Formula di Taylor col resto di Peano

Sia $f:]a, b[\to \mathbb{R}$ derivabile $n-1$ volte in $]a, b[$ e derivabile n-volte in $x_0 \in]a, b[$

Posto

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

T_n si chiama polinomio di Taylor di ordine n di centro x_0 associato ad f ed è l'unico polinomio di grado $\leq n$ t.c.

$$f(x) = T_n(x) + o((x - x_0)^n) \quad \text{OSSIA}$$

$$\lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^n} = 0$$

OSS. Se $x_0 = 0$ il polinomio di Taylor di centro 0 si chiama anche POLINOMIO DI MACLAURIN

ESEMPI DI POLINOMI DI MACLAURIN

$f(x) = e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n) =$$

$$= \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + o(x^{2n+1}) =$$

$$= \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2m+1})$$

$2! = 2$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{(-1)^m x^{2m}}{(2m)!} + o(x^{2m})$$

$$= \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2m}).$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{m-1} x^m}{m} + o(x^m)$$

$$= \sum_{k=1}^{m} (-1)^{k-1} \frac{x^k}{k} + o(x^m)$$

$$(1-x)^{-1} = \frac{1}{1-x} = \sum_{k=0}^{n} x^k + o(x^n)$$

$$(1+x)^{-1} = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{n} (-1)^k x^k + o(x^n)$$

$\alpha \in \mathbb{R}$, $\alpha \neq -1$ $\alpha \neq 0$

$f(x) = (1+x)^\alpha$ Trovare il polinomio di Maclaurin associato a f

Ho bisogno di alcune notazioni

$k \in \mathbb{N}, \; k \geq 1$ $\qquad \alpha \in \mathbb{R}, \quad \alpha \neq -1$

$\alpha \neq 0$

$$\binom{\alpha}{k} \doteq \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!}$$

$k=1$

$$\binom{\alpha}{1} = \frac{\alpha}{1!} = \alpha.$$

$k=2$

$$\binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2!} = \frac{\alpha(\alpha-1)}{2}.$$

$k=3$

$$\binom{\alpha}{3} = \frac{\alpha(\alpha-1)(\alpha-2)}{3!} = \frac{\alpha(\alpha-1)(\alpha-2)}{6}.$$

Per convenzione $\binom{\alpha}{0} = 1$

$f(x) = (1+x)^{\alpha}$

$\frac{f^{(k)}(0)}{k!} = \binom{\alpha}{k}$

$$(1+x)^{\alpha} = \sum_{k=0}^{n} \binom{\alpha}{k} x^{k} + o(x^{n})$$

ESERCIZIO

Scrivere il polinomio di Maclaurin T_1 e T_2 della funzione

$x \geq -1$

$$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$$

$$T_1(x) = \sum_{k=0}^{1} \binom{\frac{1}{2}}{k} x^{k} = \binom{\frac{1}{2}}{0} x^{0} + \binom{\frac{1}{2}}{1} x^{1} =$$

$$= 1 + \frac{1}{2} x$$

$$T_2(x) = \sum_{k=0}^{2} \binom{\frac{1}{2}}{k} x^k = \binom{\frac{1}{2}}{0} x^0 +$$

$$+ \binom{\frac{1}{2}}{1} x^1 + \binom{\frac{1}{2}}{2} x^2 =$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} x^2 =$$

$$= 1 + \frac{1}{2}x + \frac{-\frac{1}{4}}{2} x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

In definitive

$$\sqrt{1+x} = T_2(x) + o(x^2) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$$

Trovare T_1 e T_2 di ESERCIZIO

$$f(x) = \frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$$

$x > -1$

$$T_1(x) = 1 - \frac{1}{2}x$$

$$T_2(x)\ 1 - \frac{1}{2}x + \frac{3}{8}x^2$$

In particolare

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + o(x^2)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + o(x^2)$$

ESERCIZI

$$\lim_{x \to 0} \frac{e^{x^2} - \cos x - \frac{3}{2}x^2}{x^4}$$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$x \to x^2$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + o(x^4)$$

$4! = 24$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

$$\lim_{x \to 0} \frac{e^{x^2} - \cos x - \frac{3}{2}x^2}{x^4} =$$

$$= \lim_{x \to 0} \frac{\left(\cancel{1} + \cancel{x^2} + \frac{x^4}{2} + o(x^4)\right) - \cancel{1} + \cancel{\frac{x^2}{2}} - \frac{x^4}{24} + o(x^4)\Big) - \cancel{\frac{3}{2}x^2}}{x^4}$$

$$= \lim_{x \to 0} \frac{\left(\frac{1}{2} - \frac{1}{24}\right) x^4 + o(x^4)}{x^4} =$$

$$= \lim_{x \to 0} \frac{\frac{11}{24}\cancel{x^4}}{\cancel{x^4}} + \underbrace{\frac{o(x^4)}{x^4}}_{\to 0} = \frac{11}{24}$$

$$\lim_{x \to +\infty} \left(x - x^2 \lg\left(1 + \frac{1}{x}\right)\right)$$

$$\lg(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

$$\lg\left(1+\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} + o\left(\frac{1}{x^3}\right)$$

$$\lim_{x \to +\infty} \left(x - x^2 \cdot \lg\left(1 + \frac{1}{x}\right)\right) =$$

$$= \lim_{x \to +\infty} \left(x - x^2 \left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} + o\left(\frac{1}{x^3}\right)\right)\right) =$$

$$\lim_{x\to+\infty}\left(\cancel{x}-\cancel{x}+\frac{1}{2}-\frac{1}{3x}+x^2 o\left(\frac{1}{x^3}\right)\right)$$

$$=\frac{1}{2}-\underbrace{\lim_{x\to+\infty}\frac{1}{3x}}_{\downarrow\, 0}+\underbrace{\lim_{x\to+\infty} x^2 o\left(\frac{1}{x^3}\right)}_{\downarrow\, 0}=\frac{1}{2}$$

$$x^2 o\left(\frac{1}{x^3}\right)=o\left(\frac{x^2}{x^3}\right)=o\left(\frac{1}{x}\right)$$

OSS: Potevo fermarmi a $n=2$

$$\lg(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$\lg\left(1+\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)$$

$$\lim_{x\to+\infty}\left(x - x^2\lg\left(1+\frac{1}{x}\right)\right) =$$

$$= \lim_{x\to+\infty} x - x^2\left(\frac{1}{x} - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right) =$$

$$= \lim_{x\to+\infty} \cancel{x} - \cancel{x} + \frac{1}{2} - x^2 o\left(\frac{1}{x^2}\right) = \frac{1}{2} +$$

$$0 \quad - \lim_{x\to+\infty} x^2 o\left(\frac{1}{x^2}\right) = \frac{1}{2} + 0 = \frac{1}{2}$$

Se mi fossi fermato a $n=1$

$$\lg(1+x) = x + o(x)$$

$$\lg\left(1+\frac{1}{x}\right) = \frac{1}{x} + o\left(\frac{1}{x}\right)$$

$$\lim_{x\to+\infty}\left(x - x^2\lg\left(1+\frac{1}{x}\right)\right) =$$

$$= \lim_{x\to+\infty} x - x^2\left(\frac{1}{x} + o\left(\frac{1}{x}\right)\right) =$$

$$= \lim_{x\to+\infty} \cancel{x} - \cancel{x} - x^2 o\left(\frac{1}{x}\right) = \lim_{x\to+\infty} x^2 o\left(\frac{1}{x}\right)$$

???

$$\lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^3 (e^x - \cos x)}$$

$$\sin x = x - \frac{x^3}{3!} + o(x^3) = x - \frac{x^3}{6} + o(x^3)$$

- $$\sin^2 x = \left(x - \frac{x^3}{6} + o(x^3) \right)^2 =$$

$$= x^2 + \frac{x^6}{36} - \frac{1}{3} x^4 + o(x^6)$$

- $$e^x = 1 + x + o(x)$$
- $$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

$$\lim_{x\to 0} \frac{\cancel{x^2} - \cancel{x^2} + \frac{1}{3}x^4 - \frac{x^6}{36} + o(x^6)}{x^3 \cdot \left(\cancel{1} + x + o(x) - \cancel{1} + \frac{x^2}{2} + o(x^2)\right)} =$$

$$= \lim_{x\to 0} \frac{x^4\left(\frac{1}{3} - \frac{x^2}{36} + \frac{o(x^6)}{x^4}\right)}{x^3\left(x + \frac{x^2}{2} + o(x^2)\right)} =$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}\left(\frac{1}{3} - \frac{x^2}{36} + o\left(\frac{x^6}{x^4}\right)\right)}{\cancel{x^4}\left(1 + \frac{x}{2} + \frac{o(x^2)}{x}\right)} =$$

$$\frac{o(x^2)}{x} = o\left(\frac{x^2}{x}\right) = o(x).$$

$$= \lim_{x\to 0} \frac{\frac{1}{3} - \overbrace{\frac{x^2}{36}}^{\to 0} + \overbrace{o(x^2)}^{\to 0}}{1 + \underbrace{\frac{x}{2}}_{\downarrow 0} + \underbrace{o(x)}_{\downarrow 0}} = \frac{1}{3}$$

- $$\lim_{x\to 0} \frac{\sin(3x)\lg(1+2x) - 6x^2 + 6x^3}{x^4}$$

$$\sin x = x - \frac{x^3}{3!} + o(x^3) = x - \frac{x^3}{6} + o(x^3)$$

$$\sin(3x) = 3x - \frac{9}{2}x^3 + o(x^3)$$

$$\lg(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$\lg(1+2x) = 2x - 2x^2 + o(x^2)$$

$$\lim_{x \to 0} \frac{\sin(3x)\,\lg(1+2x) - 6x^2 + 6x^3}{x^4} =$$

$$= \lim_{x \to 0} \frac{\left(3x - \frac{9}{2}x^3 + o(x^3)\right)\left(2x - 2x^2 + o(x^2)\right) - 6x^2 + 6x^3}{x^4}$$

$$= \lim_{x \to 0} \frac{\cancel{6x^2} - \cancel{6x^3} - 3x\,o(x^2) - 9x^4 + 9x^5 - \frac{9}{2}x^3\,o(x^2) + 2x\,o(x^3) - 2x^2\,o(x^3) + o(x^2)\cdot o(x^3) \cancel{-6x^2} \cancel{+6x^3}}{x^4}$$

$$= \lim_{x\to 0} \frac{-9x^4 + 9x^5 + o(x^5) + o(x^4) + o(x^5) + o(x^6)}{x^4} =$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}\left(-9 + \underbrace{0x}_{\to 0} + \underbrace{\frac{o(x^5)}{x^4}}_{\to 0} + \underbrace{\frac{o(x^4)}{x^4}}_{\to 0} + \underbrace{\frac{o(x^6)}{x^4}}_{\to 0}\right)}{\cancel{x^4}}$$

$$= -9$$

$$\lim_{x\to 0} \frac{e^{3x} - 3x - \sqrt{1+9x^2}}{x^2 + x \lg(1-x)}$$

$$\lg(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$\lg(1-x) = -x - \frac{x^2}{2} + o(x^2)$$

$$x^2 + x\lg(1-x) = \cancel{x^2} - \cancel{x^2} - \frac{x^3}{2} + x\,o(x^2)$$

$$= -\frac{x^3}{2} + o(x^3)$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$$

$$e^{3x} = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + o(x^3)$$

$$\sqrt{1+x} = 1+\frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$$

$$\sqrt{1+9x^2} = 1 + \frac{9x^2}{2} - \frac{81}{8}x^4 + o(x^4)$$

$$\lim_{x\to 0} \frac{e^{3x} - 3x - \sqrt{1-9x^2}}{x^2 + x\lg(1-x)} =$$

$$= \lim_{x\to 0} \frac{\cancel{1}+\cancel{3x}+\cancel{\frac{9}{2}x^2}+\frac{9}{2}x^3+o(x^3) - \cancel{3x} - \cancel{1} - \cancel{\frac{9}{2}x^2} + \frac{81}{8}x^4 + o(x^3)}{-\frac{x^3}{2}+o(x^3)}$$

$$= \lim_{x \to 0} \frac{-\frac{9}{2}x^3 + \frac{81}{8}x^4 + o(x^3) + o(x^4)}{-\frac{x^3}{2} + o(x^3)} =$$

$$\frac{-\frac{9}{2}}{-\frac{1}{2}} = 9$$