Limiti da risolvere con le formule di Taylor

1) $$\lim_{x\to 0} \frac{\sin(x+2x^2) - \cos x - x + 1}{\log(1+x) - \sin x}$$

$$\sin x = x - \frac{x^3}{6} + o(x^3) \quad \text{per } x \to 0$$

$$\Rightarrow \sin(x+2x^2) = x+2x^2 - \frac{(x+2x^2)^3}{6} + o((x+2x^2)^3) \quad \text{per } x\to 0$$

$$= x+2x^2 - \frac{x^3}{6}(1+2x)^3 + o(x^3)$$

$$= x+2x^2 - \frac{x^3}{6}(1+6x+12x^2+8x^3) + o(x^3)$$

$$\boxed{(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3}$$

$$= x+2x^2 - \frac{x^3}{6} - x^4 - 2x^5 - \frac{4}{3}x^6 + o(x^3)$$

$$= x+2x^2 - \frac{x^3}{6} + o(x^3) + o(x^3) = x+2x^2 - \frac{x^3}{6} + o(x^3) \quad \text{per } x\to 0$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$$

$$\sin(x+2x^2) - \cos x - x + 1 = x+2x^2 - \frac{x^3}{6} + o(x^3) - 1 + \frac{x^2}{2} + o(x^3) - x + 1$$

$$= \frac{5}{2}x^2 - \frac{x^3}{6} + o(x^3) = \frac{5}{2}x^2 + o(x^2) \quad \text{per } x\to 0$$

$$\log(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\log(1+x) - \sin x = x - \frac{x^2}{2} + o(x^2) - \left(x - \frac{x^3}{6} + o(x^3)\right)$$

$$= -\frac{x^2}{2} + o(x^2) + \frac{x^3}{6} - o(x^3)$$

$$= -\frac{x^2}{2} + o(x^2)$$

$$\lim_{x\to 0} \frac{\sin(x+2x^2) - \cos x - x + 1}{\log(1+x) - \sin x} = \lim_{x\to 0} \frac{\frac{5}{2}x^2 + o(x^2)}{-\frac{x^2}{2} + o(x^2)}$$

$$= \frac{\frac{5}{2}}{-\frac{1}{2}} = -\frac{5}{2}\cdot 2 = -5$$

2) $$\lim_{x\to 0} \frac{\sin(2x) - x - x\sqrt{1-3x^2}}{x\,(\cos^2 x - e^{x^2})}$$

$$\sin x = x - \frac{x^3}{6} + o(x^3) \implies \sin(2x) = 2x - \frac{(2x)^3}{6} + o((2x)^3)$$
$$= 2x - \frac{4}{3}x^3 + o(x^3)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + o(x^2) \quad \text{per } x\to 0$$

$$\uparrow$$

$$-\frac{1}{8} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}$$

$$\sqrt{1-3x^2} = 1 + \frac{(-3x^2)}{2} - \frac{(-3x^2)^2}{8} + o((-3x^2)^2)$$
$$= 1 - \frac{3}{2}x^2 - \frac{9x^4}{8} + o(x^4) \quad \text{per } x\to 0$$

$$\sin(2x) - x - x\sqrt{1-3x^2} = 2x - \frac{4}{3}x^3 + o(x^3) - x - x\left(1 - \frac{3}{2}x^2 - \frac{9x^4}{8} + o(x^4)\right)$$
$$= 2x - \frac{4}{3}x^3 + o(x^3) - x - x + \frac{3}{2}x^3 + \frac{9}{8}x^5 + o(x^5)$$
$$= \left(-\frac{4}{3} + \frac{3}{2}\right)x^3 + o(x^3) = \frac{x^3}{6} + o(x^3) \quad \text{per } x\to 0$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2) \quad \text{per } x\to 0$$
$$\cos^2 x = \left(1 - \frac{x^2}{2} + o(x^2)\right)^2 = 1 + \frac{x^4}{4} + (o(x^2))^2 - x^2 + 2\,o(x^2) - x^2 o(x^2)$$

$$\boxed{(A+B+C)^2 = A^2 + B^2 + C^2 + 2AB + 2AC + 2BC}$$

$$= 1 - x^2 + \frac{x^4}{4} + o(x^4) + o(x^2) + o(x^4) = 1 - x^2 + \frac{x^4}{2} + o(x^2) = 1 - x^2 + o(x^2)$$

$$e^{x} = 1 + x + \frac{x^2}{2} + o(x^2) \qquad x \to 0$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + o(x^4) \qquad \text{per } x \to 0$$

$$x\left(\cos^2 x - e^{x^2}\right) = x\left(1 - x^2 + o(x^2) - 1 - x^2 + o(x^2)\right)$$

$$= x\left(-2x^2 + o(x^2)\right) = -2x^3 + o(x^3)$$

$$\lim_{x \to 0} \frac{\sin(2x) - x - x\sqrt{1-3x^2}}{x\left(\cos^2 x - e^{x^2}\right)} = \lim_{x \to 0} \frac{\frac{x^3}{6} + o(x^3)}{-2x^3 + o(x^3)} = \frac{\frac{1}{6}}{-2} = -\frac{1}{12}$$

3) $$\lim_{x \to 0^+} \frac{x \cdot \cos(2x) + e^{-x} - 1}{\left(\log(1+\sqrt{x}) - \sqrt{x}\right)^2}$$

$$\log(1+x) = x - \frac{x^2}{2} + o(x^2) \implies \log(1+x) - x = -\frac{x^2}{2} + o(x^2) \qquad \text{per } x \to 0$$

$$\implies \log(1+\sqrt{x}) - \sqrt{x} = -\frac{x}{2} + o(x) \qquad \text{per } x \to 0$$

$$\left(\log(1+\sqrt{x}) - \sqrt{x}\right)^2 = \left(-\frac{x}{2} + o(x)\right)^2 = \frac{x^2}{4} - x \cdot o(x) + (o(x))^2 = \frac{x^2}{4} + o(x^2) \qquad \text{per } x \to 0$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2) \quad \text{per } x \to 0 \implies \cos(2x) = 1 - \frac{(2x)^2}{2} + o((2x)^2)$$

$$= 1 - 2x^2 + o(x^2) \qquad \text{per } x \to 0$$

$$e^{x} = 1 + x + \frac{x^2}{2} + o(x^2) \implies e^{-x} = 1 - x + \frac{x^2}{2} + o(x^2) \qquad \text{per } x \to 0$$

$$x \cdot \cos(2x) + e^{-x} - 1 = x \cdot \left(1 - 2x^2 + o(x^2)\right) + 1 - x + \frac{x^2}{2} - 1$$

$$= x - 2x^3 + o(x^3) + 1 - x + \frac{x^2}{2} - 1$$

$$= \frac{x^2}{2} - 2x^3 + o(x^3) = \frac{x^2}{2} + o(x^2) \qquad \text{per } x \to 0$$

$$\lim_{x \to 0^+} \frac{x \cdot \cos(2x) + e^{-x} - 1}{\left(\log(1+\sqrt{x}) - \sqrt{x}\right)^2} = \lim_{x \to 0^+} \frac{\frac{x^2}{2} + o(x^2)}{\frac{x^2}{4} + o(x^2)} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$$

Studio di funzione

$$f(x) = x^2 e^{-\frac{1}{x}}$$

C.E. $x \neq 0$ $\quad$ $dom(f) = (-\infty, 0) \cup (0, +\infty)$

$f(-x) = (-x)^2 e^{-\frac{1}{-x}} = x^2 e^{\frac{1}{x}}$ $\quad$ f non è né pari né dispari

Calcoliamo gli zeri di f

$f(x) = 0 \iff x^2 e^{-\frac{1}{x}} = 0 \iff x^2 = 0 \ \vee \ e^{-\frac{1}{x}} = 0$

$\iff x = 0$ non accettabile

Studiamo il segno di f

$f(x) > 0 \iff x^2 e^{-\frac{1}{x}} > 0 \iff x^2 > 0 \iff x \neq 0$

Calcolo dei limiti nei punti di frontiera del dominio

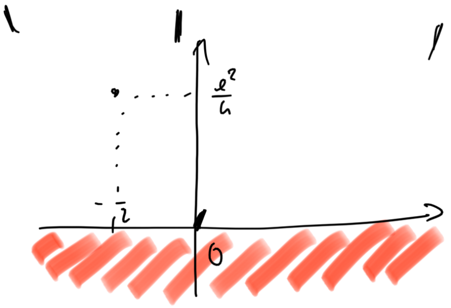
$\lim_{x \to -\infty} x^2 e^{-\frac{1}{x}} = +\infty$, $\lim_{x \to +\infty} x^2 e^{-\frac{1}{x}} = +\infty$

$\lim_{x \to 0^-} x^2 e^{-\frac{1}{x}} \overset{[0 \cdot \infty]}{=} \lim_{t \to +\infty} \frac{1}{t^2} \cdot e^t = \lim_{t \to +\infty} \frac{e^t}{t^2} = +\infty$

$t = -\frac{1}{x} \iff x = -\frac{1}{t}$

se $x \to 0^-$ allora $t \to +\infty$

$\lim_{x \to 0^+} x^2 e^{-\frac{1}{x}} = 0$



$x = 0$ è asintoto verticale sinistro

$f(x) = x^2 \cdot e^{-\frac{1}{x}}$

$f'(x) = 2x \cdot e^{-\frac{1}{x}} + x^2 \cdot e^{-\frac{1}{x}} \cdot \frac{1}{x^2} = e^{-\frac{1}{x}}(2x+1)$

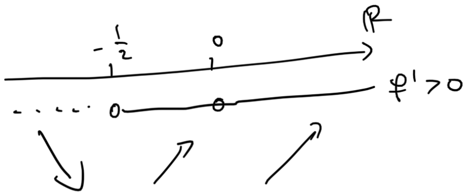
$dom f' = dom f$ $\quad$ non ci sono punti di non derivabilità

Determiniamo i punti stazionari

$f'(x) = 0 \iff e^{-\frac{1}{x}}(2x+1) = 0 \iff 2x+1 = 0 \iff x = -\frac{1}{2}$

Studiamo il segno di f'

$f'(x) > 0 \iff e^{-\frac{1}{x}}(2x+1) > 0 \iff 2x+1 > 0 \iff x > -\frac{1}{2}$



f è strettamente decrescente in $(-\infty, -\frac{1}{2})$

f è strettamente crescente in $(-\frac{1}{2}, 0)$ e in $(0, +\infty)$

$x = -\frac{1}{2}$ è un punto di minimo relativo

$f(-\frac{1}{2}) = (-\frac{1}{2})^2 e^{-\frac{1}{-\frac{1}{2}}} = \frac{e^2}{4} > 0$

Calcoliamo f''

$f'(x) = e^{-\frac{1}{x}}(2x+1)$

$f''(x) = e^{-\frac{1}{x}} \cdot \frac{1}{x^2} \cdot (2x+1) + e^{-\frac{1}{x}} \cdot 2 = \frac{e^{-\frac{1}{x}}}{x^2}(2x+1+2x^2)$

$= \frac{e^{-\frac{1}{x}}}{x^2}(2x^2+2x+1)$

Controlliamo se ci sono punti di flesso

$f''(x) = 0 \iff \frac{e^{-\frac{1}{x}}}{x^2}(2x^2+2x+1) = 0 \iff 2x^2+2x+1 = 0$

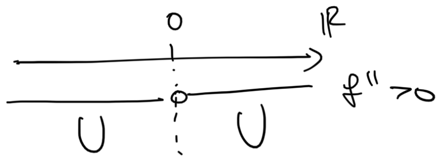
$\Delta = 4 - 4 \cdot 2 \cdot 1 = -4 < 0$

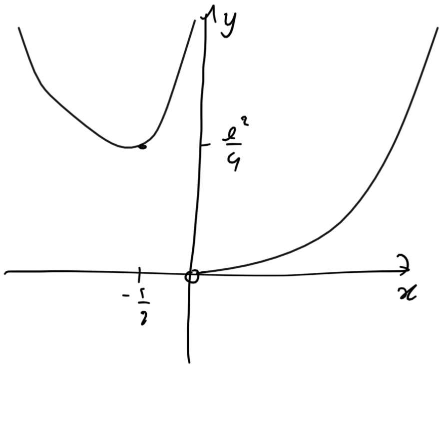
non ci sono soluzioni in $\mathbb{R}$

f non ha punti di flesso

Studiamo il segno di f''

$f''(x) > 0 \iff \frac{e^{-\frac{1}{x}}}{x^2}(2x^2+2x+1) > 0 \iff 2x^2+2x+1 > 0 \quad \forall x \in \mathbb{R}$





f è convessa in $(-\infty, 0)$ e in $(0, +\infty)$

$$f(x) = \frac{\sqrt{x}}{x^3+1}$$

$$\text{C.E.} \begin{cases} x \geq 0 \\ x^3+1 \neq 0 \Rightarrow x^3 \neq -1 \Rightarrow x \neq -1 \end{cases} \Rightarrow x \geq 0$$

$\operatorname{dom}(f) = [0, +\infty)$

Calcoliamo gli zeri di f

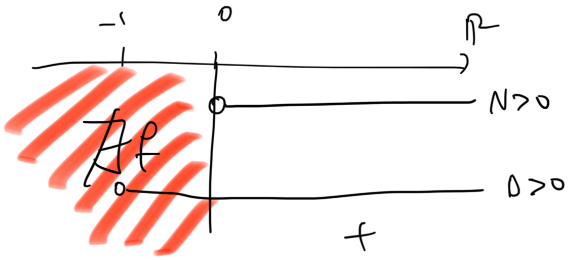
$f(x) = 0 \iff \sqrt{x} = 0 \iff x = 0$

Studiamo il segno di f

$f(x) > 0 \iff \frac{\sqrt{x}}{x^3+1} > 0$

$N > 0: \quad \sqrt{x} > 0 \iff \begin{cases} x > 0 \\ x \in \operatorname{dom}(f) \end{cases}$

$D > 0: \quad x^3+1 > 0 \iff x^3 > -1 \iff x > -1$



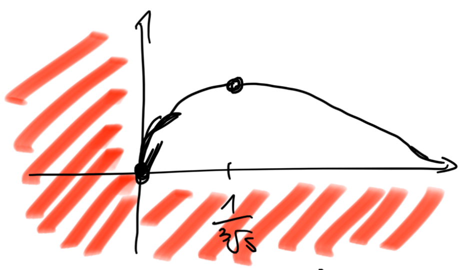
$f(x) > 0 \quad \forall x > 0$

Calcoliamo i limiti agli estremi del dominio

$\lim_{x \to 0^+} f(x) = 0$ (f è continua in 0)

$\lim_{x \to +\infty} \frac{\sqrt{x}}{x^3+1} \overset{[\frac{\infty}{\infty}]}{=} 0^+$ $y = 0$ è asintoto orizzontale per $x \to +\infty$

Calcoliamo f' $\quad f(x) = \frac{\sqrt{x}}{x^3+1}$



$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(x^3+1) - \sqrt{x} \cdot 3x^2}{(x^3+1)^2} = \frac{\frac{1}{2\sqrt{x}}\left(x^3+1 - 2\sqrt{x} \cdot \sqrt{x} \cdot 3x^2\right)}{(x^3+1)^2}$$

$$= \frac{x^3+1-6x^3}{2\sqrt{x}\,(x^3+1)^2} = \frac{1-5x^3}{2\sqrt{x}\,(x^3+1)^2}$$

$\text{dom} f' = (0, +\infty)$ $\qquad$ $x=0$ è un punto di non derivabilità

$\lim_{x \to 0^+} f'(x) = +\infty$ $\qquad$ $x=0$ è un punto a tangente verticale

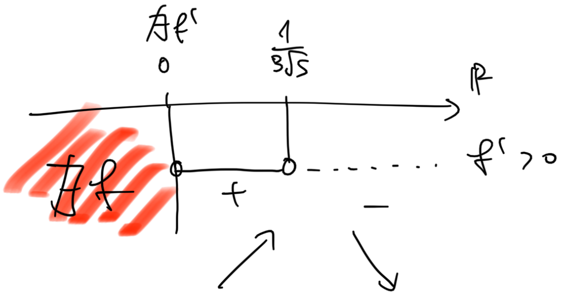
Determiniamo i punti stazionari di f

$$f'(x) = 0 \iff 1 - 5x^3 = 0 \iff x^3 = \frac{1}{5} \iff x = \sqrt[3]{\frac{1}{5}} = \frac{1}{\sqrt[3]{5}}$$

Studiamo il segno di f'

$$f'(x) > 0 \iff \frac{1-5x^3}{2\sqrt{x}\,(x^3+1)^2} > 0 \iff 1 - 5x^3 > 0 \iff x^3 < \frac{1}{5}$$

$$\iff x < \frac{1}{\sqrt[3]{5}}$$



f è strettamente crescente in $\left(0, \frac{1}{\sqrt[3]{5}}\right)$

f è strettamente decrescente in $\left(\frac{1}{\sqrt[3]{5}}, +\infty\right)$

$x = \frac{1}{\sqrt[3]{5}}$ punto di massimo assoluto