Lezione 25

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ESERCIZIO 1

Calcolare

$$\lim_{x \to 0} \frac{e^x - 1 - \sin x - \frac{1}{2} x^2 \cos x}{x^3} \qquad \text{f.i. } \frac{0}{0}.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n)$$

Ci fermiamo a $n = 3$:

$$e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4) = 1 + o(x)$$

$$e^x - 1 - \sin x - x^2 \cos x =$$

$$= 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + o(x^3) - 1 - \left(x - \frac{x^3}{6} + o(x^3)\right) - \frac{1}{2} x^2 \left(1 + o(x)\right)$$

$$= \cancel{1} + \cancel{x} + \cancel{\frac{1}{2} x^2} + \frac{1}{6} x^3 - \cancel{1} - \cancel{x} + \frac{x^3}{6} + o(x^3) - \cancel{\frac{1}{2} x^2} + o(x^3)$$

$$= \frac{x^3}{6} + \frac{x^3}{6} + o(x^3)$$

$$= \frac{1}{3} x^3 + o(x^3)$$

$$\lim_{x \to 0} \frac{e^x - 1 - \sin x - \frac{1}{2} x^2 \cos x}{x^3} = \lim_{x \to 0} \frac{\frac{1}{3} x^3 + o(x^3)}{x^3}$$

$$= \lim_{x \to 0} \frac{1}{3} + \frac{o(x^3)}{x^3} \xrightarrow[x \to 0]{} 0 \quad = \frac{1}{3}$$

Si poteva anche utilizzare de l'Hopital:

$$\lim_{x \to 0} \frac{e^x - 1 - \sin x - \frac{1}{2} x^2 \cos x}{x^3}$$

$$\overset{?}{=} \lim_{x\to 0} \frac{e^x - \cos x - \frac{1}{2}(2x\cos x - x^2 \sin x)}{3x^2}$$

$$= \lim_{x\to 0} \frac{e^x - \cos x - x\cos x + \frac{1}{2}x^2 \sin x}{3x^2} \quad \frac{0}{0} \text{ f.i.}$$

$$\overset{?}{=} \lim_{x\to 0} \frac{e^x + \sin x - (\cos x - x\sin x) + \frac{1}{2}(2x\sin x + x^2 \cos x)}{6x}$$

$$= \lim_{x\to 0} \frac{e^x + \sin x - \cos x + 2x\sin x + \frac{1}{2}x^2 \cos x}{6x} \quad \frac{0}{0} \text{ f.i.}$$

$$\overset{?}{=} \lim_{x\to 0} \frac{e^x + \cos x + \sin x + 2(\sin x + x\cos x) + \frac{1}{2}(2x\cos x - x^2 \sin x)}{6}$$

$$= \lim_{x\to 0} \frac{e^x + \cos x + 3\sin x + 3x\cos x - \frac{1}{2}x^2 \sin x}{6} = \frac{1+1}{6} = \frac{1}{3}$$

ESERCIZIO

$$\lim_{x\to 0} \frac{x^3 e^{2x^2} - \log(1+x^3)}{x^3 + x^5 - x^2 \sin x} \quad \text{f.i. } \frac{0}{0}$$

Partiamo dal denominatore:

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$x^3 + x^5 - x^2 \sin x = x^3 + x^5 - x^2\left(x - \frac{x^3}{6} + o(x^3)\right)$$

$$= \cancel{x^3} + x^5 - \cancel{x^3} + \frac{x^5}{6} + o(x^5)$$

$$= \frac{7}{6}x^5 + o(x^5)$$

Numeratore $x^3 e^{2x^2} - \log(1+x^3)$

$$e^{2x^2 \to 0} = ?$$

$$e^y = 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + o(y^3)$$ vale per $y \to 0$. prendo $y = 2x^2$.

$$e^{2x^2 \to 0} = 1 + 2x^2 + \frac{1}{2}(2x^2)^2 + \frac{1}{6}(2x^2)^3 + o((2x^2)^3)$$

$$= 1 + 2x^2 + 2x^4 + \frac{8}{6}x^6 + o(x^6)$$

$$= 1 + 2x^2 + 2x^4 + o(x^5)$$

$$x^3 e^{2x^2} = x^3 + 2x^5 + o(x^5) \quad (*)$$

$$\log(1+x^3) = ?$$

$$\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} + o(y^3)$$

$$\log(1+x^3) = x^3 + o(x^5) \quad (*)$$

$$x^3 e^{2x^2} - \log(1+x^3) = \cancel{x^3} + 2x^5 + o(x^5) - \cancel{x^3} + o(x^5)$$

$$= 2x^5 + o(x^5)$$

$$\lim_{x \to 0} \frac{x^3 e^{2x^2} - \log(1+x^3)}{x^3 + x^5 - x^2 \sin x} = \lim_{x \to 0} \frac{2x^5 + o(x^5)}{\frac{7}{6}x^5 + o(x^5)}$$

$$= \lim_{x \to 0} \frac{\cancel{x^5}\left(2 + \frac{o(x^5)}{x^5}\right)}{\cancel{x^5}\left(\frac{7}{6} + \frac{o(x^5)}{x^5}\right)} = \lim_{x \to 0} \frac{2 + \frac{o(x^5)}{x^5} \to 0}{\frac{7}{6} + \frac{o(x^5)}{x^5} \to 0} = \frac{2}{\frac{7}{6}} = \frac{12}{7}$$

Attenzione:

$$\lim_{x \to +\infty} \frac{e^{\frac{x}{2}} + \sin x}{\sqrt{4 + 5e^x}}$$

Non posso sviluppare $e^{\frac{x}{2}}$ né $\sin x$.

$$= \lim_{x\to+\infty} \frac{e^{\frac{x}{2}}\left(1+\frac{\sin x}{e^{\frac{x}{2}}}\right)}{\sqrt{e^x\left(\frac{4}{e^x}+5\right)}} = \lim_{x\to+\infty} \frac{1+\frac{\sin x}{e^{\frac{x}{2}}}}{\sqrt{\frac{4}{e^x}+5}} = \frac{1}{\sqrt{5}}$$

ESERCIZIO 3

$$\lim_{x\to+\infty} x^4\left(e^{\frac{1}{2x^2}}+\cos\frac{1}{x}-2\right) \qquad \text{f.i.} \quad +\infty\cdot 0$$

So che: $e^y = 1+y+\frac{1}{2}y^2+\underbrace{\frac{1}{6}y^3+o(y^3)}_{o(y^2)}$ per $y\to 0$.

Se $x\to+\infty$, $\frac{1}{2x^2}\to 0$ e

$$e^{\frac{1}{2x^2}} = 1+\frac{1}{2x^2}+\frac{1}{2}\frac{1}{4x^4}+o\left(\frac{1}{x^4}\right) = 1+\frac{1}{2x^2}+\frac{1}{8x^4}+o\left(\frac{1}{x^4}\right).$$

$$\cos y = 1-\frac{1}{2}y^2+\frac{1}{24}y^4+o(y^4)$$

$$\cos\frac{1}{x} = 1-\frac{1}{2}\frac{1}{x^2}+\frac{1}{24}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)$$

$$e^{\frac{1}{2x^2}}+\cos\frac{1}{x}-2 =$$

$$= \left(\cancel{1}+\cancel{\frac{1}{2x^2}}+\frac{1}{8x^4}+o\left(\frac{1}{x^4}\right)\right)+\cancel{1}-\cancel{\frac{1}{2x^2}}+\frac{1}{24}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)-\cancel{2}$$

$$= \frac{1}{6}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)$$

$$\lim_{x\to+\infty} x^4\left(e^{\frac{1}{2x^2}}+\cos\frac{1}{x}-2\right) = \lim_{x\to+\infty} x^4\left(\frac{1}{6}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)\right)$$

$$= \lim_{x\to+\infty} \frac{1}{6}+x^4 o\left(\frac{1}{x^4}\right) = \lim_{x\to+\infty} \frac{1}{6}+o(1) = \frac{1}{6}.$$

Domanda:

$$\lim_{x\to+\infty} x^4\left(e^{\frac{1}{2x^2}}+\cos\left(\frac{1}{x}\right)-2+e^{-x}\right) \quad ?$$

$e^{-x} = o\left(\frac{1}{x^\alpha}\right)$ per qualsiasi $\alpha>0$.

infatti: $\frac{e^{-x}}{\frac{1}{x^a}} = \frac{x^a}{e^x} \xrightarrow{x \to +\infty} 0$.

posso scrivere ad esempio $e^{-x} = o\left(\frac{1}{x^4}\right)$

quindi:

$$e^{\frac{1}{2x^2}} + \cos\frac{1}{x} - 2 + e^{-x} = \frac{1}{6}\frac{1}{x^4} + o\left(\frac{1}{x^4}\right).$$

il limite come primo viene $\frac{1}{6}$.

ESERCIZIO 4

$$\lim_{x \to 0} \frac{1 - x^2 - \cos^2 x}{\log^2\left(\frac{\sin x}{x}\right)} \qquad \text{f.i. } \frac{0}{0}$$

Denominatore:

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + o(x^2) \qquad (*)$$

$$\log\left(\frac{\sin x}{x}\right) = ?$$

$$\log(1+y) = y - \frac{y^2}{2} + o(y^2)$$

$$\log\Big(1 \underbrace{- \frac{x^2}{6} + o(x^2)}_{y}\Big) = -\frac{x^2}{6} + o(x^2)$$

$$\log^2\left(\frac{\sin x}{x}\right) = \left(-\frac{x^2}{6} + o(x^2)\right)^2 = \frac{x^4}{36} + o(x^4)$$

• Numeratore:

$$1 - x^2 - \cos^2 x$$

Nota
$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

$$\cos^2 x = 1 + \frac{x^4}{4} + o(x^4) + o(x^2) - x^2 + o(x^4) \Rightarrow$$

$$= 1 - x^2 + o(x^2)$$

$$1 - x^2 - \cos^2 x = o(x^2) \qquad \text{Non basta!}$$

Devo sviluppare meglio!

$$\cos x = 1 - \frac{x^2}{2} + \frac{1}{24}x^4 + o(x^4)$$

$$\cos^2 x = 1 + \frac{x^4}{4} + \frac{1}{(24)^2}x^8 + o(x^8) - x^2 + \frac{1}{12}x^4 + \boxed{o(x^4)}$$
$$+ o(x^6) + o(x^6) + o(x^8)$$

$$= 1 - x^2 + \underbrace{\frac{1}{4}x^4 + \frac{1}{12}x^4}_{\frac{1}{3}x^4} + o(x^4)$$

$$= 1 - x^2 + \frac{1}{3}x^4 + o(x^4)$$

Numeratore:

$$1 - x^2 - \cos^2 x = -\frac{1}{3}x^4 + o(x^4).$$

$$\lim_{x \to 0} \frac{1 - x^2 - \cos^2 x}{\log^2\left(\frac{\sin x}{x}\right)} = \lim_{x \to 0} \frac{-\frac{1}{3}x^4 + o(x^4)}{\frac{x^4}{36} + o(x^4)} = \frac{-\frac{1}{3}}{\frac{1}{36}} =$$

$$= -\frac{36}{3} = -12.$$

ESERCIZIO

Determinare il polinomio di Taylor di ordine 2 in $x_0 = 0$ della funzione $f(x) = \sqrt{4+x}$

• Due modi:

1) $\sqrt{4+x} = \sqrt{4\left(1+\frac{x}{4}\right)} = 2\sqrt{1+\frac{x}{4}} = 2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$

Per $x \to 0$, $\frac{x}{4} \to 0$ quindi posso utilizzare lo sviluppo di $(1+y)^\alpha$ con $y=\frac{x}{4}$.

$$(1+y)^{\frac{1}{2}} = 1 + \binom{\frac{1}{2}}{1} y + \binom{\frac{1}{2}}{2} y^2 + o(y^2) \quad (*)$$

dove $\binom{\alpha}{k} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ $\qquad \binom{\alpha}{1} = \alpha$

$$(*) = 1 + \alpha y + \frac{\alpha(\alpha-1)}{2} y^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} y^3 + o(y^3)$$

con $\alpha = \frac{1}{2}$

$$= 1 + \frac{1}{2} y - \frac{1}{8} y^2 + o(y^2)$$

$$\sqrt{1+\frac{x}{4}} = 1 + \frac{1}{2}\frac{x}{4} - \frac{1}{8}\frac{x^2}{16} + o(x^2)$$

$$2\sqrt{1+\frac{x}{4}} = \underbrace{2 + \frac{x}{4} - \frac{1}{64}x^2}_{} + o(x^2)$$

$$T_2(x) = 2 + \frac{x}{4} - \frac{x^2}{64} + o(x^2)$$

- Secondo metodo (usare la definizione)

$$f(x) = \sqrt{4+x}$$

$$T_2(x) = f(0) + f'(0)x + \frac{1}{2} f''(0) x^2$$

$$f'(x) = \frac{1}{2\sqrt{4+x}} \cdot 1 = \frac{1}{2\sqrt{4+x}} = \frac{1}{2}(4+x)^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{4}(4+x)^{-\frac{3}{2}} = -\frac{1}{4}\frac{1}{\sqrt{(4+x)^3}}$$

$$f(0) = \sqrt{4} = 2$$

$$f'(0) = \frac{1}{2}\frac{1}{\sqrt{4}} = \frac{1}{4}$$

$$f''(0) = -\frac{1}{4}\,\frac{1}{\sqrt{4^3}} = -\frac{1}{32}$$

$$T_2(x) = 2 + \frac{1}{4}x - \frac{1}{64}x^2$$

Formula di Lagrange per il resto

Sia $f : (a, b) \longrightarrow \mathbb{R}$. Sia $x_0 \in (a, b)$. Assumiamo f n volte derivabile in x_0.

Poniamo $E_n(x) = f(x) - T_n(x)$ dove $T_n(x)$ è il polinomio di Taylor di f in x_0 di ordine n.

Il teorema di Peano dice che $E_n(x) = o((x - x_0)^n)$.

TEOREMA

Sia $f : (a, b) \longrightarrow \mathbb{R}$ e sia $x_0 \in (a, b)$. Assumiamo che f sia derivabile $n+1$ volte in (a, b). Allora $\forall\, x \in (a, b)$ $\exists\, \xi \in (a, b)$, ξ compreso tra x e x_0 tale che
$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

ESEMPIO

$f(x) = e^x$, $x_0 = 0$.

$e = f(1)$.

$e = T_n(1) + E_n(1)$ dove T_n è il polinomio di Taylor di ordine n in $x_0 = 0$.

$$T_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots + \frac{1}{n!}x^n.$$

$$E_n(1) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,1^{n+1} = \frac{e^{\xi}}{(n+1)!}\cdot 1 \qquad \xi \in (0, 1).$$

$\leq \frac{3}{(n+1)!}$. Notiamo che:

$$0 \leq E_n(1) \leq \frac{3}{(n+1)!}$$ quindi $E_n(1) \xrightarrow{n \to +\infty} 0$ se $n \to +\infty$.

Se $n = 5$

$$E_n(1) \leq \frac{3}{6!} = \frac{3}{6 \cdot 5!} = \frac{1}{2 \cdot 120} = \frac{1}{240} < \frac{1}{200} = 0{,}005$$

$e = T_n(1) + E_n(1)$ e $0 \leq E_n(1) \leq 0{,}005$.

$$T_n(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} = 2{,}71\overline{6}$$