LABORATORIO DI MATEMATICA

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Derivate di funzioni elementari

- $(c)' = 0 \qquad \left(\frac{d}{dx} c = 0\right)$
- $(x)' = 1 \qquad \left(\frac{d}{dx} x = 1\right)$
- $(cx)' = c$
- $(x^2)' = 2x$
- $(x^\alpha)' = \alpha x^{\alpha - 1}$
- $(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
- $(e^x)' = e^x$
- $(\log x)' = \frac{1}{x}$
- $(\log |x|)' = \frac{1}{x}$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \tan^2 x$
- $(\arctan x)' = \frac{1}{x^2+1}$
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

Regole di derivazione

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

$$(fg)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$(f(g(x)))' = f'(g(x))\, g'(x)$$ Derivata di una composizione.

ESEMPI

1) $(e^{2x})' = e^{2x} \cdot (2x)' = e^{2x} \cdot 2$

$f(x) = e^x, \; g(x) = 2x, \; f'(x) = e^x, \; g'(x) = 2$

$(f(g(x)))' = f'(g(x)) \cdot g'(x) = e^{2x} \cdot 2$

2) $\left(\cos(x^2)\right)' = -\sin(x^2) \cdot 2x$

3) $\left(e^{\sin x}\right)' = e^{\operatorname{sen} x} \cos x$

4) $\left(\log(3x+1)\right)' = \dfrac{1}{3x+1} \cdot 3$

5) $\left(\log(\sin x)\right)' = \dfrac{1}{\operatorname{sen} x} \cdot \cos x$.

Integrale:

Calcolare $\int f(x)\,dx$ significa trovare tutte le funzioni che hanno per derivata $f(x)$.

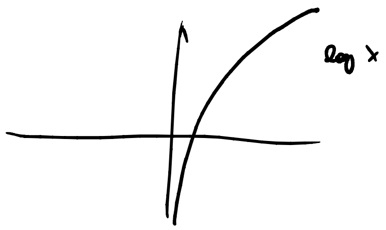
- $\int 1\,dx = x + C$
- $\int k\,dx = kx + c$
- $\int x\,dx = \frac{1}{2}x^2 + C$
- $\int x^\alpha\,dx = \dfrac{1}{1+\alpha}x^{1+\alpha} + C$ se $\alpha \neq -1$
- $\int \frac{1}{x}\,dx = \log|x| + c$

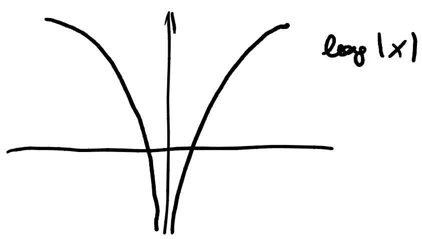
$$\log|x| = \begin{cases} \log x & \text{se } x > 0 \\ \log(-x) & \text{se } x < 0 \end{cases}$$

Allora:

se $x > 0$: $(\log x)' = \dfrac{1}{x}$

se $x < 0$: $(\log(-x))' = \dfrac{1}{-x} \cdot (-1) = \dfrac{1}{x}$





$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{1}{1-3}x^{1-3} + c = \frac{1}{-2}x^{-2} + c$$

infatti: $\left(\dfrac{x^{-2}}{-2}\right)' = \dfrac{\cancel{-2}\,x^{-3}}{\cancel{-2}} = x^{-3}$

- $\int e^x\,dx = e^x + c$

- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \frac{1}{x^2+1} \, dx = \arctan x + C$

Metodo per calcolare $\int \frac{p(x)}{q(x)} \, dx$ con $p(x)$ e $q(x)$ polinomi. Assumeremo che $\deg(p(x)) < \deg(q(x))$ e $\deg(q(x)) = 1$ o $\deg(q(x)) = 2$.

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1) Caso $\deg(q(x)) = 1$

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \log|ax+b| + C$$

$$\left(\text{infatti: } \left(\frac{1}{a} \log|ax+b|\right)' = \frac{1}{\cancel{a}} \frac{1}{ax+b} \cdot \cancel{a} = \frac{1}{ax+b} \right)$$

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$$\int \frac{1}{2x+1} \, dx = \frac{1}{2} \log|2x+1| + C$$

$$\int \frac{1}{x-3} \, dx = \log|x-3| + C$$

$$\int \frac{1}{1-x} \, dx = -\int \frac{1}{x-1} \, dx = -\log|x-1| + C$$

$$\int \frac{1}{1-4x} \, dx = -\int \frac{1}{4x-1} \, dx = -\frac{1}{4} \log|4x-1| + C$$

2) Caso in cui $\deg(q(x)) = 2$.

$$\int \frac{mx+q}{ax^2+bx+c} \, dx$$

Ci sono tre possibilità:

i) Δ del denominatore è > 0:

$ax^2+bx+c = 0$ ha due soluzioni x_1 e x_2.

L'idea è che $ax^2+bx+c = a(x-x_1)(x-x_2)$

possiamo cercare $A, B \in \mathbb{R}$ t.c.:

$$\frac{mx+q}{ax^2+bx+c} = \frac{A}{a(x-x_1)} + \frac{B}{x-x_2}$$

ESEMPI

- $$\int \frac{x+1}{x^2+x-2}\,dx \qquad x^2+x-2 = (x-1)(x+2)$$

$$\Delta = 1+8 = 9 > 0 \qquad x_{1,2} = \frac{-1\pm 3}{2} = \begin{cases} 1 \\ -2 \end{cases}$$

$$\frac{x+1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$= \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$= \frac{Ax+2A+Bx-B}{(x-1)(x+2)}$$

$$= \frac{(A+B)x+2A-B}{(x-1)(x-2)}$$

Vogliamo che $x+1 = (A+B)x + 2A - B$

$$\begin{cases} A+B = 1 \; * \\ 2A-B = 1 \; * \end{cases}$$

$$\begin{cases} B = 1-A \\ 2A-1+A = 1 \end{cases} \Rightarrow \begin{cases} B = 1-A \\ 3A = 2 \end{cases} \Rightarrow \begin{cases} B = 1-\frac{2}{3} = \frac{1}{3} \\ A = \frac{2}{3} \end{cases}$$

Abbiamo dimostrato che

$$\frac{x+1}{x^2+x-2} = \frac{\frac{2}{3}}{x-1} + \frac{\frac{1}{3}}{x+2}$$ quindi

$$\int \frac{x+1}{x^2+x-2}\,dx = \int \frac{\frac{2}{3}}{x-1} + \frac{\frac{1}{3}}{x+2}\,dx$$

$$= \frac{2}{3}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{1}{x+2}\,dx$$

$$= \frac{2}{3}\log|x-1| + \frac{1}{3}\log|x+2| + C$$

- $$\int \frac{2x-1}{3x^2-5x-2}\,dx$$

$$\Delta = 25+24 = 49 > 0 \qquad x_{1,2} = \frac{5\pm 7}{6} = \begin{cases} 2 \\ -\frac{1}{3} \end{cases}$$

Quindi

$$3x^2-5x-2 = 3(x-2)\left(x+\frac{1}{3}\right) = (x-2)(3x+1)$$

Cerchiamo $A, B \in \mathbb{R}$ t.c.

$$\frac{2x-1}{3x^2-5x-2} = \frac{A}{x-2} + \frac{B}{3x+1}$$
$$= \frac{A(3x+1)+B(x-2)}{(x-2)(3x+1)}$$

$$2x-1 = 3Ax + A + Bx - 2B$$
$$= (3A+B)x + A - 2B$$

$$\begin{cases} 3A+B=2 \\ A-2B=-1 \end{cases} \iff \begin{cases} B = 2-3A \\ A-2(2-3A)=-1 \end{cases}$$
$$\iff \begin{cases} B = 2-3A \\ A-4+6A=-1 \end{cases}$$
$$\iff \begin{cases} B = 2-3A \\ 7A = 3 \end{cases} \Rightarrow \begin{cases} B = 2-\frac{9}{7} = \frac{5}{7} \\ A = \frac{3}{7} \end{cases}$$

$$\int \frac{2x-1}{3x^2-5x-2}\,dy = \int \frac{\frac{3}{7}}{x-2} + \frac{\frac{5}{7}}{3x+1}\,dx$$
$$= \frac{3}{7}\log|x-2| + \frac{5}{7}\log|3x+1| \cdot \frac{1}{3} + C$$
$$= \frac{3}{7}\log|x-2| + \frac{5}{21}\log|3x+1| + C$$

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b) $\Delta = 0$

$ax^2+bx+c = a(x-x_0)^2$ dove x_0 è l'unica soluzione di $ax^2+bx+c=0$.

$$\int \frac{mx+q}{a(x-x_0)^2}\,dx = \frac{1}{a}\int \frac{mx+q}{(x-x_0)^2}\,dx$$

Cerchiamo A e $B \in \mathbb{R}$ tali che

$$\frac{mx+q}{(x-x_0)^2} = \frac{A}{(x-x_0)} + \frac{B}{(x-x_0)^2}$$

$$\int \frac{A}{x-x_0}\,dx = \log|x-x_0| + c$$

$$\int \frac{B}{(x-x_0)^2}\,dx = B\int (x-x_0)^{-2}\,dx = B \cdot \frac{1}{1-2}(x-x_0)^{1-2}$$
$$= -B\frac{1}{x-x_0} + c$$

ESEMPI

- $$\int \frac{x}{4x^2+4x+1}\,dx$$

$\Delta = 16 - 16 = 0$ e $x_0 = \dfrac{-4 \pm 0}{8} = -\dfrac{1}{2}$

$$4x^2+4x+1 = 4\left(x+\frac{1}{2}\right)^2$$

$$\int \frac{x}{4x^2+4x+1}\,dx = \frac{1}{4}\int \frac{x}{(x+\frac{1}{2})^2}\,dx \quad (*)$$

Cerchiamo A, B t.c.

$$\frac{x}{(x+\frac{1}{2})^2} = \frac{A}{x+\frac{1}{2}} + \frac{B}{(x+\frac{1}{2})^2} = \frac{A(x+\frac{1}{2})+B}{(x+\frac{1}{2})^2}$$

$$x = A\left(x+\frac{1}{2}\right) + B = Ax + \frac{1}{2}A + B$$

$$\begin{cases} A = 1 \\ \frac{1}{2}A + B = 0 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -\frac{1}{2} \end{cases}$$

$$\int \frac{x}{(x+\frac{1}{2})^2}\,dx = \int \frac{1}{x+\frac{1}{2}} - \frac{\frac{1}{2}}{(x+\frac{1}{2})^2}\,dx$$

$$= \log\left|x+\frac{1}{2}\right| + \frac{1}{2}\,\frac{1}{x+\frac{1}{2}} + C$$

Quindi $(*)$

$$\int \frac{x}{4x^2+4x+1}\,dx = \frac{1}{4}\log\left|x+\frac{1}{2}\right| + \frac{1}{8}\,\frac{1}{x+\frac{1}{2}} + \tilde{C}$$

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c) $\Delta < 0$

$ax^2+bx+c = 0$ ha due soluzioni complesse del tipo $\alpha \pm i\beta$

$$ax^2+bx+c = a\left((x-\alpha)^2 + \beta^2\right)$$

Si cercano A e B tali che

$$\frac{mx+q}{ax^2+bx+c} = \frac{A\,\overbrace{(2ax+b)}^{\text{derivata del denominatore}}}{ax^2+bx+c} + \frac{B}{ax^2+bx+c}$$

$$\int \frac{A(2ax+b)}{ax^2+bx+c}\,dx = A\int \frac{2ax+b}{ax^2+bx+c}\,dx$$

$$= A\log|ax^2+bx+c| + C$$

Per il secondo integrale

$$\int \frac{B}{ax^2+bx+c}\,dx = B\int \frac{1}{a(x-\alpha)^2+\beta^2}\,dx$$

$$= \frac{B}{a}\int \frac{1}{(x-\alpha)^2+\beta^2}\,dx$$

$$= \frac{B}{a}\,\frac{1}{\beta}\arctan\left(\frac{x-\alpha}{\beta}\right) + C$$

Ricordare che

$$\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C$$

e

$$\int \frac{1}{(x-\alpha)^2+\beta^2}\,dx = \frac{1}{\beta}\arctan\left(\frac{x-\alpha}{\beta}\right) + C$$

perché:

$$\left(\log|f(x)|\right)' = \frac{1}{f(x)}\cdot f'(x) = \frac{f'(x)}{f(x)}$$

$$\left(\frac{1}{\beta}\arctan\left(\frac{x-\alpha}{\beta}\right)\right)' = \frac{1}{\beta}\,\frac{1}{\left(\frac{x-\alpha}{\beta}\right)^2+1}\cdot\frac{1}{\beta}$$

$$= \frac{1}{\beta^2}\,\frac{1}{\frac{(x-\alpha)^2}{\beta^2}+1} = \frac{1}{(x-\alpha)^2+\beta^2}.$$

ESEMPIO

$$\int \frac{x+2}{x^2+x+2}\,dx$$

$\Delta = 1-8 = -7 < 0$

Radici complesse del denominatore:

$$\frac{-1\pm\sqrt{-7}}{2} = \frac{-1\pm i\sqrt{7}}{2} = -\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$$

$$\alpha = -\frac{1}{2}, \quad \beta = \frac{\sqrt{7}}{2}$$

$$x^2+x+2 = \left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2$$

$$\frac{x+2}{x^2+x+2} = \frac{A(2x+1)+B}{x^2+x+2}$$

$$x+2 = 2Ax + A + B$$

$$\begin{cases} 2A = 1 \\ A+B = 2 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = 2-\frac{1}{2} = \frac{3}{2} \end{cases}$$

$$\int \frac{x+2}{x^2+x+2}\,dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+2}\,dx + \int \frac{\frac{3}{2}}{x^2+x+2}\,dx$$

$$= \frac{1}{2}\log|x^2+x+2| + \frac{3}{2}\int \frac{1}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{7}}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\log|x^2+x+2| + \frac{3}{2}\,\frac{1}{\frac{\sqrt{7}}{2}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)+C$$

$$= \frac{1}{2}\log|x^2+x+2| + \frac{3}{\sqrt{7}}\arctan\left(\frac{2x+1}{\sqrt{7}}\right)+C$$

ESEMPIO 2

$$\int \frac{x-3}{x^2+4}$$

$$\Delta = 0-16 = -16 < 0$$

$$z = \frac{0 \pm \sqrt{-16}}{2} = \pm\frac{4i}{2} = \pm 2i \qquad \alpha = 0,\ \beta = 2$$

$$x^2+4 = x^2+2^2$$

$$\frac{x-3}{x^2+4} = \frac{A(2x)+B}{x^2+4} = A\,\frac{2x}{x^2+4} + B\cdot\frac{1}{x^2+4}$$

$$x-3 = 2Ax + B \quad \Rightarrow \begin{cases} 2A = 1 \\ B = -3 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = -3 \end{cases}$$

$$\int \frac{x-3}{x^2+4}\,dx = \frac{1}{2}\int \frac{2x}{x^2+4}\,dx - 3\int \frac{1}{x^2+4}\,dx$$

$$= \frac{1}{2}\log|x^2+4| - 3\int \frac{1}{x^2+(2)^2}\,dx$$

$$= \frac{1}{2}\log(x^2+4) - 3\,\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

$$= \frac{1}{2}\log(x^2+4) - \frac{3}{2}\arctan\left(\frac{x}{2}\right) + C.$$

In generale

$$\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C$$

$$\int \frac{1}{(x-\alpha)^2+\beta^2} = \frac{1}{\beta}\arctan\left(\frac{x-\alpha}{\beta}\right) + C$$

se $\alpha = 0$

$$\int \frac{1}{x^2+\beta^2}\,dx = \frac{1}{\beta}\arctan\left(\frac{x}{\beta}\right) + C$$

se $p=2$ $\int \frac{1}{x^2+(2)^2}\,dx = \frac{1}{2}\arctan\left(\frac{x}{2}\right)+C.$

ESERCIZI

① $\int \frac{1}{2x^2-5x-3}\,dx$

② $\int \frac{1}{2x^2-x-1}\,dx$

③ $\int \frac{x}{x^2+2x+5}\,dx$

④ $\int \frac{x}{x^2+6x+9}\,dx$

⑤ $\int \frac{1}{2x^2+x+1}\,dx$

Svolgimento:

① $\int \frac{1}{2x^2-5x-3}\,dx$

$\Delta = 25+24=49 \qquad x_{1,2} = \frac{5\pm 7}{4} = \begin{cases} 3 \\ -\frac{1}{2} \end{cases}$

$2x^2-5x-3 = 2(x-3)\left(x+\frac{1}{2}\right) = (x-3)(2x+1)$

Cerchiamo A e B tali che:

$$\frac{1}{2x^2-5x-3} = \frac{A}{x-3}+\frac{B}{2x+1} = \frac{A(2x+1)+B(x-3)}{(x-3)(2x+1)}$$

$$1 = x(2A+B)+A-3B$$

$$\begin{cases} 2A+B=0 \\ A-3B=1 \end{cases} \Rightarrow \begin{cases} B=-2A \\ 7A=1 \end{cases} \Rightarrow \begin{cases} B=-\frac{2}{7} \\ A=\frac{1}{7} \end{cases}$$

$$\int \frac{1}{2x^2-5x-3}\,dx = \frac{1}{7}\int\frac{1}{x-3}\,dx - \frac{2}{7}\int\frac{1}{2x+1}\,dx =$$

$$= \frac{1}{7}\log|x-3| - \frac{2}{7}\cdot\frac{1}{2}\log|2x+1|+C$$

$$= \frac{1}{7}\log|x-3| - \frac{1}{7}\log|2x+1|+C$$

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② $\int \frac{1}{2x^2-x-1}\,dx$

$\Delta = 1+8=9 \qquad x_{1,2} = \frac{1\pm 3}{4} = \begin{cases} 1 \\ -\frac{1}{2} \end{cases}$

$$2x^2 - x - 1 = 2(x-1)\left(x+\frac{1}{2}\right) = (x-1)(2x+1)$$

Cerchiamo $A, B \in \mathbb{R}$ tali che:

$$\frac{1}{2x^2-x-1} = \frac{A}{x-1} + \frac{B}{2x+1} = \frac{A(2x+1)+B(x-1)}{(x-1)(2x+1)}$$

$$1 = A(2x+1) + B(x-1)$$

$$1 = x(2A+B) + A - B$$

$$\begin{cases} 2A + B = 0 \\ A - B = 1 \end{cases} \qquad \begin{cases} B = -2A \\ 3A = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{3} \\ B = -\frac{2}{3} \end{cases}$$

$$\int \frac{1}{2x^2-x-1}\,dx = \frac{1}{3}\int \frac{1}{x-1} - \frac{2}{3}\int \frac{1}{2x+1}\,dx =$$

$$= \frac{1}{3}\log|x-1| - \frac{1}{3}\log|2x+1| + C$$

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3) $\displaystyle\int \frac{x}{x^2+2x+5}\,dx$

$$\Delta = 4 - 20 = -16 \qquad x_{1,2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 + 2i$$

$$x^2 + 2x + 5 = (x+1)^2 + 2^2$$

Cerchiamo $A, B \in \mathbb{R}$ tali che

$$\frac{x}{x^2+2x+5} = \frac{A(2x+2)}{x^2+2x+5} + \frac{B}{x^2+2x+5} = \frac{2Ax + 2A + B}{x^2+2x+5}$$

$$x = 2Ax + 2A + B$$

$$\begin{cases} 2A = 1 \\ 2A + B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = -2A = -1 \end{cases}$$

$$\int \frac{x}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx - \int \frac{1}{x^2+2x+5}\,dx$$

$$= \frac{1}{2}\log(x^2+2x+5) - \int \frac{1}{(x+1)^2+2^2}\,dx$$

$$= \frac{1}{2}\log(x^2+2x+5) - \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C$$

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4) $\displaystyle\int \frac{x}{x^2+6x+9}\,dx$

$\Delta = 36-36 = 0$: $\qquad x^2+6x+9 = (x+3)^2$

Cerchiamo $A, B \in \mathbb{R}$ t.c.

$$\frac{x}{x^2+6x+9} = \frac{A}{x+3} + \frac{B}{(x+3)^2} = \frac{A(x+3)+B}{(x+3)^2}$$

$$x = Ax + 3A + B$$

$$\begin{cases} A = 1 \\ 3A + B = 0 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -3 \end{cases}$$

$$\int \frac{x}{x^2+6x+9}\,dx = \int \frac{1}{x+3} - \frac{3}{(x+3)^2}\,dx$$

$$= \log|x+3| - 3\int (x+3)^{-2}\,dx$$

$$= \log|x+3| + \frac{3}{x+3} + C$$

5) $\displaystyle\int \frac{1}{2x^2+x+1}\,dx$

$\Delta = 1 - 8 = -7 < 0 \qquad x_{1,2} = \dfrac{-1 \pm \sqrt{-7}}{4} = \dfrac{-1 \pm \sqrt{7}\,i}{4} = -\dfrac{1}{4} \pm \dfrac{\sqrt{7}}{4} i$

$$2x^2+x+1 = 2\left(\left(x+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2\right)$$

$$\int \frac{1}{2x^2+x+1}\,dx = \int \frac{1}{2\left(\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2\right)}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}\,dx$$

$$= \frac{1}{2}\cdot\frac{4}{\sqrt{7}} \arctan\left(\frac{x+\frac{1}{4}}{\frac{\sqrt{7}}{4}}\right) + C$$

$$= \frac{2}{\sqrt{7}} \arctan\left(\frac{4x+1}{\sqrt{7}}\right) + C$$