LEZIONE 26

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ESERCIZIO 1

Calcolare $\lim_{x \to 0} \frac{e^x - 1 - \sin x - \frac{1}{2} x^2 \cos x}{x^3}$ f.i. $\frac{0}{0}$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

$$e^x - 1 - \sin x - \frac{1}{2}x^2 \cos x =$$

$$= \cancel{1} + \cancel{x} + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3) - \cancel{1} - (\cancel{x} - \frac{x^3}{6} + o(x^3)) - \frac{1}{2}x^2\left(1 - \frac{x^2}{2} + o(x^2)\right)$$

$$= \cancel{\frac{1}{2}}x^2 + \frac{1}{6}x^3 + o(x^3) + \frac{x^3}{6} - \cancel{\frac{1}{2}}x^2$$

$$= \frac{1}{6}x^3 + \frac{1}{6}x^3 + o(x^3) = \frac{1}{3}x^3 + o(x^3)$$

Quindi: $\lim_{x \to 0} \frac{\frac{1}{3}x^3 + o(x^3)}{x^3} = \lim_{x \to 0} \frac{1}{3} + \frac{o(x^3)}{x^3} = \frac{1}{3}$

Domanda? Si può risolvere usando De L'Hospital?

$$\lim_{x \to 0} \frac{e^x - 1 - \sin x - \frac{1}{2}x^2 \cos x}{x^3}$$

$$\overset{D.L.H}{=} \lim_{x \to 0} \frac{e^x - \cos x - \frac{1}{2}(2x \cos x - x^2 \sin x)}{3x^2} \quad \text{f.i. } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{e^x - \cos x - x\cos x + \frac{1}{2}x^2 \sin x}{3x^2}$$

$$\overset{D.L.H.}{=} \lim_{x \to 0} \frac{e^x + \sin x - \cos x + x \sin x + \frac{1}{2}(2x \sin x + x^2 \cos x)}{6x}$$

$$= \lim_{x \to 0} \frac{e^x + \sin x - \cos x + 2x \sin x + \frac{1}{2}x^2 \cos x}{6x} \quad \text{f.i. } \frac{0}{0}$$

$$\overset{D.L.H}{=} \lim_{x \to 0} \frac{e^x + \cos x + \sin x + 2\sin x + 2x\cos x + x\cos x - \frac{1}{2}x^2 \cos x}{6}$$

$$= \frac{1+1+0}{6} = \frac{2}{3} = \frac{1}{3}$$

ESERCIZIO 2

$$\lim_{x\to 0} \frac{x^3 e^{2x^2} - \log(1+x^3)}{x^3+x^5-x^2\sin x} \qquad \text{f.i. } \frac{0}{0}$$

DENOM:

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$x^3+x^5-x^2\sin x = x^3+x^5-x^2\left(x-\frac{x^3}{6}+o(x^3)\right)$$

$$= \cancel{x^3} + x^5 - \cancel{x^3} + \frac{x^5}{6} + o(x^5)$$

$$= x^5 + \frac{x^5}{6} + o(x^5)$$

$$= \frac{7}{6}x^5 + o(x^5)$$

Numeratore: $x^3 e^{2x^2} - \log(1+x^3)$

$$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$$e^{2x^2} = 1 + 2x^2 + \frac{1}{2}(2x^2)^2 + o((2x^2)^2)$$

$$= 1 + 2x^2 + 2x^4 + o(x^4)$$

$$x^3 e^{2x^2} = x^3\left(1+2x^2+2x^4+o(x^4)\right)$$

$$= x^3 + 2x^5 + o(x^5) \quad *$$

$$\log(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$\log(1+x^3) = x^3 - \frac{(x^3)^2}{2} + o((x^3)^2) = x^3 + o(x^5) \quad (a)$$

NUM: $x^3 e^{2x^2} - \log(1+x^3) = \cancel{x^3} + 2x^5 + o(x^5) - \cancel{x^3}$

$$= 2x^5 + o(x^5).$$

Conclusione:

$$\lim_{x\to 0} \frac{2x^5+o(x^5)}{\frac{7}{6}x^5+o(x^5)} = \frac{2}{\frac{7}{6}} = \frac{12}{7}$$

Attenzione: Non tutti i limiti si possono calcolare con gli sviluppi di Taylor.

ESEMPIO

$$\lim_{x\to+\infty} \frac{e^{\frac{x}{2}} + \sin x}{\sqrt{4 + e^x}}$$

f.i. $\frac{+\infty}{+\infty}$

Noi conosciamo lo sviluppo di e^x e $\sin x$ per $x \to 0$ mentre qui abbiamo $x \to +\infty$.

$$\frac{e^{\frac{x}{2}}\left(1 + \frac{\sin x}{e^{\frac{x}{2}}}\right)}{\sqrt{e^x\left(\frac{4}{e^x} + 1\right)}} = \frac{\cancel{e^{\frac{x}{2}}}\left(1 + o(1)\right)}{\cancel{e^{\frac{x}{2}}}\sqrt{\frac{4}{e^x} + 1}} = \frac{1 + o(1)}{\sqrt{\frac{4}{e^x} + 1}} \xrightarrow{x\to 0} 1$$

ESERCIZIO 3

$$\lim_{x\to+\infty} x^4\left(e^{\frac{1}{2x^2}} + \cos\frac{1}{x} - 2\right)$$

f.i. $+\infty \cdot 0$

$e^y = 1 + y + \frac{1}{2}y^2 + o(y^2)$ per $y \to 0$.

Se $x \to +\infty$, $\frac{1}{2x^2} \to 0$ quindi posso prendere $y = \frac{1}{2x^2}$

$$e^{\frac{1}{2x^2}} = 1 + \frac{1}{2x^2} + \frac{1}{2}\left(\frac{1}{2x^2}\right)^2 + o\left(\left(\frac{1}{2x^2}\right)^2\right)$$

$$= 1 + \frac{1}{2x^2} + \frac{1}{8x^4} + o\left(\frac{1}{x^4}\right)$$

$\cos y = 1 - \frac{y^2}{2} + \frac{y^4}{24} + o(y^4)$ per $y \to 0$.

Se $x \to +\infty$, $\frac{1}{x} \to 0$ e prendendo $y = \frac{1}{x}$ si trova:

$$\cos\frac{1}{x} = 1 - \frac{1}{2x^2} + \frac{1}{24}\frac{1}{x^4} + o\left(\frac{1}{x^4}\right)$$

$$\lim_{x\to\infty} x^4\left(e^{\frac{1}{2x^2}} + \cos\frac{1}{x} - 2\right) =$$

$$\lim_{x\to\infty} x^4\left(\cancel{1}+\cancel{\frac{1}{2}\frac{1}{x^2}}+\frac{1}{8}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)+\cancel{1}-\cancel{\frac{1}{2}\frac{1}{x^2}}+\frac{1}{24}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)\cancel{-2}\right)$$

$$=\lim_{x\to\infty} x^4\left(\frac{1}{6}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)\right)$$

$$=\lim_{x\to\infty}\frac{1}{6}+x^4\,o\left(\frac{1}{x^4}\right)=\lim_{x\to+\infty}\frac{1}{6}\quad\frac{1}{\frac{1}{x^4}}\,o\left(\frac{1}{x^4}\right)=\frac{1}{6}$$

ESERCIZIO 4

$$\lim_{x\to+\infty} x^4\left(e^{\frac{1}{2x^2}}+\cos\frac{1}{x}-2+e^{-x}\right)$$

non riesco a sviluppare e^{-x}

$$=\lim_{x\to+\infty} x^4\left(e^{\frac{1}{2x^2}}+\cos\frac{1}{x}-2\right)+x^4e^{-x}$$

$=\frac{x^4}{e^x}\to 0$ per $x\to+\infty$

$$=\frac{1}{6}+0=\frac{1}{6}$$

ESERCIZIO 5

$$\lim_{x\to0}\frac{e^{x-x^3}-\frac{1}{2}x^2-\operatorname{sen}x-1}{\log^2(1+2x)-4x^2}$$

Nota:

$(a+b+c)^2=$

$=((a+b)+c)^2$

$=(a+b)^2+c^2+2(a+b)c$

$=a^2+b^2+2ab+c^2+2ac+$
$+2bc$

$=a^2+b^2+c^2+2ab+2ac$
$+2bc.$

DENOM:

$$\log(1+x)=x-\frac{x^2}{2}+o(x^2)$$

$$\log(1+2x)=2x-\frac{(2x)^2}{2}+o(x^2)$$

$$=2x-2x^2+o(x^2)$$

$$\log^2(1+2x)=\left(2x-2x^2+o(x^2)\right)^2$$

$$=4x^2+4x^4+o(x^4)-8x^3+o(x^3)+o(x^4)$$

$$=4x^2-8x^3+o(x^3)$$

$$\log^2(1+2x)-4x^2=-8x^3+o(x^3).$$

NUMERATORE: $e^{x-x^3} - \frac{1}{2}x^2 - \sin x - 1$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$$

$$e^{x-x^3} = 1 + x - x^3 + \frac{1}{2}(x-x^3)^2 + \frac{1}{6}(x-x^3)^3 + o((x-x^3)^3)$$

$$= 1 + x - x^3 + \frac{1}{2}(x^2 + o(x^3)) + \frac{1}{6}(x^3 + o(x^3)) + o(x^3)$$

$$= 1 + x + \frac{1}{2}x^2 - x^3 + \frac{1}{6}x^3 + o(x^3)$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{5}{6}x^3 + o(x^3)$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

NUM: $\cancel{1} + \cancel{x} + \cancel{\frac{1}{2}x^2} - \frac{5}{6}x^3 + o(x^3) - \cancel{\frac{1}{2}x^2} - \cancel{x} + \frac{x^3}{6} - \cancel{1}$

$$= -\frac{2}{3}x^3 + o(x^3)$$

Conclusione: $\lim_{x \to 0} \frac{-\frac{2}{3}x^3 + o(x^3)}{-8x^3 + o(x^3)} = \frac{-\frac{2}{3}}{-8} = \frac{2}{3} \cdot \frac{1}{8} = \frac{1}{12}$

ESEMPIO

$$\lim_{x \to 0} \frac{\cos(\sqrt{x}) - e^{-\frac{x}{2}}}{x^2}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

$$\cos\sqrt{x} = 1 - \frac{(\sqrt{x})^2}{2} + \frac{(\sqrt{x})^4}{24} + o((\sqrt{x})^4)$$

$$= 1 - \frac{x}{2} + \frac{x^2}{24} + o(x^2)$$

$$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$$e^{-\frac{x}{2}} = 1 - \frac{x}{2} + \frac{1}{2}\left(-\frac{x}{2}\right)^2 + o\left(\left(\frac{x}{2}\right)^2\right)$$

$$= 1 - \frac{x}{2} + \frac{1}{8}x^2 + o(x^2)$$

NUM: $\cos\sqrt{x} - e^{\frac{x}{2}} = \cancel{1} - \cancel{\frac{x}{2}} + \frac{x^2}{24} + o(x^2) - \left(\cancel{1} - \cancel{\frac{x}{2}} + \frac{1}{8}x^2 + o(x^2)\right)$

$$= \frac{x^2}{24} - \frac{1}{8}x^2 + o(x^2) = -\frac{1}{12}x^2 + o(x^2)$$

Conclusione: $\lim_{x\to 0} \frac{-\frac{1}{12}x^2 + o(x^2)}{x^2} = -\frac{1}{12}$.

ESERCIZIO

$$\lim_{x\to 0} \frac{1 - x^2 - \cos^2 x}{\log^2\left(\frac{\sin x}{x}\right)} \qquad \text{f.i. } \frac{0}{0}$$

DENOM:

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + o(x^2)$$

$$\log\left(\frac{\sin x}{x}\right) = \log\Big(1 \underbrace{- \frac{x^2}{6} + o(x^2)}_{y}\Big)$$

$$\log(1+y) = y + o(y)$$

$$\log\left(1 - \frac{x^2}{6} + o(x^2)\right) = -\frac{x^2}{6} + o(x^2) + o\left(-\frac{x^2}{6} + o(x^2)\right)$$

$$= -\frac{x^2}{6} + o(x^2)$$

Allora:

$$\log^2\left(\frac{\sin x}{x}\right) = \left(-\frac{x^2}{6} + o(x^2)\right)^2 = \frac{1}{36}x^4 + o(x^4)$$

NUM: $1 - x^2 - \cos^2 x = \sin^2 x - x^2$

$$\sin x = x - \frac{x^3}{6} + o(x^4)$$

$$\sin^2 x = x^2 + o(x^4) - 2\frac{x^4}{6} = x^2 - \frac{1}{3}x^4 + o(x^4)$$

$$\sin^2 x - x^2 = -\frac{1}{3}x^4 + o(x^4)$$

Conclusione: $\lim_{x\to 0} \frac{-\frac{1}{3}x^4 + o(x^4)}{\frac{1}{36}x^4 + o(x^4)} = \frac{-\frac{1}{3}}{\frac{1}{36}} = -\frac{36}{3} = -12$

Formula di Lagrange per il resto di Taylor

Sia $f: (a,b) \to \mathbb{R}$ con $a, b \in \mathbb{R}^*$, $a < b$.
Sia $x_0 \in (a,b)$ con f n volte derivabile in x_0.
Denotiamo $E_n(x) := f(x) - T_n(x)$ dove
$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.
Il teorema di Peano dice che: $E_n(x) = o((x-x_0)^n)$
per $x \to 0$

TEOREMA (FORMULA DI LAGRANGE PER E_n)
Sia $f: (a,b) \to \mathbb{R}$ con $a, b \in \mathbb{R}^*$, $a < b$.
Sia $x_0 \in (a,b)$ e assumiamo f $n+1$ volte in x_0.
Allora $\exists\, c \in (a,b)$ compreso tra x e x_0 tale che

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

ESEMPIO
La formula consente di approssimare numeri irrazionali ad esempio e:

$f(x) = e^x$, $x_0 = 0$.
$e = f(1)$
$e = T_n(1) + E_n(1)$ dove T_n è il polinomio di Taylor di ordine n di $f(x)$ in $x_0 = 0$.

La formula di Lagrange dice che

$$E_n(1) = \frac{f^{(n+1)}(c)}{(n+1)!}\cdot 1^n = \frac{e^c}{(n+1)!} \quad \text{dove } c\in[0,1]$$

In particolare $0 \leq E_n(1) \leq \frac{e}{(n+1)!} \leq \frac{3}{(n+1)!}$

Dunque: $E_n(1) \xrightarrow{n\to+\infty} 0$

Se $n = 5$

$$T_n(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} = 2{,}71\bar{6}$$

e $E_n(1) \leq \frac{3}{6!} = \frac{3}{720} = \frac{3}{5!\,6} = \frac{1}{120\cdot 2} = \frac{1}{240} < \frac{1}{200} = 0{,}005$

Possiamo scrivere: $e \simeq 2{,}71\bar{6}$ con errore al più $0{,}005$.
