x_0)^2}\,dx = -\frac{c}{a}\frac{1}{(x-x_0)^2} + c_2$$

Non serve cercare A e B:

$$\frac{c}{a(x-x_0)^2} = \frac{A}{(x-x_0)} + \frac{B}{(x-x_0)^2} \Rightarrow \begin{cases} A = 0 \\ B = \frac{c}{a} \end{cases}$$

$$\int \frac{x}{9x^2+6x+1}\,dx:$$

$$q(x) = 9x^2+6x+1 \qquad \Delta = 36 - 4\cdot 9 = 0.$$

$$x_1 = -\frac{1}{3}$$

$$q(x) = 9\left(x+\frac{1}{3}\right)^2$$

$$\int \frac{x}{9x^2+6x+1}\,dx = \frac{1}{9}\int \frac{x}{\left(x+\frac{1}{3}\right)^2}\,dx$$

Cerchiamo A e B tali che $\frac{x}{(x+\frac{1}{3})^2} = \frac{A}{x+\frac{1}{3}} + \frac{B}{(x+\frac{1}{3})^2}$

$$= \frac{A(x+\frac{1}{3}) + B}{(x-\frac{1}{3})^2} = \frac{Ax + \frac{A}{3} + B}{(x-\frac{1}{3})^2}$$

$$\begin{cases} A = 1 \\ +\frac{A}{3} + B = 0 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -\frac{1}{3} \end{cases}$$

$$\int \frac{x}{9x^2+6x+1}\,dx = \frac{1}{9}\int \frac{x}{(x+\frac{1}{3})^2}\,dx$$

$$= \frac{1}{9}\int \frac{1}{x+\frac{1}{3}} - \frac{\frac{1}{3}}{(x+\frac{1}{3})^2}\,dx + C$$

$$= \frac{1}{9}\log\left|x+\frac{1}{3}\right| + \frac{1}{9}\cdot\frac{1}{3}\,\frac{1}{x+\frac{1}{3}} + C$$

$$= \frac{1}{9}\log\left|x+\frac{1}{3}\right| + \frac{1}{27}\,\frac{1}{x+\frac{1}{3}} + C.$$

$$\int \frac{1}{(4x-1)^2}\,dx = \int \frac{1}{16(x-\frac{1}{4})^2}\,dx$$

$$= \frac{1}{16}\int \frac{1}{(x-\frac{1}{4})^2}\,dx = -\frac{1}{16}\,\frac{1}{x-\frac{1}{4}} + C.$$

$$= -\frac{1}{16x-4} + C.$$

ESEMPIO

$$\int \frac{x-2}{x^2+4}\,dx \qquad q(x) = x^2+4 \qquad \Delta = -16 < 0.$$

$$\frac{A\,q'(x)}{q(x)} + \frac{B}{q(x)}$$

$$\frac{x-2}{x^2+4} = \frac{A\cdot 2x}{x^2+4} + \frac{B}{x^2+4}$$

$$= \frac{2Ax \quad + \quad B}{x^2+4}$$

$$\begin{cases} 2A = 1 \\ B = -2 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = -2 \end{cases}$$

$$\int \frac{x-2}{x^2+4}\,dx = \int \frac{1}{2}\,\frac{2x}{x^2+4} - 2\,\frac{1}{x^2+4}\,dx$$

$$= \frac{1}{2}\log(x^4+4) - 2\underbrace{\int \frac{1}{x^2+4}\,dx}_{(*)}$$

(*) $\displaystyle\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4\left(\frac{x^2}{4}+1\right)}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$ *

$y = \frac{x}{2}$

$dy = \frac{1}{2}dx$

$dx = 2\,dy$

$$= \frac{1}{4}\int \frac{1}{y^2+1}\,2\,dy = \frac{1}{2}\int \frac{1}{y^2+1}\,dy$$

$$= \frac{1}{2}\arctan y + C$$

$$= \frac{1}{2}\arctan\frac{x}{2} + C.$$

Quindi:

$$\int \frac{x-2}{x^2+4}\,dx = \frac{1}{2}\log(x^2+4) - \arctan\frac{x}{2} + C.$$

ESEMPIO

$$\int \frac{x+3}{x^2+2x+5}\,dx$$

$q(x) = x^2+2x+5 \qquad \Delta = 4-20 = -16 < 0.$

Cerchiamo A, B tali che

$$\frac{x+3}{x^2+2x+5} = \frac{A(2x+2)}{x^2+2x+5} + \frac{B}{x^2+2x+5}$$

$$= \frac{2Ax + 2A + B}{x^2+2x+5} \Rightarrow \begin{cases} 2A = 1 \\ 2A+B = 3 \end{cases}$$

$$\Rightarrow \begin{cases} A = \frac{1}{2} \\ B = 2 \end{cases}$$

$$\int \frac{x+3}{x^2+2x+5}\,dx = \int \frac{1}{2}\,\frac{2x+2}{x^2+2x+5} + \frac{2}{x^2+2x+5}\,dx$$

$$= \frac{1}{2}\log(x^2+2x+5) + 2\underbrace{\int \frac{1}{x^2+2x+5}\,dx}_{(*)}$$

$$ax^2+bx+c = a\left(\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right)$$

$$x^2+2x+5 = (x+1)^2+4$$

$$(*) = \int \frac{1}{x^2+2x+5}\,dx = \int \frac{1}{(x+1)^2+4}\,dx$$

$$= \frac{1}{4}\int \frac{1}{\frac{(x+1)^2}{4}+1}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x+1}{2}\right)^2+1}\,dx$$

$$\boxed{\begin{aligned} y &= \frac{x+1}{2} \\ dy &= \frac{1}{2}dx \\ dx &= 2dy \end{aligned}}$$

$$= \frac{1}{4}\int \frac{1}{y^2+1}\,2dy = \frac{1}{2}\arctan y + C$$

$$= \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C$$

$$\int \frac{x+3}{x^2+2x+5}\,dx = \frac{1}{2}\log(x^2+2x+5) - 2\left(\frac{1}{2}\arctan\left(\frac{x+1}{2}\right)\right) + c$$

$$= \frac{1}{2}\log(x^2+2x+5) - \arctan\left(\frac{x+1}{2}\right) + C.$$

$$\int \frac{4x}{x^2-4x+13}\,dx$$

$$q(x) = x^2-4x+13 \qquad \Delta = 16-52 = -36$$

$$\frac{4x}{x^2-4x+13} = \frac{A(2x-4)}{x^2-4x+13} + \frac{B}{x^2-4x+13}$$

$$= \frac{2Ax-4A+B}{x^2-4x+13} \qquad \begin{cases} 2A = 4 \\ -4A+B = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} A = 2 \\ B = 8 \end{cases}$$

Allora

$$\int \frac{4x}{x^2-4x+13}\,dx = \int 2\,\frac{(2x-4)}{x^2-4x+13} + \frac{8}{x^2-4x+13}\,dx$$

$$= 2\log(x^2-4x+13) + 8\underbrace{\int \frac{1}{x^2-4x+13}\,dx}_{(*)}$$

$$x^2-4x+13 = (x-2)^2+9$$

(*) $$\int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{(x-2)^2+9}\,dx$$

$$= \frac{1}{9}\int \frac{1}{\left(\frac{x-2}{3}\right)^2+1}\,dx$$

$$= \frac{1}{3}\arctan\left(\frac{x-2}{3}\right) + c.$$

$$\int \frac{4x}{x^2-4x+13}\,dx = 2\log(x^2-4x+13) + \frac{8}{3}\arctan\left(\frac{x-2}{3}\right) + c.$$

Attenzione: Non sempre occorre cercare A e B:

$$\int \frac{6x-7}{3x^2-7x+1}\,dx = \log(3x^2-7x+1) + C.$$

$$\int \frac{1}{x^2+6}\,dx = \int \frac{1}{6\left(\frac{x^2}{6}+1\right)}\,dx = \frac{1}{6}\int \frac{1}{\left(\frac{x}{\sqrt{6}}\right)^2+1}\,dx$$

$$= \frac{1}{6}\arctan\left(\frac{x}{\sqrt{6}}\right)\cdot\sqrt{6} + C$$

$$= \frac{\sqrt{6}}{6}\arctan\left(\frac{x}{\sqrt{6}}\right) + C$$

In entrambi i casi non serve cercare A e B.

Nota:

La scomposizione $\frac{p(x)}{q(x)} = \frac{A\,q'(x)}{q(x)} + \frac{B}{q(x)}$ non serve se:

- $p(x) = q'(x)$ $\quad (A = 1$ e $B = 0)$.
- $p(x) = C$ è costante: $\quad (A = 0,\ B = c)$.