LABORATORIO DI MATEMATICA 5

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ESERCIZIO

$$\lim_{x \to 0} \frac{\sqrt{1+x^2} - e^{\frac{x^2}{2}}}{2\cos x - 2 + x^2}$$

$\frac{0}{0}$ f.i.

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$$

$$D(x) = 2\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)\right) - 2 + x^2$$

$$= \cancel{2} - \cancel{x^2} + \frac{1}{12}x^4 + \underbrace{2o(x^4)}_{o(x^4)} - \cancel{2} + \cancel{x^2}$$

$$= \frac{1}{12}x^4 + o(x^4)$$

$$\sqrt{1+x^2} \overset{y=x^2}{=} \sqrt{1+y} = (1+y)^{\frac{1}{2}}$$

$(1+x)^\alpha$ con $\alpha = \frac{1}{2}$

$$= 1 + \alpha y + \frac{\alpha(\alpha-1)}{2!}y^2 + o(y^2)$$

$$= 1 + \frac{1}{2}y + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}y^2 + o(y^2)$$

$$= 1 + \frac{1}{2}y - \frac{1}{8}y^2 + o(y^2)$$

$$= 1 + \frac{1}{2}x^2 - \frac{1}{8}\underbrace{x^4}_{(x^2)^2 = x^4} + o(x^4)$$

$$\boxed{\frac{1}{2} - 1 = -\frac{1}{2} \qquad \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{2} = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}}$$

$$e^{\frac{x^2}{2}} \overset{y=\frac{x^2}{2}}{=} 1 + y + \frac{y^2}{2} + o(y^2)$$

$$= 1 + \frac{x^2}{2} + \frac{\left(\frac{x^2}{2}\right)^2}{2} + o\left(\left(\frac{x^2}{2}\right)^2\right)$$

$$= 1 + \frac{x^2}{2} + \frac{1}{2}\cdot\frac{x^4}{4} + o(x^4)$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{8} + o(x^4)$$

$$\boxed{\frac{1}{2}\left(\frac{x^2}{2}\right)^2 = \frac{1}{2}\,\frac{x^4}{4} = \frac{x^4}{8}}$$

$$N(x) = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4) - \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + o(x^4)\right)$$

$$= \cancel{1} + \cancel{\frac{1}{2}x^2} - \frac{1}{8}x^4 + o(x^4) - \cancel{1} - \cancel{\frac{x^2}{2}} - \frac{x^4}{8} + o(x^4)$$

$$= -\frac{1}{4}x^4 + o(x^4)$$

$$\left(-\frac{1}{8} - \frac{1}{8} = -\frac{1}{4}\right)$$

$$\lim_{x\to 0} \frac{-\frac{1}{4}x^4 + o(x^4)}{\frac{1}{12}x^4 + o(x^4)} = \lim_{x\to 0} \frac{-\frac{1}{4}x^4}{\frac{1}{12}x^4} = \frac{-\frac{1}{4}}{\frac{1}{12}} = -\frac{1}{4}\cdot 12 = -3$$

ESERCIZIO

$$\lim_{x\to 0} \frac{\sin^2 x - x\arctan x}{e^{2x^2}\cos x - 1 - \frac{3}{2}x^2}$$

Numeratore:

$$(a+b)^2 = a^2+b^2+2ab$$

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\sin^2 x = \left(x - \frac{x^3}{6} + o(x^3)\right)^2$$

$$= x^2 + \frac{x^6}{36} + o(x^6) + 2x\cdot\left(-\frac{x^3}{6}\right) + 2x\, o(x^3) - \frac{2x^3}{6}o(x^3)$$

$$= x^2 + \frac{x^6}{36} + o(x^6) - \frac{x^4}{3} + o(x^4) + o(x^6)$$

$$= x^2 + \frac{x^6}{36} - \frac{x^4}{3} + o(x^4)$$

$$= x^2 - \frac{x^4}{3} + o(x^4) + \underbrace{\frac{x^6}{36}}_{=o(x^4)}$$

$$= x^2 - \frac{x^4}{3} + o(x^4)$$

$$-x\arctan x = -x\left(x - \frac{x^3}{3} + o(x^3)\right)$$

$$= -x^2 + \frac{x^4}{3} + o(x^4)$$

$$N(x) = \cancel{x^2} - \cancel{\frac{x^4}{3}} + o(x^4) - \cancel{x^2} + \cancel{\frac{x^4}{3}} + o(x^4)$$

$$= o(x^4)$$

Non basta!

$$\sin^2 x = \left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5) \right)^2$$

$$= x^2 + \frac{x^6}{36} - \frac{x^4}{3} + \frac{x^6}{60} + o(x^6)$$

$$= x^2 - \frac{x^4}{3} + \frac{2}{45} x^6 + o(x^6) \qquad \left(\frac{1}{36} + \frac{1}{60} = \frac{2}{45} \right)$$

$$-x \arctan x = -x\left(x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^5) \right)$$

$$= -x^2 + \frac{x^4}{3} - \frac{x^6}{5} + o(x^6)$$

$$N(x) = \frac{2}{45} x^6 - \frac{1}{5} x^6 + o(x^6) = -\frac{7}{45} x^6 + o(x^6)$$

$$D(x) = e^{2x^2} \cos x - 1 - \frac{3}{2} x^2$$

$$e^{2x^2} \overset{y = 2x^2}{=} 1 + y + \frac{1}{2} y^2 + o(y^2)$$

$$= 1 + 2x^2 + \frac{1}{2}(2x^2)^2 + o((2x^2)^2)$$

$$= 1 + 2x^2 + 2x^4 + o(x^4)$$

$$\cos x = 1 - \frac{1}{2} x^2 + \frac{x^4}{24} + o(x^4)$$

$$e^{2x^2} \cos x = \left(1 + 2x^2 + 2x^4 + o(x^4) \right) \left(1 - \frac{1}{2} x^2 + \frac{x^4}{24} + o(x^4) \right)$$

$$= 1 - \frac{1}{2} x^2 + \frac{x^4}{24} + o(x^4) + 2x^2 - x^4 + 2x^4$$

$$= 1 + \frac{3}{2} x^2 + \frac{25}{24} x^4 + o(x^4)$$

$$D(x) = \frac{25}{24} x^4 + o(x^4)$$

$$\lim_{x \to 0} \frac{-\frac{7}{45} x^6 + o(x^6)}{\frac{25}{24} x^4 + o(x^4)} = \lim_{x \to 0} \frac{-\frac{7}{45} x^{\cancel{6}\,2}}{\frac{25}{24} \cancel{x^4}} = \lim_{x \to 0} \frac{-\frac{7}{45}}{\frac{25}{24}} x^2$$

$$= 0.$$

ESERCIZIO

- $$\lim_{x\to 0} \frac{\arctan(2x) - 2\sqrt{1+2x} + 2}{\log(1+5x) - 5\sin x}$$ (Risultato $-\frac{2}{25}$)

- $$\lim_{x\to 0} \frac{(e^{2x}-1)^2 - 4x\sin x}{x\cos x - \sin x}$$ (Risultato -24)

- $$\lim_{x\to 0} \frac{x\cos(2x) + e^{-x} - 1}{(\ln(1+\sqrt{x}) - \sqrt{x})^2}$$ (Risultato: 2)

Nota sull'inversione delle frazioni:

$$\frac{-\frac{1}{4}}{2} = -\frac{1}{4}\cdot\frac{1}{2} = -\frac{1}{8}$$

$$\frac{\frac{a}{b}}{c} = \frac{a}{b}\cdot\frac{1}{c} = \frac{a}{bc} \quad \text{mentre} \quad \frac{a}{\frac{b}{c}} = a\cdot\frac{c}{b} = \frac{ac}{b}$$

ESEMPI

- $$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{1}{2}\cdot 3$$

- $$\frac{\frac{1}{2}}{3} = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$$

- $$\frac{\frac{4}{3}}{\frac{1}{2}} = \frac{4}{3}\cdot 2 = \frac{8}{3}$$

- $$\frac{\frac{4}{3}}{2} = \frac{4}{3}\cdot\frac{1}{2} = \frac{4}{6} = \frac{2}{3}$$

- $$\frac{1}{-\frac{25}{2}} = 1\cdot\left(-\frac{2}{25}\right) = -\frac{2}{25}$$

Soluzioni degli esercizi

1) $$\lim_{x\to 0} \frac{\arctan(2x) - 2\sqrt{1+2x} + 2}{\ln(1+5x) - 5\sin x}$$

- Denominatore $D(x) = \ln(1+5x) - 5\sin x$

$$\ln(1+5x) \overset{y=5x}{=} y - \frac{1}{2}y^2 + o(y^2) = 5x - \frac{1}{2}(5x)^2 + o((5x)^2)$$

$$= 5x - \frac{25}{2}x^2 + o(x^2)$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$D(x) = 5x - \frac{25}{2}x^2 + o(x^2) - 5\left(x - \frac{x^3}{6} + o(x^3)\right)$$

$$= \cancel{5x} - \frac{25}{2}x^2 + o(x^2) - \cancel{5x} + \frac{5}{6}x^3 + o(x^3)$$

$$= -\frac{25}{2}x^2 + o(x^2) + \underbrace{\frac{5}{6}x^3 + o(x^3)}_{\text{Trascurabili rispetto a } x^2 \ (=o(x^2))}$$

$$= -\frac{25}{2}x^2 + o(x^2)$$

- Numeratore $N(x) = \arctan(2x) - 2\sqrt{1+2x} + 2$

$$\arctan(2x) \overset{y=2x}{=} y - \frac{y^3}{3} + o(y^3) = 2x - \frac{(2x)^3}{3} + o((2x)^3)$$

$$= 2x - \frac{8}{3}x^3 + o(x^3)$$

$$\sqrt{1+2x} \overset{y=2x}{=} \sqrt{1+y}$$

$$= 1 + \frac{1}{2}y - \frac{1}{8}y^2 + o(y^2)$$

$$= 1 + \frac{1}{2}\cdot 2x - \frac{1}{8}\cdot(2x)^2 + o((2x)^2)$$

$$= 1 + x - \frac{1}{8}\cdot 4x^2 + o(x^2)$$

$$= 1 + x - \frac{1}{2}x^2 + o(x^2)$$

$$N(x) = 2x - \frac{8}{3}x^3 + o(x^3) - 2\left(1 + x - \frac{1}{2}x^2 + o(x^2)\right) + 2$$

$$= \cancel{2x} - \frac{8}{3}x^3 + o(x^3) \cancel{-2} \cancel{-2x} + x^2 + o(x^2) \cancel{+2}$$

$$= -\frac{8}{3}x^3 + o(x^3) + x^2 + o(x^2)$$

$$= x^2 + o(x^2)$$

- $$\lim_{x\to 0} \frac{x^2 + o(x^2)}{-\frac{25}{2}x^2 + o(x^2)} = \lim_{x\to 0} \frac{x^2}{-\frac{25}{2}x^2} = \frac{1}{-\frac{25}{2}} = -\frac{2}{25}$$

ESERCIZIO 2

$$\lim_{x\to 0} \frac{(e^{2x}-1)^2 - 4x\sin x}{x\cos x - \sin x}$$

- Denominatore $(D(x) = x\cos x - \operatorname{sen} x)$

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$$

$$x\cos x = x - \frac{1}{2}x^3 + o(x^3)$$

$$\operatorname{sen} x = x - \frac{1}{6}x^3 + o(x^3)$$

$$D(x) = x - \frac{1}{2}x^3 + o(x^3) - \left(x - \frac{1}{6}x^3 + o(x^3)\right)$$

$$= \cancel{x} - \frac{1}{2}x^3 + o(x^3) - \cancel{x} + \frac{1}{6}x^3 + o(x^3)$$

$$= -\frac{1}{2}x^3 + \frac{1}{6}x^3 + o(x^3)$$

$$= -\frac{1}{3}x^3 + o(x^3)$$

- Numeratore:

$$e^{2x} \overset{y=2x}{=} 1 + y + \frac{1}{2}y^2 + o(y^2)$$

$$= 1 + 2x + \frac{1}{2}(2x)^2 + o(x^2)$$

$$= 1 + 2x + 2x^2 + o(x^2)$$

$$(e^{2x}-1)^2 = (2x + 2x^2 + o(x^2))^2$$

$$= 4x^2 + 4x^4 + o(x^4) + 8x^3 + o(x^3) + o(x^4)$$

$$= 4x^2 + 8x^3 + o(x^3)$$

$$\text{sen}\, x = x - \frac{1}{6}x^3 + o(x^3)$$

$$-4x\,\text{sin}\, x = -4x^2 + \frac{2}{3}x^4 + o(x^4)$$

$$N(x) = 4x^2 + 8x^3 + o(x^3) - 4x^2 + \frac{2}{3}x^4 + o(x^4)$$

$$= 8x^3 + o(x^3)$$

- Conclusione:

$$\lim_{x\to 0} \frac{N(x)}{D(x)} = \lim_{x\to 0} \frac{8x^3 + o(x^3)}{-\frac{1}{3}x^3 + o(x^3)} = \lim_{x\to 0} \frac{8x^3}{-\frac{1}{3}x^3} = \frac{8}{-\frac{1}{3}} -$$

$$= -8 \cdot 3 = -24.$$

ESERCIZIO 3

$$\lim_{x\to 0} \frac{x\cos(2x) + ce^{-x} - 1}{(\ln(1+\sqrt{x}) - \sqrt{x})^2}$$

- Denominatore $(D(x))$:

$$\ln(1+\sqrt{x}) \overset{y=\sqrt{x}}{=} \ln(1+y)$$

$$= y - \frac{1}{2}y^2 + o(y^2)$$

$$= \sqrt{x} - \frac{1}{2}(\sqrt{x})^2 + o(\sqrt{x}^2)$$

$$= \sqrt{x} - \frac{1}{2}x + o(x)$$

$$D(x) = \left(\cancel{\sqrt{x}} - \frac{1}{2}x + o(x) - \cancel{\sqrt{x}}\right)^2$$

$$= \left(-\frac{1}{2}x + o(x)\right)^2$$

$$= \frac{1}{4}x^4 + o(x^2) + o(x^2)$$

$$= \frac{1}{2}x^4 + o(x^4)$$

- Numeratore:

$$\cos 2x \overset{y=2x}{=} \cos y = 1 - \frac{1}{2}y^2 + o(y^2)$$

$$= 1 - \frac{1}{2}(2x)^2 + o(x^2)$$

$$= 1 - \frac{1}{2}\cdot 4x^2 + o(x^2)$$

$$= 1 - 2x^2 + o(x^2)$$

$$e^{-x} \overset{y=-x}{=} e^y = 1 + y + \frac{1}{2}y^2 + o(y^2)$$

$$= 1 - x + \frac{1}{2}(-x)^2 + o((-x)^2)$$

$$= 1 - x + \frac{1}{2}x^2 + o(x^2)$$

$$N(x) = x\left(1 - 2x^2 + o(x^2)\right) + 1 - x + \frac{1}{2}x^2 + o(x^2) - 1$$

$$= \cancel{x} - 2x^3 + o(x^3) \cancel{+1} - \cancel{x} + \frac{1}{2}x^2 + o(x^2) \cancel{-1}$$

$$= -2x^3 + o(x^3) + \frac{1}{2}x^2 + o(x^2)$$

$$= \frac{1}{2}x^2 + o(x^2)$$

$$\lim_{x\to 0} \frac{N(x)}{D(x)} = \lim_{x\to 0} \frac{\frac{1}{2}x^2 + o(x^2)}{\frac{1}{4}x^2 + o(x^2} = \lim_{x\to 0} \frac{\frac{1}{2}x^2}{\frac{1}{4}x^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = \frac{1}{2}\cdot 4 = 2$$