Lezione 28

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Itegrazione di funzioni razionali:

Funzioni razionali:

$\frac{p(x)}{q(x)}$ con $p(x)$ e $q(x)$ polinomi

$\left(p(x) = a_0 + a_1 x + \dots + a_n x^n \right.$
se $a_n \neq 0$ si scrive che $\left. \deg(p(x)) = n \right)$

Vogliamo calcolare

$$\int \frac{p(x)}{q(x)} \, dx$$

- **Caso in cui $\deg(q(x)) = 1$** (cioè $q(x) = ax + b$)

Ci sono due possibilità:

1) $p(x)$ è costante:

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \log(|ax+b|) + C$$

Si può dimostrare per sostituzione

$y = ax + b$

$dy = a\,dx$ cioè $dx = \frac{1}{a} dy$

$$\int \frac{1}{ax+b} \, dx = \int \frac{1}{y} \frac{1}{a} \, dy = \frac{1}{a} \int \frac{1}{y} \, dy$$

$$= \frac{1}{a} \log|y| + C$$

$$= \frac{1}{a} \log|ax+b| + C$$

2) Se $\deg(p(x)) \geq 1 = \deg(q(x))$

Si divide $p(x)$ per $q(x)$ e ci si riconduce al caso 1)

ESEMPIO 1

$$\int \frac{x^2}{2x-1}\,dx$$

grado 2

grado 1

Dividiamo x^2 per $2x-1$.

$$\begin{array}{rrr|l}
x^2 & +\,0x & +\,0 & 2x-1 \\
\cline{4-4}
x^2 & -\,\frac{x}{2} & & \frac{x}{2}+\frac{1}{4} \\
\cline{1-3}
// & \frac{x}{2} & +\,0 & \\
 & \frac{x}{2} & -\,\frac{1}{4} & \\
\cline{2-3}
 & // & \frac{1}{4} &
\end{array}$$

$$x^2 = (2x-1)\left(\frac{x}{2}+\frac{1}{4}\right) + \frac{1}{4}$$

quindi

$$\frac{x^2}{2x-1} = \frac{x}{2} + \frac{1}{4} + \frac{\frac{1}{4}}{2x-1}$$

$$\int \frac{x^2}{2x-1}\,dx = \int \frac{x}{2} + \frac{1}{4} + \frac{\frac{1}{4}}{2x-1}\,dx$$

$$= \frac{x^2}{4} + \frac{1}{4}x + \frac{1}{4}\int \frac{1}{2x-1}\,dx$$

$$= \frac{x^2}{4} + \frac{1}{4}x + \frac{1}{8}\log(|2x-1|) + C$$

ESEMPIO 2

$$\int \frac{4x^3-2x+1}{2-3x}\,dx = -\int \frac{4x^3-2x+1}{3x-2}\,dx$$

$$\begin{array}{rrrr|l}
4x^3 & +0\cdot x^2 & -2x & +1 & 3x-2 \\
\cline{5-5}
4x^3 & -\frac{8}{3}x^2 & & & \frac{4}{3}x^2+\frac{8}{9}x-\frac{2}{27} \\
\hline
// & \frac{8}{3}x^2 & -2x & +1 & \\
 & \frac{8}{3}x^2 & -\frac{16}{9}x & & \\
\cline{2-4}
 & // & -\frac{2}{9}x & +1 & \\
 & & -\frac{2}{9}x & +\frac{4}{27} & \\
\cline{3-4}
 & & // & \frac{23}{27} &
\end{array}$$

$$4x^3-2x+1 = (3x-2)\left(\frac{4}{3}x^2+\frac{8}{9}x-\frac{2}{27}\right)+\frac{23}{27}$$

$$\frac{4x^3-2x+1}{3x-2} = \frac{4}{3}x^2+\frac{8}{9}x-\frac{2}{27}+\frac{\frac{23}{27}}{3x-2}$$

$$\int \frac{4x^3-2x+1}{3x-2}\,dx = \frac{4}{9}x^3+\frac{4}{9}x^2-\frac{2}{27}x+\frac{23}{27}\int\frac{1}{3x-2}\,dx$$

$$= \frac{4}{9}x^3+\frac{4}{9}x^2-\frac{2}{27}x+\frac{23}{81}\log(|3x-2|)+C$$

Quindi

$$\int \frac{4x^3-2x+1}{2-3x}\,dx = -\frac{4}{9}x^3 - \frac{4}{9}x^2 + \frac{2}{27}x - \frac{23}{81}\log(|3x-2|) + \tilde{C}$$

ESEMPIO 3

$$\int \frac{2x+5}{3x-1}\,dx = \int \frac{2x}{3x-1}\,dx + \int \frac{5}{3x-1}\,dx$$

$$= \frac{2}{3}\int \frac{3x}{3x-1}\,dx + \int \frac{5}{3x-1}\,dx$$

$$= \frac{2}{3}\int \frac{3x-1+1}{3x-1}\,dx + \int \frac{5}{3x-1}\,dx$$

$$= \frac{2}{3}\int 1\,dx + \frac{2}{3}\int \frac{1}{3x-1} + 5\int \frac{1}{3x-1}\,dx$$

$$= \frac{2}{3}x + \frac{17}{3}\int \frac{1}{3x-1}\,dx$$

$$= \frac{2}{3}x + \frac{17}{9}\log|3x-1| + C.$$

Caso $q(x)$ di grado 2.

$$q(x) = ax^2 + bx + c$$

1) Se $\deg(p(x)) \geq 2$ posso dividere $p(x)$ per $q(x)$ e ricondurmi a un integrale di una funzione razionale in cui il numeratore

ha grado 0 o 1.

Supponiamo quindi che $\deg(p(x)) \leq 1$.

2) Si cerca di scomporre $q(x)$.
Ci sono tre possibilità:

i) $\Delta > 0$. In questo caso $q(x)$ ha due radici reali x_1, x_2 e

$$q(x) = a(x-x_1)(x-x_2)$$

In questo caso si possono cercare $A, B \in \mathbb{R}$ tali che

$$\frac{p(x)}{q(x)} = \frac{A}{x-x_1} + \frac{B}{x-x_2} \qquad (*)$$

Allora:

$$\int \frac{p(x)}{q(x)}\,dx = A\int \frac{1}{x-x_1}\,dx + B\int \frac{1}{x-x_2}\,dx$$
$$= A \log|x-x_1| + B \log|x-x_2| + C.$$

ii) $\Delta = 0$.

$q(x) = a(x-x_0)^2$ dove x_0 è l'unica radice reale.

Si cercano A, B tali che

$$\frac{p(x)}{q(x)} = \frac{A\,q'(x)}{q(x)} + \frac{B}{q(x)} \quad (*)$$

$$\int \frac{p(x)}{q(x)}\,dx = A\int \frac{q'(x)}{q(x)}\,dx + B\int \frac{1}{q(x)}\,dx$$

e $$\int \frac{q'(x)}{q(x)}\,dx = \log|q(x)| + c$$

$$\int \frac{1}{q(x)}\,dx = \int \frac{1}{a(x-x_0)^2}\,dx = \frac{1}{a}\int \frac{1}{(x-x_0)^2}\,dx$$

$y = x - x_0$
$dy = dx$

$$= \frac{1}{a}\int \frac{1}{y^2}\,dy = \frac{1}{a}\left(\frac{1}{-1}y^{-1}\right) + c$$

$$= -\frac{1}{a\,y} + c$$

$$= -\frac{1}{a(x-x_0)} + c.$$

OSS

$q(x) = a(x-x_0)^2$

$q'(x) = 2a(x-x_0)$

$$\frac{q'(x)}{q(x)} = \frac{2}{(x-x_0)}$$

Stiamo dicendo che A e B vanno cercati in modo da avere

$$\frac{p(x)}{q(x)} = \frac{2A}{x-x_0} + \frac{B}{a(x-x_0)^2}$$

È equivalente cercare A e B tali che

$$\frac{p(x)}{q(x)} = \frac{A}{x-x_0} + \frac{B}{(x-x_0)^2}$$

iii) $\Delta < 0$

Come prima si cercano A e B t.c.

$$\frac{p(x)}{q(x)} = \frac{A\,q'(x)}{q(x)} + \frac{B}{q(x)}$$

e $\displaystyle\int \frac{q'(x)}{q(x)}\,dx = \log|q(x)| + c.$

Dobbiamo poi calcolare $\displaystyle\int \frac{1}{q(x)}\,dx$

Questo integrale si riconduce a quello di $\frac{1}{1+x^2}$

Idea: $x_0 = -\frac{b}{2a}$

$$q(x) = a\,(x-x_0)^2 - \frac{\Delta}{4a} = a\left((x-x_0)^2 + \left(\frac{-\Delta}{4a^2}\right)\right)$$

$K > 0$

$$\int \frac{1}{q(x)}\,dx = \int \frac{1}{a((x-x_0)^2 + K)}\,dx$$

$$= \frac{1}{aK}\int \frac{1}{\frac{(x-x_0)^2}{K} + 1}\,dx \qquad y = \frac{x-x_0}{\sqrt{K}}$$

$$dy = \frac{1}{\sqrt{K}}\,dx$$

$$= \frac{1}{aK}\int \frac{1}{y^2+1}\sqrt{K}\,dy \qquad dx = \sqrt{K}\,dy$$

$$= \frac{1}{a\sqrt{K}}\operatorname{arctg} y + c$$

$$= \frac{1}{a\sqrt{K}}\operatorname{arctg}\left(\frac{x-x_0}{\sqrt{K}}\right) + c.$$

$$q(x) = a\left(\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right)$$

ESEMPIO 1

$$\int \frac{1}{x^2-1}\,dx$$

$$x^2-1 = (x-1)(x+1)$$

Cerco A e B t.c.

$$(*)\quad \frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$= \frac{A(x+1)+B(x-1)}{x^2-1} = \frac{Ax+A+Bx-B}{x^2-1} = \frac{(A+B)x+A-B}{x^2-1} \quad (*)$$

$$\begin{cases} A+B = 0 \\ A-B = 1 \end{cases} \Rightarrow \begin{cases} 2A = 1 \\ B = -A \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{2} \end{cases}$$

Quindi

$$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{2}\log|x+1| + C.$$

ESEMPIO 2

$$\int \frac{12-5x}{3x^2+x-10}\,dx$$

$$q(x) = 3x^2+x-10 \quad , \quad \Delta = 1+120 = 121 > 0$$

$$q(x) = 0 \iff x = \frac{-1\pm\sqrt{121}}{6} \begin{cases} -2 \\ \frac{5}{3} \end{cases}$$

$$q(x) = 3\left(x-\frac{5}{3}\right)(x+2)$$

$$= (3x-5)(x+2)$$

Cerco A e B tali che

$$\frac{12-5x}{3x^2+x-10} = \frac{A}{3x-5} + \frac{B}{x+2}$$

$$= \frac{A(x+2)+B(3x-5)}{3x^2+x-10} = \frac{Ax+2A+3Bx-5B}{3x^2+x-10}$$

$$= \frac{(A+3B)x+2A-5B}{3x^2+x-10}$$

$$\begin{cases} A+3B = -5 \\ 2A-5B = 12 \end{cases} \Rightarrow \begin{cases} A = -5-3B \\ -10-6B-5B = 12 \end{cases}$$

$$\Rightarrow \begin{cases} A = 1 \\ B = -2 \end{cases}$$

$$\int \frac{12-5x}{3x^2+x-10}\,dx = \int \frac{1}{3x-5} - \frac{2}{x+2}\,dx$$

$$= \frac{1}{3}\log|3x-5| - 2\log|x+2| + C$$

ESEMPIO 3

$$\int \frac{5x^3-9x^2}{x^2-2x+1}\,dx$$

Dato che il numeratore ha grado 3, partiamo facendo la divisione tra i due polinomi:

$$
\begin{array}{rrrr|l}
5x^3 & -9x^2 & +0x & +0 & x^2-2x+1 \\
\cline{5-5}
5x^3 & -10x^2 & +5x & & 5x+1 \\
\hline
// & x^2 & -5x & +0 & \\
 & x^2 & -2x & +1 & \\
\cline{2-4}
 & // & -3x & -1 &
\end{array}
$$

$$5x^3 - 9x^2 = (x^2 - 2x + 1)(5x+1) - (3x+1)$$

$$\frac{5x^3 - 9x^2}{x^2 - 2x + 1} = 5x + 1 - \frac{3x+1}{x^2-2x+1}$$

$$\int \frac{5x^3 - 9x^2}{x^2 - 2x + 1}\,dx = \frac{5}{2}x^2 + x - \int \frac{3x+1}{x^2-2x+1}\,dx$$

Calcoliamo

$$\int \frac{3x+1}{x^2-2x+1}\,dx$$

$$q(x) = x^2 - 2x + 1 \qquad \Delta = 0$$

$$q'(x) = 2x - 2 = 2(x-1)$$

Cerco A e B t.c.

$$\frac{3x+1}{x^2-2x+1} = \frac{2A(x-1)}{x^2-2x+1} + \frac{B}{x^2-2x+1}$$

$$= \frac{2A(x-1) + B}{x^2-2x+1} = \frac{2Ax - 2A + B}{x^2-2x+1}$$

$$\begin{cases} 2A = 3 \\ -2A + B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{3}{2} \\ B = 1+2A = 4 \end{cases}$$

Quindi:

$$\frac{3x+1}{x^2-2x+1} = \frac{3(x-1)}{x^2-2x+1} + \frac{4}{x^2-2x+1}$$

$$= \frac{3}{x-1} + \frac{4}{(x-1)^2}$$

$$\int \frac{3x+1}{x^2-2x+1}\,dx = 3\int \frac{1}{x-1} + 4\int \frac{1}{(x-1)^2}\,dx$$

$$= 3\log|x-1| - \frac{4}{x-1} + C \quad (*)$$

$$\int \frac{5x^3-9x^2}{x^2-2x+1}\,dx = \frac{5}{2}x^2 + x - 3\log|x-1| + \frac{4}{x-1} + C$$

Nota

Quando il numeratore è costante e $\Delta = 0$

Il passaggio con A e B si può saltare

$$\int \frac{1}{a(x-x_0)^2}\,dx = \frac{1}{a}\int \frac{1}{(x-x_0)^2}\,dx = -\frac{1}{a(x-x_0)} + C.$$

- ESEMPIO 4

$$\int \frac{x-2}{x^2+4}\,dx$$

$$q(x) = x^2+4 \qquad \Delta = -16$$

Cerco A e B t.c.

$$\frac{x-2}{x^2+4} = \frac{A\,2x}{x^2+4} + \frac{B}{x^2+4} = \frac{2Ax+B}{x^2+4}$$

$$\begin{cases} 2A = 1 \\ B = -2 \end{cases}$$

$$\frac{x-2}{x^2+4} = \frac{1}{2}\,\frac{2x}{x^2+4} - \frac{2}{x^2+4}$$

$$\int \frac{x-2}{x^2+4}\,dx = \frac{1}{2}\int \frac{2x}{x^2+4}\,dx - 2\int \frac{1}{x^2+4}\,dx$$

$$= \frac{1}{2}\log(x^2+4) - 2\int \frac{1}{x^2+4}\,dx$$

$$\int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{\frac{x^2}{4}+1}\,dx$$

$$= \frac{1}{4}\int \frac{1}{(\frac{x}{2})^2+1}\,dx$$

$$= \frac{1}{4}\,\operatorname{arctg}\left(\frac{x}{2}\right)\cdot 2 \quad + C$$

$$= \frac{1}{2}\,\operatorname{arctg}\left(\frac{x}{2}\right) \quad + C$$

$$\int \frac{x-2}{x^2+4}\,dx = \frac{1}{2}\log(x^2+4) - \operatorname{arctg}\left(\frac{x}{2}\right) + C$$

ESEMPIO 5

$$\int \frac{4x}{x^2-4x+13}\,dx$$

$$q(x) = x^2 - 4x + 13$$

$$\Delta = 16 - 52 = -36 < 0$$

$$\frac{4x}{x^2-4x+13} = \frac{A(2x-4)}{x^2-4x+13} + \frac{B}{x^2-4x+13}$$

$$= \frac{2Ax - 4A + B}{x^2 - 4x + 13}$$

$$\begin{cases} 2A = 4 \\ -4A + B = 0 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = 8 \end{cases}$$

$$\frac{4x}{x^2-4x+13} = \frac{2\,(2x-4)}{x^2-4x+13} + \frac{8}{x^2-4x+13}$$

$$\int \frac{4x}{x^2-4x+13}\,dx = 2 \log(x^2-4x+13) + 8 \int \frac{1}{x^2-4x+13}\,dx$$

$$q(x) = a\left(\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right)$$

$$x^2 - 4x + 13 = (x-2)^2 + 9 = 9\left(\frac{(x-2)^2}{9} + 1\right)$$

$$= 9\left(\left(\frac{x-2}{3}\right)^2 + 1\right)$$

$$\int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{9\left(\left(\frac{x-2}{3}\right)^2+1\right)}\,dx$$

$$= \frac{1}{9}\int \frac{1}{\left(\frac{x-2}{3}\right)^2+1}\,dx$$

$$= \frac{1}{9}\,\operatorname{arctg}\left(\frac{x-2}{3}\right)\cdot 3 + C$$

$$= \frac{1}{3}\,\operatorname{arctg}\left(\frac{x-2}{3}\right) + C$$

$$\int \frac{4x}{x^2-4x+13}\,dx = 2\log(x^2-4x+13) + \frac{8}{3}\operatorname{arctg}\left(\frac{x-2}{3}\right) + C$$