Lezione 29

giovedì 2 dicembre 2021 09:04

ESERCIZIO 1

$$\int \underbrace{x}_{f'} \underbrace{\ln(2+3x)}_{g}\, dx$$

$$= \frac{1}{2}x^2 \log(2+3x) - \int \frac{1}{2}x^2 \frac{1}{2+3x}\cdot 3 \, dx$$

$$= \frac{1}{2}x^2 \log(2+3x) - \frac{3}{2}\underbrace{\int \frac{x^2}{2+3x}\, dx}$$

$$\begin{array}{l|l}
x^2 + 0x + 0 & 3x + 2 \\
x^2 + \frac{2}{3}x & \frac{x}{3} - \frac{2}{9} \\
\hline
// \quad -\frac{2}{3}x + 0 & \\
\quad -\frac{2}{3}x - \frac{4}{9} & \\
\hline
\quad // \quad \frac{4}{9} &
\end{array}$$

$$x^2 = (3x+2)\left(\frac{x}{3} - \frac{2}{9}\right) + \frac{4}{9}$$

$$\frac{x^2}{3x+2} = \frac{x}{3} - \frac{2}{9} + \frac{\frac{4}{9}}{3x+2}$$

$$\int \frac{x^2}{3x+2}\, dx = \frac{x^2}{6} - \frac{2}{9}x + \frac{4}{9}\int \frac{1}{3x+2}\, dx = \frac{x^2}{6} - \frac{2}{9}x + \frac{4}{27}\log|3x+2| + C$$

$$\int x \log(2+3x)\, dx = \frac{1}{2}x^2 \log(2+3x) - \frac{3}{2}\left(\frac{x^2}{6} - \frac{2}{9}x + \frac{4}{27}\log(3x+2) + C\right)$$

$$= \frac{1}{2}x^2 \log(2+3x) - \frac{x^2}{4} + \frac{x}{3} - \frac{2}{9}\log(3x+2) + \hat{C}$$

ESERCIZIO 2

$$\int \frac{1}{2e^x + e^{-x} + 1}\, dx$$

$$= \int \frac{1}{2e^x + \frac{1}{e^x} + 1}\, dx = \int \frac{1}{\frac{2e^{2x} + 1 + e^x}{e^x}}\, dx$$

$$= \int \frac{e^x}{2e^{2x}+e^x+1}\,dx$$

$$t = e^x \qquad x = \log t$$
$$dt = e^x\,dx \qquad dx = \frac{1}{t}\,dt$$
$$dt = t\,dx$$
$$dx = \frac{1}{t}\,dt$$

$$= \int \frac{\cancel{t}}{2t^2+t+1}\,\frac{1}{\cancel{t}}\,dt$$

$$= \int \frac{1}{2t^2+t+1}\,dt$$

$$q(t) = 2t^2+t+1 \qquad \Delta = 1-8 = -7 < 0$$

$$at^2+bt+c = a\left(\left(t+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right)$$

$$2t^2+t+1 = 2\left(\left(t+\frac{1}{4}\right)^2 + \frac{7}{16}\right)$$

$$t^2+\frac{t}{2}+\frac{1}{16}+\frac{7}{16} = t^2+\frac{t}{2}+\frac{1}{2}$$

$$2t^2+t+1 = 2\cdot\frac{7}{16}\left(\frac{16}{7}\left(t+\frac{1}{4}\right)^2+1\right)$$

$$= \frac{7}{8}\left(\left(\frac{4}{\sqrt{7}}\cdot\left(t+\frac{1}{4}\right)\right)^2+1\right)$$

$$= \frac{7}{8}\left(\left(\frac{4t+1}{\sqrt{7}}\right)^2+1\right)$$

$$\int \frac{1}{2t^2+t+1}\,dt = \frac{8}{7}\int \frac{1}{\left(\frac{4t+1}{\sqrt{7}}\right)^2+1}\,dt$$

$$= \frac{8}{7}\,\operatorname{arctg}\left(\frac{4t+1}{\sqrt{7}}\right)\frac{\sqrt{7}}{4} + C$$

$$= \frac{2}{\sqrt{7}}\,\operatorname{arctg}\left(\frac{4t+1}{\sqrt{7}}\right) + C$$

$$\int \frac{1}{2e^x+e^{-x}+1}\,dx = \frac{2}{\sqrt{7}}\,\operatorname{arctg}\left(\frac{4e^x+1}{\sqrt{7}}\right) + C$$

- ESERCIZIO 3

$$\int \frac{x^3-8x+4}{x^2+x-6}\,dx$$

Soluzione:

Per prima cosa facciamo la divisione tra numeratore e denominatore

$$\begin{array}{l|l}
x^3+0x^2-8x+4 & x^2+x-6 \\ \cline{2-2}
x^3+x^2-6x & x-1 \\ \cline{1-1}
// \quad -x^2-2x+4 & \\
\quad -x^2-x+6 & \\ \cline{1-1}
\quad // \quad -x-2 &
\end{array}$$

Allora $x^3-8x+4 = (x^2+x-6)(x-1) - x - 2$

$$\frac{x^3-8x+4}{x^2+x-6} = x-1 - \frac{x+2}{x^2+x-6}$$

Quindi $\displaystyle\int \frac{x^3-8x+4}{x^2+x-6}\,dx = \frac{1}{2}x^2 - x - \int \frac{x+2}{x^2+x-6}\,dx$

Calcoliamo $\displaystyle\int \frac{x+2}{x^2+x-6}\,dx$

$$x^2+x-6=0 \iff x = \frac{-1\pm\sqrt{1+24}}{2} = \begin{cases} 2 \\ -3 \end{cases}$$

$$x^2+x-6 = (x-2)(x+3)$$

Cerchiamo A, B tali che:

$$\frac{x+2}{x^2+x-6} = \frac{A}{x-2} + \frac{B}{x+3}$$

$$= \frac{A(x+3)+B(x-2)}{x^2+x-6} = \frac{x(A+B)+3A-2B}{x^2+x-6}$$

$$\begin{cases} A+B=1 \\ 3A-2B=2 \end{cases} \Rightarrow \begin{cases} B=1-A \\ 3A-2+2A=2 \end{cases} \Rightarrow \begin{cases} B=\frac{1}{5} \\ A=\frac{4}{5} \end{cases}$$

$$\frac{x+2}{x^2+x-6} = \frac{4}{5}\,\frac{1}{x-2} + \frac{1}{5}\,\frac{1}{x+3}$$

$$\int \frac{x+2}{x^2+x-6}\,dx = \frac{4}{5}\int\frac{1}{x-2}\,dx + \frac{1}{5}\int\frac{1}{x+3}\,dx$$

$$= \frac{4}{5}\log|x-2| + \frac{1}{5}\log|x+3| + C$$

Quindi:

$$\int \frac{x^3-8x+4}{x^2+x-6}\,dx = \frac{1}{2}x^2 - x - \int \frac{x+2}{x^2+x-6}\,dx$$
$$= \frac{1}{2}x^2 - x - \frac{4}{5}\log|x-2| - \frac{1}{5}\log|x+3| + C$$

Caso $\deg(q(x)) \geq 3$

$$\int \frac{p(x)}{q(x)}\,dx$$

Dopo aver fatto la divisione (se serve) tra $p(x)$ e $q(x)$ possiamo supporre che $\deg(p(x)) < \deg(Q(x))$

La scomposizione di q è del tipo

$$q(x) = a(x-x_1)^{\alpha_1}(x-x_2)^{\alpha_2} \cdots (x-x_n)^{\alpha_n} q_1(x)^{\beta_1} \cdots q_s(x)^{\beta_s}$$

dove $q_1(x), \dots, q_s(x)$ sono polinomi irriducibili di grado 2.

Si può scrivere:

$$\frac{p(x)}{q(x)} = \frac{A_{11}}{(x-x_1)} + \frac{A_{12}}{(x-x_1)^2} + \cdots + \frac{A_{1\alpha_1}}{(x-x_1)^{\alpha_1}} +$$
$$+ \frac{A_{21}}{(x-x_2)} + \frac{A_{22}}{(x-x_2)^2} + \cdots + \frac{A_{2\alpha_2}}{(x-x_2)^{\alpha_2}}$$
$$\vdots$$
$$+ \frac{A_{n1}}{(x-x_n)} + \frac{A_{n2}}{(x-x_n)^2} + \cdots + \frac{A_{\alpha_n n}}{(x-x_n)^{\alpha_n}}$$
$$+ \frac{C_{11}\, q_1'(x) + D_{11}}{q_1(x)} + \frac{C_{12}\, q_1'(x) + D_{12}}{q_1(x)^2} + \cdots + \frac{C_{1\beta_1}\, q_1'(x) + D_{1\beta}}{q_1(x)^{\beta_1}}$$
$$\vdots$$
$$+ \frac{C_{s1}\, q_s'(x) + D_{s1}}{q_s(x)} + \cdots + \frac{C_{s\beta_s}\, q_s'(x) + D_{s\beta_s}}{q_s(x)^{\beta_s}}$$

ESEMPIO

$$\int \frac{1}{(2x+1)\underbrace{(x^2+x+1)}_{\Delta<0}}\,dx$$

$$\frac{1}{(2x+1)(x^2+x+1)} = \frac{A}{2x+1} + \frac{C(2x+1)+D}{x^2+x+1}$$

$$= \frac{A(x^2+x+1)+C(2x+1)^2+D(2x+1)}{(2x+1)(x^2+x+1)}$$

$$= \frac{A(x^2+x+1)+C(4x^2+4x+1)+2Dx+D}{(2x+1)(x^2+x+1)}$$

$$= \frac{x^2(A+4C)+x(A+4C+2D)+A+C+D}{(2x+1)(x^2+x+1)}$$

$$\begin{cases} A+4C=0 \\ A+4C+2D=0 \\ A+C+D=1 \end{cases} \Rightarrow \begin{cases} A=-4C \\ D=0 \\ -4C+C=1 \end{cases}$$

$$\Rightarrow \begin{cases} A=\frac{4}{3} \\ C=-\frac{1}{3} \\ D=0 \end{cases}$$

$$\frac{1}{(2x+1)(x^2+x+1)} = \frac{4}{3}\,\frac{1}{2x+1} - \frac{1}{3}\,\frac{2x+1}{x^2+x+1} + 0$$

$$\int \frac{1}{(2x+1)(x^2+x+1)}\,dx = \frac{4}{3}\int \frac{1}{2x+1}\,dx - \frac{1}{3}\int \frac{2x+1}{x^2+x+1}\,dx$$

$$= \frac{4}{3}\log|2x+1|\cdot\frac{1}{2} - \frac{1}{3}\log(x^2+x+1)+c$$

$$= \frac{2}{3}\log|2x+1| - \frac{1}{3}\log(x^2+x+1)+c$$

ESERCIZIO 10

$$\int \frac{6x}{(x-2)^2(x^2+2)}\,dx$$

$$\frac{6x}{\underbrace{(x-2)}_{q_1(x)}{}^2(x^2+2)} = \underbrace{\frac{A}{x-2}+\frac{B}{(x-2)^2}}_{\frac{A\,q_1(x)+B}{q(x)}} + \frac{2Cx+D}{x^2+2}$$

$$= \frac{A(x-2)(x^2+2) + B(x^2+2) + (2Cx+D)(x-2)^2}{(x-2)^2(x^2+2)}$$

$$= \frac{A(x^3-2x^2+2x-4) + B(x^2+2) + (2Cx+D)(x^2-4x+4)}{(x-2)^2(x^2+2)}$$

$$= \frac{A(x^3-2x^2+2x-4) + B(x^2+2) + (2Cx^3+Dx^2-8Cx^2-4Dx+8Cx+4D)}{(x-2)^2(x^2+2)}$$

$$= \frac{x^3(A+2C) + x^2(-2A+B+D-8C) + x(2A-4D+8C) - 4A+2B+4D}{(x-2)^2(x^2+2)}$$

$$\begin{cases} A+2C=0 \\ -2A+B+D-8C=0 \\ 2A-4D+8C=6 \\ -4A+2B+4D=0 \end{cases} \Leftrightarrow \begin{cases} A+2C=0 \\ -2A+\overset{-2D}{B}+D-8C=0 \\ A-2D+4C=3 \\ -2A+B+2D=0 \end{cases}$$

$$\begin{cases} A=-2C \\ -D-8C=0 \\ A-2D+4C=3 \\ -2A+B+2D=0 \end{cases} \Rightarrow \begin{cases} A=-2C \\ D=-8C \\ -2C+16C+4C=3 \\ +4C+B-16C=0 \end{cases}$$

$$\begin{cases} C = \frac{1}{6} \\ A = -\frac{1}{3} \\ D = -\frac{4}{3} \\ B = 12C = 2 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{3} \\ B = 2 \\ C = \frac{1}{6} \\ D = -\frac{4}{3} \end{cases}$$

$$\frac{6x}{(x-2)^2(x^2+2)} = -\frac{1}{3}\frac{1}{x-2} + \frac{2}{(x-2)^2} + \frac{1}{6}\frac{2x}{x^2+2} - \frac{4}{3}\cdot\frac{1}{x^2+2}$$

$$\int \frac{6x}{(x-2)^2(x^2+2)}\,dx = -\frac{1}{3}\log|x-2| - \frac{2}{x-2} + \frac{1}{6}\log(x^2+2) - \frac{4}{3}\underbrace{\int\frac{1}{x^2+2}\,dx}$$

$$\int \frac{1}{x^2+2}\,dx = \frac{1}{2}\int\frac{1}{\frac{x^2}{2}+1}\,dx = \frac{1}{2}\int\frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,dx$$

$$= \frac{1}{2}\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right)\cdot\sqrt{2} + C$$

$$= \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right) + C$$

$$\int \frac{6x}{(x-2)^2(x^2+2)}\,dx = -\frac{1}{3}\log|x-2| - \frac{2}{x-2} + \frac{1}{6}\log(x^2+2) - \frac{4}{3\sqrt{2}}\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right) + C$$

ESERCIZIO

$$\int \frac{1}{2+3\sqrt{x}}\,dx$$

$$t = \sqrt{x}$$
$$x = t^2$$
$$dx = 2t\,dt$$

$$= \int \frac{1}{2+3t}\,2t\,dt$$

$$= \int \frac{2t}{2+3t}\, dt = \frac{2}{3}\int \frac{3t}{2+3t}\, dt = \frac{2}{3}\int \frac{3t+2-2}{2+3t}\, dt$$

$$= \frac{2}{3}\int 1 - \frac{2}{2+3t}\, dt$$

$$= \frac{2}{3}t - \frac{2}{3}\int \frac{2}{2+3t}\, dt = \frac{2}{3}t - \frac{4}{3}\int \frac{1}{2+3t}\, dt$$

$$= \frac{2}{3}t - \frac{4}{3}\log(|2+3t|)\cdot\frac{1}{3} + C$$

$$= \frac{2}{3}\sqrt{x} - \frac{4}{9}\log(|2+3\sqrt{x}|) + C$$

ESERCIZIO

$$\int \frac{1}{2\cos x + 5}\, dx$$

$$t = \operatorname{tg}\frac{x}{2} \qquad \operatorname{sen} x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\operatorname{arctg} t = \frac{x}{2}$$

$$x = 2\operatorname{arctg} t$$

$$dx = \frac{2}{1+t^2}\, dt$$

$$\int \frac{1}{2\cos x + 5}\, dx = \int \frac{1}{2\cdot\frac{1-t^2}{1+t^2}+5}\cdot\frac{2}{1+t^2}\, dt$$

$$= \int \frac{1+t^2}{2(1-t^2)+5(1+t^2)}\cdot\frac{2}{1+t^2}\, dt$$

$$= \int \frac{\cancel{1+t^2}}{3t^2+7} \cdot \frac{2}{\cancel{1+t^2}}\, dt$$

$$= \int \frac{2}{3t^2+7}\, dt$$

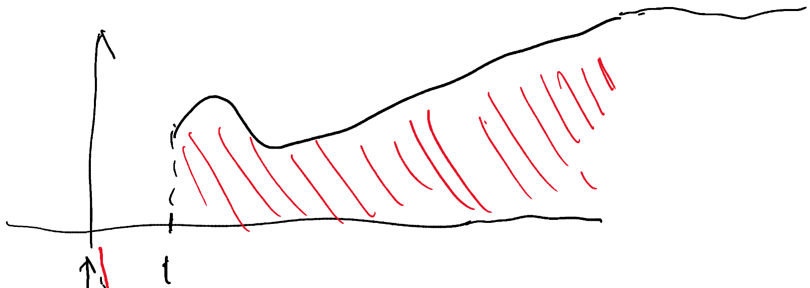
$$= \frac{2}{7} \int \frac{1}{\frac{3}{7}t^2+1}\, dt$$

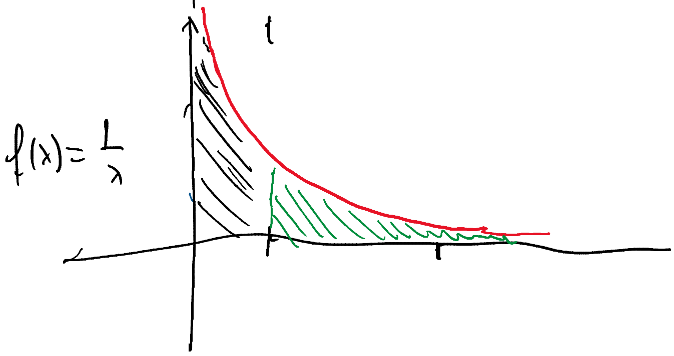
$$= \frac{2}{7} \int \frac{1}{\left(\sqrt{\frac{3}{7}}t\right)^2+1}\, dt$$

$$= \frac{2}{7} \operatorname{arctg}\left(\sqrt{\frac{3}{7}}t\right) \sqrt{\frac{7}{3}} + C$$

$$= \frac{2}{\sqrt{21}} \operatorname{arctg}\left(\sqrt{\frac{3}{7}}t\right) + C$$

$$= \frac{2}{\sqrt{21}} \operatorname{arctg}\left(\sqrt{\frac{3}{7}} \operatorname{tg}\left(\frac{x}{2}\right)\right) + C.$$





$$\int_1^{+\infty} \frac{1}{x}\, dx = ?$$

$$\int_0^1 \frac{1}{x}\, dx = ?$$

Integrali impropri

Def: Sia $f : [a, b) \longrightarrow \mathbb{R}$, $a \in \mathbb{R}$ e $b \in \mathbb{R} \cup \{+\infty\}$, $b > a$.

Si dice che f è INTEGRABILE IN SENSO IMPROPRIO IN $[a, b)$ se

- f è integrabile in $[a, c]$ $\forall\ c \in (a, b)$
- esiste finito il limite $\lim_{c \to b^-} \int_a^c f(x)\,dx$ (se $b = +\infty$ il limite è per $c \to +\infty$)

Terminologia:

- Se f è integrabile in senso improprio si dice che

$$\int_a^b f(x)\,dx \text{ è CONVERGENTE}$$

- Se $\lim_{c \to b^-} \int_a^c f(x)\,dx = +\infty$ o $-\infty$ si dice che l'integrale $\int_a^b f(x)\,dx$ è DIVERGENTE
- Se $\lim_{c \to b^-} \int_a^c f(x)\,dx \ \nexists$ si dice che $\int_a^b f(x)\,dx$ è INDEFINITO.

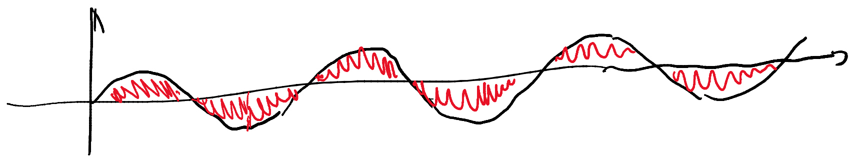
ESEMPI

- $f(x) = \sin x$ in $[0, +\infty)$

$$\int_0^{+\infty} \sin x\,dx \text{ è indefinito.}$$

$$\int_0^c \sin x\,dx = -\cos x \Big|_0^c = -\cos c + 1 = 1 - \cos c$$

e $\lim_{c \to +\infty} \int_0^c \sin x\,dx = \lim_{c \to +\infty} 1 - \cos c \ \nexists$

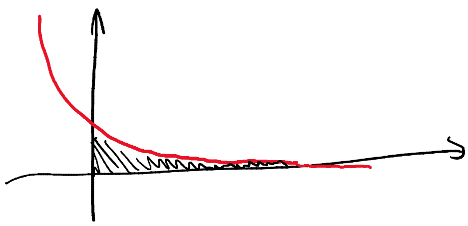


- $f(x) = e^{-x}$ in $[0, +\infty)$

$$\boxed{\int e^{ax}\,dx = \frac{e^{ax}}{a} + C}$$

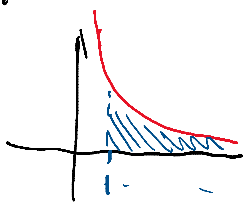
$$\int_0^{+\infty} e^{-x}\,dx = \lim_{c\to+\infty}\int_0^c e^{-x}\,dx = \lim_{c\to+\infty} -e^{-x}\Big|_0^c = \lim_{c\to+\infty} -e^{-c} + e^0$$
$$= \lim_{c\to+\infty} 1 - e^{-c} = 1$$

$$\int_0^{+\infty} e^{-x}\,dx = 1$$ l'integrale è convergente



- $f(x) = \frac{1}{x}$ in $[1, +\infty)$

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{c\to+\infty}\int_1^c \frac{1}{x}\,dx = \lim_{c\to+\infty} \log x\Big|_1^c = \lim_{c\to+\infty} \log c$$
$$= +\infty.$$



area $= +\infty$ l'integrale è divergente

- $\int_1^{+\infty} \frac{1}{x^\alpha}\,dx$?

- Per $\alpha = 1$ l'integrale è divergente
- Se $\alpha \neq 1$

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \lim_{c\to+\infty} \int_1^c \frac{1}{x^\alpha}\,dx = \lim_{c\to+\infty} \int_1^c x^{-\alpha}\,dx$$

$$= \lim_{c\to+\infty} \frac{1}{1-\alpha} x^{1-\alpha} \Big|_1^c = \lim_{c\to+\infty} \frac{1}{1-\alpha} c^{1-\alpha} - \frac{1}{1-\alpha}$$

$$= \lim_{c\to+\infty} \frac{1}{1-\alpha}\left(c^{1-\alpha} - 1\right) = \begin{cases} +\infty & \text{se } \alpha < 1 \\ \frac{1}{\alpha-1} & \text{se } \alpha > 1. \end{cases}$$

Riassumendo

$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx$ è convergente (e vale $\frac{1}{\alpha-1}$) se $\alpha > 1$

è divergente (vale $+\infty$) se $\alpha \leq 1$.