LEZIONE 28

lunedì 28 novembre 2022 09:05

Integrali:

Dati $a, b \in \mathbb{R}$ e $f: [a,b] \longrightarrow \mathbb{R}$ integrabile in $[a,b]$ abbiamo definito:

- $\int_a^b f(x)\,dx$ Area (con segno) della regione di piano compresa tra l'asse x e il grafico di f.

- $\int f(x)\,dx$ insieme delle primitive di f (cioè di tutte le funzioni la cui derivata è f)

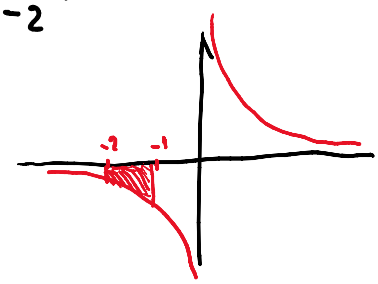
Se F è una primitiva di f, allora:

$$\int f(x)\,dx = F(x) + C$$

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

ESEMPI

- $\int x\,dx = \frac{1}{2}x^2 + C$
- $\int_0^4 x\,dx = \frac{1}{2}x^2\Big|_0^4 = \frac{1}{2}\cdot 4^2 - \frac{1}{2}\cdot 0^2 = 8$
- $\int_{-2}^{-1} \frac{1}{x}\,dx = \ln|x|\Big|_{-2}^{-1} = \log|-1| - \log|-2|$
$= \log 1 - \log 2$
$= -\log 2$



Primitive delle funzioni elementari:

$$\int a\,dx = a\,x + c \qquad \forall a \in \mathbb{R}.$$

$$\int x\,dx = \frac{1}{2}x^2 + C$$

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1} + C \qquad \text{se } \alpha \neq -1$$

$$\int \frac{1}{x}\,dx = \log|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\log a} + C \qquad \forall\ a > 0 \text{ e } a \neq 1$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + C$$

NOTA

$\arcsin x = \frac{\pi}{2} - \arccos x$

$\forall\ x \in [-1, 1]$.

PROPRIETÀ DEGLI INTEGRALI INDEFINITI

1) $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

2) $\int a\,f(x)\,dx = a\int f(x)\,dx$

$\left(\forall \alpha, \beta \in \mathbb{R}:\quad \int \alpha f(x) + \beta g(x)\,dx = \alpha\int f(x)\,dx + \beta\int g(x)\,dx\right)$

LINEARITÀ DEGLI INTEGRALI INDEFINITI

ESEMPIO

$$\int e^x - 3\sin x \, dx = e^x - 3(-\cos x) + C$$
$$= e^x + 3\cos x + C$$

$$\int_0^\pi e^x - 3\sin x \, dx = \left[e^x + 3\cos x\right]_0^\pi$$
$$= e^\pi + 3\cos\pi - (e^0 + 3\cos 0)$$
$$= e^\pi - 3 - (1+3)$$
$$= e^\pi - 7.$$

FATTO UTILE:

Se $\int f(x)\,dx = F(x) + C$ allora:

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$$

$\left(\text{Infatti: } \left(\frac{1}{a}F(ax+b)\right)' = \frac{1}{\cancel{a}} F'(ax+b)\cdot\cancel{a} = f(ax+b) \right)$

ESEMPI

- $\int e^{2x}\,dx = \frac{e^{2x}}{2} + C$
- $\int \cos(4x)\,dx = \sin(4x)\cdot\frac{1}{4} + C = \frac{1}{4}\sin(4x) + C$
- $\int \sin\left(\frac{x}{2}\right)dx = -\cos\left(\frac{x}{2}\right)/\frac{1}{2} + C = -2\cos\left(\frac{x}{2}\right) + C$
- $\int \frac{1}{4x-1}\,dx = \log|4x-1| \cdot \frac{1}{4} + C = \frac{1}{4}\log|4x-1| + C$
- $\int \frac{1}{x+3}\,dx = \log|x+3| + C$
- $\int \frac{1}{2-x}\,dx = \log|2-x| \cdot \frac{1}{-1} + C = -\log|2-x| + C$

$$\int e^{-3x} + \sqrt{x}\, dx = \frac{e^{-3x}}{-3} + \int x^{\frac{1}{2}}\, dx$$
$$= -\frac{1}{3} e^{-3x} + \frac{1}{1+\frac{1}{2}} x^{1+\frac{1}{2}} + c$$
$$= -\frac{1}{3} e^{-3x} + \frac{2}{3} \underbrace{x^{\frac{3}{2}}}_{= \sqrt{x^3}} + c$$

Attenzione: usare solo per i polinomi di I grado.

- $\int \frac{1}{1+x^2}\, dx = \arctan x + c$
- $\int \frac{1}{2+x^2}\, dx = \int \frac{1}{2\left(1+\frac{x^2}{2}\right)}\, dx = \frac{1}{2} \int \frac{1}{1+\left(\frac{x}{\sqrt{2}}\right)^2}\, dx$

$$= \frac{1}{2} \arctan\left(\frac{x}{\sqrt{2}}\right) \Big/ \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \arctan\left(\frac{x}{\sqrt{2}}\right) + c.$$

Non sempre è possibile calcolare esplicitamente la le primitive di una funzione.

e^{x^2} l'unico modo per scrivere una primitiva è

$$F(x) = \int_0^x e^{t^2}\, dt$$

Anche di e^{-x^2} non è nota un'espressione esplicita per la primitiva.

ALTRI ESEMPI DI CALCOLO DI INTEGRALI :

$$\int \frac{x-1}{x}\, dx = \int \frac{1}{x} - \frac{1}{x^2}\, dx$$
$$= \int \frac{1}{x}\, dx - \int x^{-2}\, dx$$
$$= \log|x| - \frac{1}{-2+1} x^{-2+1} + c$$

$$= \log|x| - \frac{1}{-1} x^{-1} + C$$
$$= \log|x| + \frac{1}{x} + C$$

$$\int \sin^2 x \, dx$$

Ricordare :

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

Quindi:

$$\int \sin^2 x \, dx = \int \frac{1}{2} - \frac{1}{2}\cos(2x) \, dx$$
$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin(2x)}{2} + C$$
$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.$$

FATTO UTILE

Siccome $\left(\log|f(x)|\right)' = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$

possiamo dire che

$$\int \frac{f'(x)}{f(x)} \, dx = \log|f(x)| + C.$$

ESEMPI

- $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx$

  $= -\int \frac{(\cos x)'}{\cos x} dx = -\log|\cos x| + C.$

- $\int \frac{x}{1+x^2} dx = \frac{1}{2}\int \frac{2x}{1+x^2} dx = \frac{1}{2}\log|1+x^2| + C.$

  $= \frac{1}{2}\log(1+x^2) + C.$

- $$\int \frac{x-2}{2x^2-8x+1}\,dx =$$

$$(2x^2-8x+1)' = 4x-8 = 4(x-2)$$

$$= \frac{1}{4}\int \frac{4(x-2)}{2x^2-8x+1}\,dx = \frac{1}{4}\log|2x^2-8x+1| + C.$$

- $$\int \frac{1}{\sin x\cos x}\,dx = \int \frac{\cos x}{\sin x\cos^2 x}\,dx$$

$$= \int \frac{1}{\cos^2 x}\cdot\frac{1}{\tan x}\,dx = \int \frac{(\tan x)'}{\tan x}\,dx.$$

$$= \log|\tan x| + C.$$

- $$\int \frac{1}{2x-3}\,dx = \frac{1}{2}\int \frac{2}{2x-3}\,dx = \frac{1}{2}\log|2x-3| + C.$$

Formule di integrazione per parti

Def: Sia $I \subseteq \mathbb{R}$ un intervallo e sia $f: I \to \mathbb{R}$. Si dice che f È DI CLASSE C^1 IN I se f è derivabile in I e f' è continua su I.
L'insieme delle funzioni di classe C^1 si indica con $C^1(I)$.

Def: Più in generale si dice che f è di CLASSE C^k IN I se f è derivabile k volte in I e le derivate $f', f'', \dots, f^{(k)}$ sono continue in I.
L'insieme delle funzioni C^k in I si indica con $C^k(I)$.

TEOREMA (FORMULA DI INTEGRAZIONE PER PARTI)

Siano $f, g \in C^1([a,b])$ con $a, b \in \mathbb{R}$, $a \le b$.
Allora:

1) $\int_a^b f'(x)\,g(x)\,dx = \underbrace{f(x)g(x)\Big|_a^b}_{f(b)g(b)-f(a)g(a).} - \int_a^b f(x)\,g'(x)\,dx$

2) $\int f'(x)\,g(x)\,dx = f(x)\,g(x) - \int f(x)\,g'(x)\,dx$

DIM

2) $(f(x)\,g(x))' = f'(x)\,g(x) + f(x)\,g'(x)$

$$f'(x)\,g(x) = (f(x)\,g(x))' - f(x)\,g'(x)$$

$$\int f'(x)\,g(x)\,dx = \int (f(x)g(x))'\,dx - \int f(x)\,g'(x)\,dx$$

$$= f(x)\,g(x) - \int f(x)\,g'(x)\,dx$$

1) $f'(x)\,g(x) = (f(x)\,g(x))' - f(x)\,g'(x)$

$$\int_a^b f'(x)\,g(x)\,dx = \int_a^b (f(x)\,g(x))'\,dx - \int_a^b f(x)g'(x)\,dx$$

$$= f(x)\,g(x)\Big|_a^b - \int_a^b f(x)\,g'(x)\,dx.$$

ESEMPI

$$\boxed{\begin{array}{ll} f'(x) = e^x & f(x) = e^x \\ g(x) = x & g'(x) = 1 \end{array}}$$

1) $\int \underbrace{x}_{g}\,\underbrace{e^x}_{f'}\,dx$

$= e^x x - \int e^x \cdot 1\,dx = e^x x - \int e^x\,dx$

$= e^x x - e^x + C.$

2) $\int x\,\underbrace{\cos(2x)}_{f'}\,dx = \frac{\sin(2x)}{2}\cdot x - \int \frac{\sin(2x)}{2}\cdot 1\,dx$

$= \frac{x}{2}\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx$

$= \frac{x}{2}\sin(2x) - \frac{1}{2}\left(\frac{-\cos(2x)}{2}\right) + C$

$= \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C.$

Può capitare di dover utilizzare la formula più volte per calcolare un integrale:

$$\int x^2 \underbrace{e^{-x}}_{f'} dx = -e^{-x}x^2 - \int (-e^{-x}) \cdot 2x \, dx$$

$$= -e^{-x}x^2 + 2\int \underbrace{e^{-x}}_{f'} \cdot x \, dx$$

$$= -e^{-x}x^2 + 2\left(-e^{-x} \cdot x - \int (-e^{-x}) \cdot 1 \, dx\right)$$

$$= -e^{-x}x^2 - 2e^{-x}x + 2\int e^{-x} dx$$

$$= -e^{-x}x^2 - 2e^{-x}x - 2e^{-x} + C.$$

$$\int \sin x \cos x \, dx = \frac{1}{2}\int 2\sin x \cos x \, dx = \frac{1}{2}\int \sin(2x) \, dx$$

$$= -\frac{1}{4}\cos(2x) + C.$$

Si può anche ottenere il risultato integrando per parti:

$$\int \sin x \underbrace{\cos x}_{f'} \, dx = \sin x \cdot \sin x - \int \sin x \cdot \cos x \, dx$$

$$= \sin^2 x - \int \sin x \cos x \, dx$$

$$2\int \sin x \cos x \, dx = \sin^2 x + C$$

$$\int \sin x \cos x \, dx = \frac{\sin^2 x}{2} + C.$$

NOTA:

$$-\frac{1}{4}\cos(2x) = -\frac{1}{4}(1 - 2\sin^2 x) = -\frac{1}{4} + \frac{1}{2}\sin^2 x$$

Le due primitive trovate differiscono per una costante.

Dunque i due risultati sono coerenti tra loro

$$\int \cos^2 x \, dx = \int \cos x \underbrace{\cos x}_{f'} \, dx$$

$$= \sin x \cdot \cos x - \int \sin x (-\sin x) \, dx$$

$$= \sin x \cos x + \int \sin^2 x \, dx$$

$$= \sin x \cos x + \int 1 - \cos^2 x \, dx$$

$$= \sin x \cos x + x - \int \cos^2 x \, dx$$

$$2 \int \cos^2 x \, dx = \sin x \cos x + x + C$$

$$\int \cos^2 x \, dx = \frac{\sin x \cos x}{2} + \frac{x}{2} + C$$

$$\int \log x \, dx = \int \underbrace{1}_{f'} \cdot \log x \, dx$$

$$= x \log x - \int x \cdot \frac{1}{x} \, dx$$

$$= x \log x - \int 1 \, dx$$

$$= x \log x - x + C.$$

$$\int \arctan x \, dx = \int \underbrace{1}_{f'} \cdot \arctan x \, dx$$

$$= x \arctan x - \int x \frac{1}{1+x^2} \, dx$$

$$= x \arctan x - \int \frac{x}{1+x^2} \, dx$$

$$= x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$

$$= x \arctan x - \frac{1}{2} \log(1+x^2) + C.$$

$$\int \sqrt{1-x^2} \, dx = \int \underbrace{1}_{f'} \cdot \sqrt{1-x^2} \, dx$$

$$= x \sqrt{1-x^2} - \int x \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) \, dx$$

$$= x \sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}$$

$$= x \sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}} \, dx$$

$$= x \sqrt{1-x^2} - \int \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\, dx + \int \frac{1}{\sqrt{1-x^2}}\, dx$$

$$2\int \sqrt{1-x^2}\, dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= x\sqrt{1-x^2} + \arcsin x + C$$

$$\int \sqrt{1-x^2}\, dx = \frac{x\sqrt{1-x^2} + \arcsin x}{2} + C.$$

$$\int \underbrace{x^7}_{f'} \log x \, dx$$

$$= \frac{x^8}{8} \log x - \int \frac{x^8}{8} \cdot \frac{1}{x}\, dx$$

$$= \frac{x^8}{8} \log x - \frac{1}{8} \int x^7 \, dx$$

$$= \frac{x^8}{8} \log x - \frac{x^8}{64} + C.$$

$$\int \underbrace{e^{3x}}_{f'} \sin(2x)\, dx = \frac{1}{3} e^{3x} \sin(2x) - \int \frac{1}{3} e^{3x} \cos(2x) \cdot 2 \, dx$$

$$= \frac{1}{3} e^{3x} \sin(2x) - \frac{2}{3} \int \underbrace{e^{3x}}_{f'} \cos(2x)\, dx$$

$$= \frac{1}{3} e^{3x} \sin(2x) - \frac{2}{3} \Big[\frac{1}{3} e^{3x} \cos(2x) - \int \frac{1}{3} e^{3x} (-\sin(2x) \cdot 2)\, dx$$

$$= \frac{1}{3} e^{3x} \sin(2x) - \frac{2}{3} \Big[\frac{1}{3} e^{3x} \cos(2x) + \frac{2}{3} \int e^{3x} \sin(2x)\, dx \Big]$$

$$= \frac{1}{3} e^{3x} \operatorname{sen}(2x) - \frac{2}{9} e^{3x} \cos(2x) - \frac{4}{9} \int e^{3x} \operatorname{sen}(2x)\, dx$$

$$\int e^{3x} \operatorname{sen}(2x)\, dx + \frac{4}{9} \int e^{3x} \sin(2x)\, dx = \frac{1}{3} e^{3x} \sin(2x) - \frac{2}{9} e^{3x} \cos(2x) + C$$

$$\frac{13}{9} \int e^{3x} \sin(2x)\, dx = \frac{1}{3} e^{3x} \sin(2x) - \frac{2}{9} e^{3x} \cos(2x) + C$$

$$\int e^{3x} \sin(2x)\, dx = \frac{3}{13} e^{3x} \sin(2x) - \frac{2}{13} e^{3x} \cos(2x) + C$$

$$\int_0^{\frac{\pi}{2}} x^2 \underbrace{\sin x}_{f'}\, dx = -\cos x \cdot x^2 \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x)\cdot 2x\, dx$$

$$= 0 - 0 + \int_0^{\frac{\pi}{2}} \underbrace{(\cos x)}_{f'} \cdot 2x\, dx$$

$$= \sin x \cdot 2x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \cdot 2\, dx$$

$$= 1\cdot\pi - 0 - 2\int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$= \pi - 2\left[-\cos x\right]_0^{\frac{\pi}{2}}$$

$$= \pi - 2\left[0 + 1\right] = \pi - 2.$$

Altro metodo:

$$\int x^2 \sin x = -x^2\cos x + 2x\sin x + 2\cos x + c$$

$$\int_0^{\pi} x^2 \sin x = \left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\frac{\pi}{2}}$$

$$= \left[0 + \pi\cdot 1 + 0 - (0 + 0 + 2)\right]$$

$$= \pi - 2.$$

Integrazione per sostituzione

Idea: Se F è una primitiva di f $(F' = f)$

Allora
$$(F(\varphi(t)))' = F'(\varphi(t))\,\varphi'(t) = f(\varphi(t))\,\varphi'(t)$$

$$\int f(\varphi(t))\,\varphi'(t)\, dt = F(\varphi(t)) + C = \left(\int f(x)\, dx\right)\Big|_{x=\varphi(t)}$$

In altri termini:

$$\int f(\varphi(x))\,\varphi'(x)\, dx \overset{y=\varphi(x)}{=} \left(\int f(y)\, dy\right)\Big|_{y=\varphi(x)}$$

ESEMPIO

$$\int e^{x^2}\cdot x\,dx = \frac{1}{2}\int e^{x^2}2x\,dx \overset{y=x^2}{=} \frac{1}{2}\int e^y\,dy$$

$$= \frac{1}{2}e^y + c = \frac{1}{2}e^{x^2} + c.$$

Come ricordare la formula:

$y = \varphi(x)$, " $dy = \varphi'(x)\,dx$ "

$$\int_a^b f(y)\,dy \overset{y=\varphi(x)}{=} \int_\alpha^\beta f(\varphi(x))\,\varphi'(x)$$

dove $\alpha, \beta \in \mathbb{R}$ sono tali che $\varphi(\alpha) = a$ e $\varphi(\beta) = b$.

TEOREMA (FORMULA DI CAMBIAMENTO DI VARIABILE)

Siano $\alpha, \beta \in \mathbb{R}$ con $\alpha \le \beta$. Sia $\varphi : [\alpha, \beta] \longrightarrow \mathbb{R}$ di classe C^1 tale che $\varphi(\alpha) = a$ e $\varphi(\beta) = b$. Allora

$$\int_\alpha^\beta f(\varphi(x))\,\varphi'(x)\,dx = \int_a^b f(y)\,dy$$

La formula si può utilizzare in diversi modi

1) $\int f(x)\,dx$ $\quad$ $x = \varphi(y)$, $dx = \varphi'(y)\,dy$

$= \int f(\varphi(y))\,\varphi'(y)\,dy$

2) $\int f(\varphi(x))\,\varphi'(x)\,dx$ $\quad$ $y = \varphi(x)$, $dy = \varphi'(x)\,dx$

$= \int f(y)\,dy$

3) $\int f(x)\,dx$ $\quad$ $y = \varphi(x)$ $\quad$ $x = \varphi^{-1}(y)$, $dx = (\varphi^{-1})'(y)\,dy$

$= \int f(\varphi^{-1}(y))(\varphi^{-1})'(y)\,dy.$

ESEMPI DI SOSTITUZIONI (TIPO 1)

- $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$. $\quad x = y^2$ (conviene assumere $y \geq 0$)
$\quad dx = 2y\,dy$

$$= \int \frac{1}{1+\sqrt{y^2}}\,2y\,dy = \int \frac{2y}{1+y}\,dy$$
$$= 2\int \frac{y}{1+y}\,dy.$$
$$= 2\int \frac{y+1-1}{1+y}\,dy$$
$$= 2\int 1 - \frac{1}{1+y}\,dy$$
$$= 2\big(y - \log(1+y)\big) + C$$

$x = y^2$
$y = \sqrt{x}$

$$= 2\big(\sqrt{x} - \log(1+\sqrt{x})\big) + C$$
$$= 2\sqrt{x} - 2\log(1+\sqrt{x}) + C.$$

- $\displaystyle\int \frac{1}{x^2+3}\,dx$ $\quad x = \sqrt{3}\,y$ $\quad \left(y = \frac{x}{\sqrt{3}}\right)$
$\quad dx = \sqrt{3}\,dy$

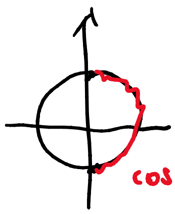
$$= \int \frac{1}{(\sqrt{3}y)^2+3}\sqrt{3}\,dy = \sqrt{3}\int \frac{1}{3y^2+3}\,dy = \frac{\sqrt{3}}{3}\int \frac{1}{y^2+1}\,dy$$
$$= \frac{\sqrt{3}}{3}\arctan y + C$$
$$= \frac{\sqrt{3}}{3}\arctan\left(\frac{x}{\sqrt{3}}\right) + C.$$

- $\displaystyle\int_{-1}^{1}\sqrt{1-x^2}\,dx$ $\quad x = \sin y$
$\quad dx = \cos y\,dy$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\sin^2 y}\,\cos y\,dy$$

$x = 1 \Rightarrow y = \frac{\pi}{2}$
$x = -1 \Rightarrow y = -\frac{\pi}{2}$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos^2 y}\,\cos y\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|\cos y|\cos y\,dy$$



$\cos y \geq 0$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 y \, dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2y)}{2} \, dy =$$

$$= \left[\frac{y}{2} + \frac{1}{4} \sin(2y) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{4} \cdot 0 - \left(-\frac{\pi}{4} + \frac{1}{4} \cdot 0 \right)$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} .$$

In alternativa si può calcolare $\int \sqrt{1-x^2}\,dx$ e sostituire negli estremi solo all'ultimo passaggio.

$$\int \sqrt{1-x^2}\,dx = \frac{x\sqrt{1-x^2} + \arcsin x}{2} + C$$

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \left[\frac{x\sqrt{1-x^2} + \arcsin x}{2} \right]_{-1}^{1} = 0 + \frac{\pi}{4} - \left(0 - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}$$

ESEMPI DI SOSTITUZIONE (TIPO 2)

- $\int \sin^2 x \underbrace{\cos x \, dx}_{dy}$ $\qquad y = \sin x$, $\quad dy = \cos x \, dx$

$$= \int y^2 \, dy = \frac{1}{3} y^3 + C = \frac{1}{3} \sin^3 x + C .$$

- $\int \frac{\arctan x}{1+x^2}\,dx$ $\qquad y = \arctan x$, $\quad dy = \frac{1}{1+x^2}\,dx$

$$= \int \arctan x \cdot \frac{1}{1+x^2}\,dx$$

$$= \int y \, dy = \frac{1}{2} y^2 + C = \frac{1}{2} \arctan^2 x + C$$

- Abbiamo detto che $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C$

Infatti se $y = f(x)$, $dy = f'(x)\,dx$ quindi

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{y}\,dy = \log|y| + C = \log|f(x)| + C$$

ESEMPIO DI SOSTITUZIONE (TIPO 3)

$$\int \frac{x+2}{\sqrt{1+x}}\,dx$$

$$y = \sqrt{1+x}$$

$$dy = \frac{1}{2\sqrt{1+x}}\,dx \Rightarrow dx = 2\sqrt{1+x}\,dy = 2y\,dy$$

$$y^2 = 1+x$$

$$x = y^2 - 1$$

Per sostituire occorre scrivere tutto in termini di y.

Quindi:

$$\int \frac{x+2}{\sqrt{1+x}}\,dx =$$

$$= \int \frac{y^2-1+2}{\cancel{y}} \cdot 2\cancel{y}\,dy = 2\int y^2+1\,dy$$

$$= 2\left(\frac{1}{3}y^3 + y\right) + C$$

$$= \frac{2}{3}y^3 + 2y + C$$

$$= \frac{2}{3}\left(\sqrt{x+1}\right)^3 + 2\sqrt{x+1} + C.$$