LABORATORIO DI MATEMATICA

mercoledì 21 febbraio 2024 16:27

ESERCIZIO 1

1) Determinare l'insieme delle soluzioni della disequazione

$$\frac{|4x-1|-3}{3-2x} \geq 0$$

2) Determinare l'insieme delle soluzioni della disequazione

$$\frac{|4e^x-1|-3}{3-2e^x} \geq 0$$

---

1) $$\frac{|4x-1|-3}{3-2x} \geq 0$$

Numeratore: $|4x-1|-3 \geq 0 \iff |4x-1| \geq 3$

$\iff 4x-1 \geq 3 \quad \vee \quad 4x-1 \leq -3$

$\iff 4x \geq 4 \quad \vee \quad 4x \leq -2$

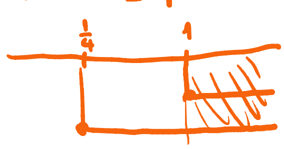
$\iff x \geq 1 \quad \vee \quad x \leq -\frac{1}{2}$

Metodo alternativo:

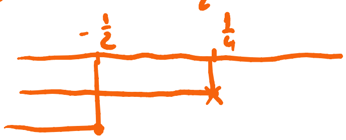
$|4x-1|-3 \geq 0$

$$\begin{cases} 4x-1 \geq 0 \\ 4x-1-3 \geq 0 \end{cases} \quad \vee \quad \begin{cases} 4x-1 < 0 \\ -4x+1-3 \geq 0 \end{cases}$$

$$\begin{cases} x \geq \frac{1}{4} \\ x \geq 1 \end{cases} \quad \vee \quad \begin{cases} x < \frac{1}{4} \\ x \leq -\frac{1}{2} \end{cases}$$



$\vee$

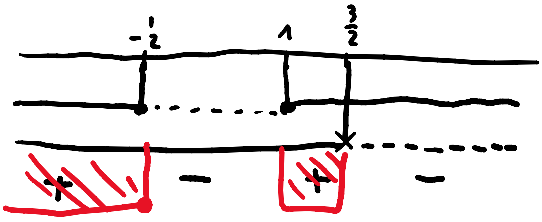


$x \geq 1 \qquad\qquad x \leq -\frac{1}{2}$

$x \geq 1 \quad \vee \quad x \leq -\frac{1}{2}$.

Denominatore: $3-2x \geq 0 \iff 2x \leq 3 \iff x \leq \frac{3}{2}$

Studio del segno della frazione:



soluzioni:

$x \leq -\frac{1}{2} \quad \vee \quad 1 \leq x < \frac{3}{2}$

2) $$\frac{|4e^x - 1| - 3}{2 - 3e^x} \geq 0$$

Poniamo $t = e^x$. Per il punto 1) si ha

$$\frac{|4e^x - 1| - 3}{2 - 3e^x} \geq 0 \iff \frac{|4t - 1| - 3}{2 - 3t} \geq 0$$

$$\iff t \leq -\frac{1}{2} \quad \vee \quad 1 \leq t < \frac{3}{2}$$

$$\iff e^x \leq -\frac{1}{2} \quad \vee \quad 1 \leq e^x < \frac{3}{2}$$

$$\iff 1 \leq e^x < \frac{3}{2}$$

$$\iff 0 \leq x < \log \frac{3}{2}$$

ESERCIZIO 2

a) Studiare la funzione $f(x) = \frac{x}{x+2} e^{\frac{1}{x^2}}$

b) Determinare l'immagine di f.

a) $f(x) = \frac{x}{x+2} e^{\frac{1}{x^2}}$

- **Dominio**: $\begin{cases} x+2 \neq 0 \\ x^2 \neq 0 \end{cases} \iff \begin{cases} x \neq -2 \\ x \neq 0 \end{cases}$

$$\mathrm{Dom}(f) = \mathbb{R} \setminus \{0,-2\} = (-\infty,-2) \cup (-2,0) \cup (0,+\infty)$$

- **Simmetrie**

Dom(f) non è simmetrico rispetto a 0. f non è pari né dispari.

- Int. con assi

$0 \notin \mathrm{Dom}(f) \Rightarrow$ il grafico non interseca l'asse y

$f(x)=0 \iff \dfrac{x e^{\frac{1}{x^2}}}{x+2} = 0 \iff x e^{\frac{1}{x^2}} = 0 \iff x=0$ ma $0 \notin \mathrm{Dom}(f)$

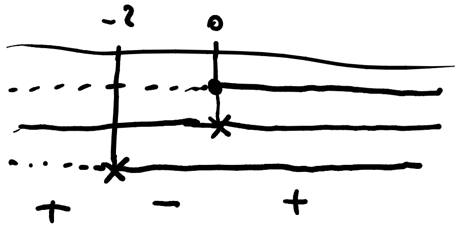
Il grafico non interseca gli assi cartesiani

- Segno: $\dfrac{x e^{\frac{1}{x}}}{x+2} > 0$

$x > 0 \iff x > 0$

$e^{\frac{1}{x}} > 0 \quad \forall x \in \mathbb{R} \setminus \{0\}$

$x+2 > 0 \iff x > -2$



$f(x) > 0 \iff x < -2 \lor x > 0$

- **Limiti**:

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \dfrac{x}{x+2} e^{\frac{1}{x^2}} = 1 \cdot e^0 = 1$

$\lim_{x \to 0} f(x)$ $\quad 0 \cdot +\infty$ f.i.

$\lim_{x \to 0} f(x) = \lim_{x \to 0^+} \dfrac{1}{x+2} x e^{\frac{1}{x^2}} = \dfrac{1}{2} \lim_{x \to 0^+} x e^{\frac{1}{x^2}} =$

$= \dfrac{1}{2} \lim_{x \to 0^+} \dfrac{e^{\frac{1}{x^2}}}{\frac{1}{x}} \overset{(D.L.H)}{=} \dfrac{1}{2} \lim_{x \to 0^+} \dfrac{e^{\frac{1}{x^2}} \cdot \left(-\frac{2}{x^3}\right)}{-\frac{1}{x^2}}$

$= \dfrac{1}{2} \lim_{x \to 0^+} e^{\frac{1}{x^2}} \cdot \dfrac{2}{x^3} \cdot x^2 = \dfrac{1}{2} \lim_{x \to 0^+} e^{\frac{1}{x^2}} \cdot \dfrac{2}{x}$

$$= \frac{1}{2} \cdot +\infty \cdot \frac{2}{0^+} = +\infty$$

In modo simile

$$\lim_{x \to 0^-} f(x) = \frac{1}{2} \lim_{x \to 0^-} e^{\frac{1}{x^2}} \cdot \frac{2}{x} = \frac{1}{2} \cdot +\infty \cdot \frac{2}{0^-} = -\infty$$

- $\lim_{x \to -2} f(x) = \lim_{x \to (-2)^-} \frac{x}{x+2} e^{\frac{1}{x^2}} = \frac{-2 e^{\frac{1}{4}}}{0^-} = +\infty$
- $\lim_{x \to (-2)^+} f(x) = \lim_{x \to (-2)^+} \frac{x}{x+2} e^{\frac{1}{x^2}} = \frac{-2 e^{\frac{1}{4}}}{0^+} = -\infty$
- $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x}{x+2} e^{\frac{1}{x^2}} = 1 \cdot e^0 = 1.$

- Derivata:

$$f(x) = \frac{x}{x+2} \cdot e^{\frac{1}{x^2}}$$

$$f'(x) = \left(\frac{x}{x+2}\right)' e^{\frac{1}{x^2}} + \frac{x}{x+2}\left(e^{\frac{1}{x^2}}\right)'$$

$$= \frac{x+2-x}{(x+2)^2} e^{\frac{1}{x^2}} + \frac{x}{x+2} e^{\frac{1}{x^2}} \cdot \left(-\frac{2}{x^3}\right)$$

$$= e^{\frac{1}{x^2}} \left(\frac{2}{(x+2)^2} - \frac{2}{(x+2)x^2} \right)$$

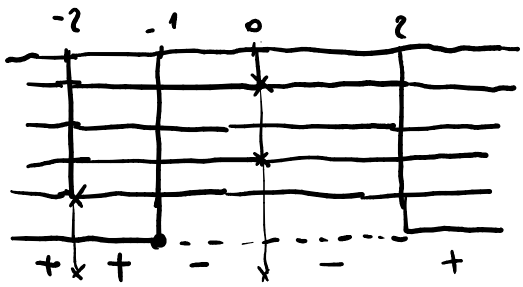
$$= 2e^{\frac{1}{x^2}} \frac{x^2-(x+2)}{x^2(x+2)^2} = \frac{2e^{\frac{1}{x^2}}(x^2-x-2)}{x^2(x+2)^2}$$

- Segno di f':

$2e^{\frac{1}{x^2}} > 0 \quad \forall x \in \mathbb{R} \setminus \{0\}$

$x^2 > 0 \quad \forall x \in \mathbb{R} \setminus \{0\}$

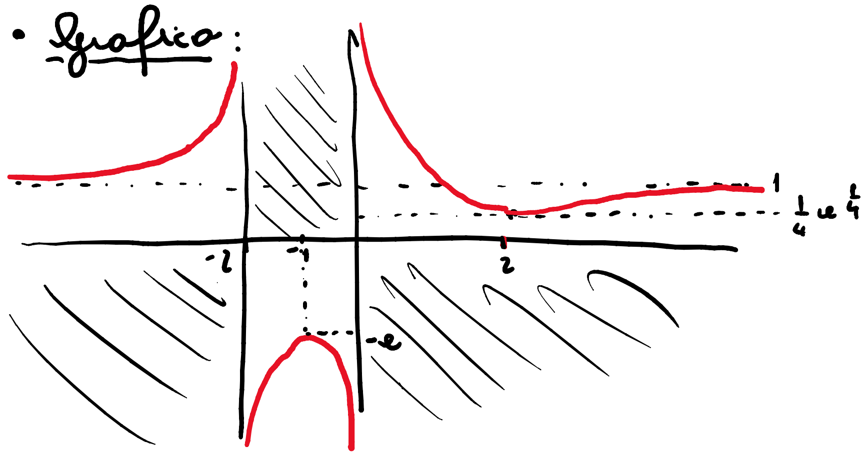
$(x+2)^2 > 0 \quad \forall x \in \mathbb{R} \setminus \{-2\}$



$x^2 - x - 2 \geq 0 \iff x \geq 2 \vee x \leq -1$

$\left(x_{1,2} = \frac{1 \pm \sqrt{9}}{2} \begin{matrix} \nearrow 2 \\ \searrow -1 \end{matrix} \right)$

• Grafico:



$f(2) = \frac{e^{\frac{1}{4}}}{4}$

$f(-1) = -e$

2) Dal grafico deduciamo che $Im(f) = (-\infty, -e) \cup (\frac{1}{4}e^{\frac{1}{4}}, +\infty)$

ES 3

Calcolare $\lim_{x \to 0} \frac{\operatorname{sen}(2x)\arctan x - 2x^2}{\sqrt{1+x^2} - e^{\frac{x^2}{2}}}$

Per $x \to 0$:

$\operatorname{sen} x = x - \frac{x^3}{6} + o(x^3)$

$\operatorname{sen} 2x = 2x - \frac{8}{6}x^3 + o(x^3) = 2x - \frac{4}{3}x^3 + o(x^3)$

$\arctan x = x - \frac{x^3}{3} + o(x^3)$

$\operatorname{sen}(2x)\arctan x = \left(2x - \frac{4}{3}x^3 + o(x^3)\right)\left(x - \frac{x^3}{3} + o(x^3)\right)$

$= 2x^2 - \frac{2}{3}x^4 + o(x^4) - \frac{4}{3}x^4 + o(x^4)$

$= 2x^2 - 2x^4 + o(x^4)$

Numeratore:

$\operatorname{sen}(2x)\arctan x = \cancel{2x^2} - 2x^4 + o(x^4) - \cancel{2x^2} = -2x^4 + o(x^4)$

$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$

$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4)$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$e^{\frac{x^2}{2}} = 1 + \frac{x^2}{2} + \frac{x^4}{8} + o(x^4)$$

<u>Denominatore</u>:

$$\sqrt{1+x^2} - e^{\frac{x^2}{2}} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4) - 1 - \frac{x^2}{2} - \frac{1}{8}x^4 + o(x^4) = -\frac{1}{4}x^4 + o(x^4)$$

Quindi:

$$\lim_{x\to 0} \frac{\sin(2x)\arctan x - 2x^2}{\sqrt{1+x^2} - e^{\frac{x^2}{2}}} = \lim_{x\to 0} \frac{-2x^4 + o(x^4)}{-\frac{1}{4}x^4 + o(x^4)} = \lim_{x\to 0} \frac{-2x^4}{-\frac{1}{4}x^4} = \frac{-2}{-\frac{1}{4}} = 8.$$

ES 4

a) $\displaystyle\int \frac{1+6x}{4x^2-3x-10}\,dx$

b) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+6\sin x)\cos x}{4\sin^2 x - 3\sin x - 10}\,dx$

a) Denominatore: $4x^2 - 3x - 10$

$$\Delta = 9 + 160 = 169 = 13^2 > 0$$

$$x_{1,2} = \frac{3 \pm 13}{8} = \begin{cases} 2 \\ -\frac{5}{4} \end{cases}$$

Quindi $4x^2 - 3x - 10 = 4(x-2)\left(x+\frac{5}{4}\right) = (x-2)(4x+5)$

Cerchiamo $A, B \in \mathbb{R}$ tali che:

$$\frac{1+6x}{4x^2-3x-10} = \frac{A}{x-2} + \frac{B}{4x+5} =$$

$$= \frac{A(4x+5) + B(x-2)}{(x-2)(4x+5)}$$

$$1+6x = 4Ax + 5A + Bx - 2B$$

$$\begin{cases} 4A+B=6 \\ 5A-2B=1 \end{cases} \Rightarrow \begin{cases} B = 6-4A \\ 5A-12+8A=1 \end{cases}$$

$$\Rightarrow \begin{cases} B = 6-4A \\ 13A = 13 \end{cases}$$

$$\Rightarrow \begin{cases} A = 1 \\ B = 6-4=2 \end{cases}$$

$$\int \frac{1+6x}{4x^2-3x-10}\,dx = \int \frac{1}{x-2}\,dx + 2\int \frac{1}{4x+5}\,dx$$

$$= \log|x-2| + \frac{1}{2}\log|4x+5| + C$$

b) $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+6\sin x)\cos x}{4\sin^2 x - 3\sin x - 10}\,dx$$

$$\int \frac{1+6\sin x}{4\sin^2 x - 3\sin x - 10}\,dx \overset{\substack{dy=\cos x\,dx \\ y=\sin x}}{=}$$

$$= \int \frac{1+6y}{4y^2-3y-10}\,dy$$

$$= \log|y-2| + \frac{1}{2}\log|4y+5| + C$$

$$= \log|\sin x - 2| + \frac{1}{2}\log|4\sin x + 5| + C$$

Quindi

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+6\sin x)\cos x}{4\sin^2 x - 3\sin x - 10}\,dx = \Big[\log|\sin x - 2| + \frac{1}{2}\log|4\sin x + 5|\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \log|1-2| + \frac{1}{2}\log|4+5| - \log|-1-2| - \frac{1}{2}\log|4-5|$$

$$= 0 + \frac{1}{2}\log 9 - \log 3 - 0 = 0$$

ESERCIZIO 5

Determinare la sol. generale dell'eq. differenziale

$$y'' - 4y' + 20y = 5x + 4$$

- Risolviamo l'eq. omogenea

$$y'' - 4y' + 20y = 5x + 4$$

$$p(t) = t^2 - 4t + 20$$

$$\Delta = 16 - 80 = -64 \qquad t_{1,2} = \frac{4 \pm \sqrt{-64}}{2} = \frac{4 \pm 8i}{2} = 2 \pm 4i$$

$$y_0(x) = C_1 e^{2x}\cos(4x) + C_2 e^{2x}\sin(4x)$$

- Cerchiamo una sol. particolare $\bar{y}(x) = Ax + B$

$$\bar{y}'(x) = A$$

$$\bar{y}''(x) = 0$$

$$0 - 4A + 20(Ax + B) = 5x + 4$$

$$-4A + 20Ax + 20B = 5x + 4$$

$$\begin{cases} 20A = 5 \\ -4A + 20B = 4 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{4} \\ -1 + 20B = 4 \end{cases}$$

$$\Rightarrow \begin{cases} A = \frac{1}{4} \\ 20B = 5 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{4} \\ B = \frac{1}{4} \end{cases}$$

$$\bar{y}(x) = \frac{1}{4}x + \frac{1}{4}$$

• la soluzione generale dell'eq. completa è

$$y(x) = \frac{1}{4}x + \frac{1}{4} + C_1 e^{2x}\cos(4x) + C_2 e^{2x}\sin(4x).$$

ESERCIZIO 6

1) Determinare la forma algebrica del numero complesso

$$w = \frac{7i-1}{1+3i}$$

2) Calcolare $(w-2+\sqrt{3})^7$

---

1) $$w = \frac{7i-1}{1+3i} = \frac{(7i-1)(1-3i)}{(1+3i)(1-3i)} = \frac{7i-1-21i^2+3i}{1+9}$$

$$= \frac{20+10i}{10} = 2+i$$

2) $$w-2+\sqrt{3} = \sqrt{3}+i$$

$$|w-2+\sqrt{3}| = \sqrt{3+1} = \sqrt{4} = 2$$

argomento: $\cos\theta = \frac{\sqrt{3}}{2}$, $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$$(w-2+\sqrt{3})^7 = 2^7\left(\cos\left(\frac{7}{6}\pi\right) + i\sin\left(\frac{7}{6}\pi\right)\right)$$

$$= 128\left(-\frac{\sqrt{3}}{2} - i\,\frac{1}{2}\right)$$

$$= -64\sqrt{3} - 64i.$$