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# Multiplicity results for critical growth problems involving the *p*-Grushin operator

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**Abstract**
We prove the existence of multiple nontrivial solutions for degenerate elliptic critical growth problems of the form

$$\begin{cases} -\Delta_{\gamma,p} u = \lambda |u|^{r-2} u + |u|^{p_\gamma^*-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{\gamma,p}$ denotes the Grushin $p$-Laplace operator, $\Omega \subset \mathbb{R}^N$ is a bounded domain intersecting the degeneration set of the operator, $\lambda > 0$ and $p < r < p_\gamma^*$, where $p_\gamma^*$ is the Grushin critical exponent. We also investigate critical Grushin $(p, q)$-Laplacian type problems, that is, Dirichlet critical growth problems governed by the operator

$$\mathcal{L} := -\Delta_{\gamma,p} - \Delta_{\gamma,q}.$$

In particular, we prove that the considered problems have arbitrarily many solutions for all sufficiently large $\lambda$. The results are based on the application of the abstract multiplicity theorems established by Perera in [27].

**Keywords** Variational methods · $p$-Grushin operator · Critical nonlinearity · Multiplicity result · $(p, q)$-Grushin Laplacian problem

**Mathematics Subject Classification** 35A15 · 35J70 · 35B38

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## 1 Introduction

In this paper we provide existence of multiple nontrivial solutions for a class of degenerate-elliptic critical growth equations involving the so-called Grushin $p$-Laplace operator. By analogous methods, we also investigate critical $(p, q)$-Grushin Laplacian type problems. The results are obtained by applying the abstract multiplicity results established by Perera in [27], which are based on the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [10].

Our first main result concerns the following degenerate-elliptic critical problem

$$\begin{cases} -\Delta_{\gamma,p} u = \lambda |u|^{r-2}u + |u|^{p_\gamma^*-2}u & \text{in } \Omega \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{\gamma,p}$ denotes the so-called Grushin $p$-Laplace operator, $1 < p < N_\gamma$, defined as

$$\Delta_{\gamma,p} u := \sum_{i=1}^{N} X_i(|\nabla_\gamma u|^{p-2} X_i u), \tag{1.2}$$

being $\nabla_\gamma = (X_1, \dots, X_N)$ the system of vector fields

$$X_i = \frac{\partial}{\partial x_i}, i = 1, \dots, m, \quad X_{m+j} = |x|^\gamma \frac{\partial}{\partial y_j}, j = 1, \dots, n, \quad \gamma > 0, \tag{1.3}$$

and $N_\gamma$ denotes the homogeneous space dimension naturally attached to the Grushin geometry, i.e.

$$N_\gamma = m + (1+\gamma)n.$$

Here, $\Omega \subset \mathbb{R}^N$ is a bounded domain intersecting the degeneration set of the operator, i.e. such that $\Omega \cap \{x = 0\} \neq \emptyset$, the exponent

$$p_\gamma^* = \frac{pN_\gamma}{N_\gamma - p}$$

is the critical Sobolev exponent in this context, $\lambda$ is a positive real parameter and $p < r < p_\gamma^*$.

The Grushin (or Baouendi-Grushin operator) represents an interesting example of degenerate-elliptic operator, and it has been receiving increasing interest in the recent literature. Concerning the study of nonlinear Grushin equations for the case $p = 2$, we quote the recent contributions in [1–3, 9, 12, 19–21, 24, 30]; for the case $1 < p < N_\gamma$, we refer to [4, 11, 15, 22].

Problem (1.1) has been recently investigated by Gandal-Loiudice-Tyagi in [11]. In particular, existence results have been proved both for the case $r = p$ and $p < r < p_\gamma^*$, extending to the Grushin setting the well-known Euclidean results established for the ordinary $p$-Laplacian operator by Garcìa Azorero and Peral Alonso in [13, 14]. We recall that the

corresponding semilinear problem, that is the classical Brezis-Nirenberg problem [6] for the Grushin operator, was previously settled and deeply investigated by Alves et al. in [3].

Concerning problem (1.1), by means of a nontrivial adaptation of the Brezis-Nirenberg type arguments relying on a refined qualitative analysis of the $p$-Sobolev Grushin minimizers, in [11] it is proved that:

- if $r = p$, problem (1.1) has a nontrivial nonnegative solution for all $\lambda \in (\lambda_*, \lambda_1)$ for some positive $\lambda_*$ when $N_\gamma < p^2$ and for any $\lambda \in (0, \lambda_1)$ when $N_\gamma \geq p^2$, where $\lambda_1$ is the first Dirichlet eigenvalue of $-\Delta_{\gamma,p}$ on $\Omega$;
- if $p < r < p^*_\gamma$, problem (1.1) has a nontrivial nonnegative solution for all sufficiently large $\lambda > 0$ when $N_\gamma < p^2$ and for all $\lambda > 0$ when $N_\gamma \geq p^2$.

We notice that bifurcation results for problem (1.1) for the case $r = p$ have been proved by Malanchini et al. [22], while infinitely many solutions for the case $1 < r < p$ has been recently obtained in [23] by means of the Krasnosel'skii genus.

In the present paper, we further investigate the existence of solutions for problem (1.1). Precisely, by applying the abstract multiplicity results established by Perera in [27], we show that, when $p < r < p^*_\gamma$, given any $k \in \mathbb{N}$, problem (1.1) has $k$ distinct pairs of nontrivial solutions for all sufficiently large $\lambda > 0$, with an explicit lower bound on $\lambda$. This lower bound is expressed in terms of an unbounded sequence $\{\lambda_k\}$ of Dirichlet eigenvalues of $-\Delta_{\gamma,p}$ in $\Omega$, introduced in a general functional setting by Perera [25], whose construction is based on the $\mathbb{Z}_2$-cohomological index by Fadell and Rabinowitz [10] (see Theorem 2.2 below). We emphasize that these multiplicity results strongly rely on the piercing property of the cohomological index, which is not shared by the genus.

Let us introduce our results. We define the space $\mathring{W}^{1,p}_\gamma(\Omega)$ as the completion of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\| := \left( \int_\Omega |\nabla_\gamma u|^p dz \right)^{\frac{1}{p}}.$$

We look for weak solutions of problem (1.1), i.e. functions $u \in \mathring{W}^{1,p}_\gamma(\Omega)$ such that

$$\int_\Omega |\nabla_\gamma u|^{p-2} \nabla_\gamma u \cdot \nabla_\gamma \varphi \, dz = \lambda \int_\Omega |u|^{r-2} u \, \varphi \, dz + \int_\Omega |u|^{p^*_\gamma - 2} u \, \varphi \, dz, \quad \forall \varphi \in \mathring{W}^{1,p}_\gamma(\Omega).$$

We emphasize that the Grushin problem (1.1) with exponent $p^*_\gamma = \frac{pN_\gamma}{N_\gamma - p}$ turns out to be "critical" in the usual sense of the Sobolev embeddings on bounded domains $\Omega$ intersecting the degeneration set $\Gamma = \{x = 0\}$, since for these domains the Sobolev embedding

$$\mathring{W}^{1,p}_\gamma(\Omega) \hookrightarrow L^q(\Omega)$$

is compact for $q < p^*_\gamma$ (see [16]), but fails to be compact for $q = p^*_\gamma$, due to the invariance with respect to the following rescalings

$$u_t(z) = t^{\frac{N_\gamma - p}{p}} u(tx, t^{1+\gamma} y), \quad t > 0, \tag{1.4}$$

and translations in the $y$-variable.

We recall that a global $p$-Sobolev inequality for the Grushin gradient holds (see [16]; see also [18, Prop. 17]): there exists a positive constant $C = C(N_\gamma, p) > 0$ such that

$$\left(\int_{\mathbb{R}^N} |u|^{p^*_\gamma} dz\right)^{p/p^*_\gamma} \leq C \int_{\mathbb{R}^N} |\nabla_\gamma u|^p \, dz, \quad \forall u \in C_0^\infty(\mathbb{R}^N). \tag{1.5}$$

If $\Omega$ is any open subset of $\mathbb{R}^N$, we define

$$S_{\gamma,p}(\Omega) = \inf_{u \in \mathring{W}^{1,p}_\gamma(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_\gamma u|^p dz}{\left(\int_\Omega |u|^{p^*_\gamma} dz\right)^{p/p^*_\gamma}},$$

that is, the best constant in the Grushin Sobolev embedding on $\Omega$. When $\Omega = \mathbb{R}^N$, we will simply denote by $S_{\gamma,p} = S_{\gamma,p}(\mathbb{R}^N)$ the best constant in (1.5). We recall that, as proved in [11, Prop. 3.5], if $\Omega \subset \mathbb{R}^N$ is a bounded domain intersecting the degeneration set of the operator, i.e. $\Omega \cap \{x = 0\} \neq \emptyset$, then

$$S_{\gamma,p}(\Omega) = S_{\gamma,p}(\mathbb{R}^N) = S_{\gamma,p}.$$

Our main result for problem (1.1) can be stated as follows.

**Theorem 1.1** *If $1 < p < N_\gamma$ and $p < r < p^*_\gamma$, for any given $k \in \mathbb{N}\setminus\{0\}$, problem (1.1) has $k$ distinct pairs of nontrivial solutions $\pm u^\lambda_1, \dots, \pm u^\lambda_k$ for all*

$$\lambda > r|\Omega|^{r/p-1} \sup_{\tau>0} \left[\frac{\lambda_k}{p\tau^{r-p}} - \frac{S^{N_\gamma/p}_{\gamma,p}}{N_\gamma \tau^r} - \frac{\tau^{p^*_\gamma - r}}{p^*_\gamma |\Omega|^{p/(N_\gamma - p)}}\right], \tag{1.6}$$

*where $|\Omega|$ denotes the Lebesgue measure of $\Omega$ and the $\lambda_k$'s are the eigenvalues defined in (2.19). In particular, the number of solutions of problem (1.1) goes to infinity as $\lambda \to \infty$.*

Concerning the above result, we observe that the quantity

$$\frac{\lambda_k}{p\tau^{r-p}} - \frac{S^{N_\gamma/p}_{\gamma,p}}{N_\gamma \tau^r} - \frac{\tau^{p^*_\gamma - r}}{p^*_\gamma |\Omega|^{p/(N_\gamma - p)}} \to -\infty, \quad \text{as } \tau \to 0, \infty$$

hence the supremum in (1.6) is finite, and it goes to infinity as $k \to \infty$ since $\lambda_k \to \infty$.

As a further application of the abstract multiplicity theorems in Perera [27], we also investigate the multiplicity of solutions for a critical Grushin $(p, q)$-Laplacian type problem, whose study is a novelty in the Grushin scenario. Precisely, we consider the problem

$$\begin{cases} -\Delta_{\gamma,p} u - \Delta_{\gamma,q} u = \lambda |u|^{r-2} u + |u|^{p^*_\gamma - 2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $1 < q < p < N_\gamma$, $p \leq r < p^*_\gamma$, $\lambda > 0$ is a real parameter and, as before, $\Omega \subset \mathbb{R}^N$ is a bounded open set such that $\Omega \cap \{x = 0\} \neq \emptyset$.

We recall that, in the ordinary $(p, q)$-Laplacian case, that is the case $\gamma = 0$ in (1.7), existence results have been proved in Candito et al. [8], Yin-Yang [29]. To state our result, let, as before, $\{\lambda_k\}$ denote the sequence of Dirichlet eigenvalues of the $p$-Grushin Laplacian in $\Omega$, whose definition is recalled in (2.19). Here we prove the following result.

**Theorem 1.2** *For any given $k \in \mathbb{N} \setminus \{0\}$, problem (1.7) has $k$ distinct pairs of nontrivial solutions $\pm u_1^\lambda, \dots, \pm u_k^\lambda$ for all*

$$\lambda > r|\Omega|^{r/p-1} \sup_{\tau>0} \left[ \frac{\lambda_k}{p\tau^{r-p}} + \frac{|\Omega|^{1-q/p}}{q\tau^{r-q}} \lambda_k^{q/p} - \frac{S_{\gamma,p}^{N_\gamma/p}}{N_\gamma \tau^r} - \frac{\tau^{p_\gamma^*-r}}{p_\gamma^* |\Omega|^{p/(N_\gamma-p)}} \right], \tag{1.8}$$

*in each of the following cases:*
(i) $p < r < p_\gamma^*$,
(ii) $r = p \le q_\gamma^*$.

In particular, the number of solutions of problem (1.7) goes to infinity as $\lambda \to \infty$.

As before, we note that the supremum in (1.8) is finite and goes to infinity as $k \to \infty$.

Moreover, extending to the Grushin context new results by Candito-Livrea-Perera [7] for the ordinary $(p, q)$-Laplace problem, we can deduce a multiplicity result for a variant of problem (1.7), depending on an additional parameter $\mu > 0$, in the case of a $p$-linear perturbation. The problem is the following

$$\begin{cases} -\Delta_{\gamma,p} u - \mu \Delta_{\gamma,q} u = \lambda |u|^{p-2} u + |u|^{p_\gamma^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $\Omega$ is a bounded domain, $\Omega \cap \{x = 0\} \neq \emptyset$, $1 < q < p < N_\gamma$ and $\lambda, \mu > 0$.

In this case, we can prove that, for all $\lambda \ge \lambda_k$, the above equation has $k$ distinct pairs of nontrivial solutions if either $\mu$ or the measure of $\Omega$ is sufficiently small. Precisely, we have the following result:

**Theorem 1.3** *If $\lambda \ge \lambda_k$, problem (1.9) has $k$ distinct pairs of nontrivial solutions $\pm u_1^\mu, \dots, \pm u_k^\mu$ for all $\mu > 0$ such that*

$$|\Omega|^{1-\frac{q}{p_\gamma^*}} \lambda_k^{q/p} \mu < \left[ \frac{pq S_{\gamma,p}^{N_\gamma/p}}{N_\gamma p - (N_\gamma - p)q} \right]^{1-q/p_\gamma^*}. \tag{1.10}$$

*Moreover, each $u_j^\mu \to 0$ as $\mu \searrow 0$.*

In particular, for the case $\mu = 1$ we get the following corollary.

**Corollary 1.4** *If $\lambda \ge \lambda_k$ and*

$$|\Omega|^{1-\frac{q}{p_\gamma^*}} \lambda_k^{q/p} < \left[ \frac{pq S_{\gamma,p}^{N_\gamma/p}}{N_\gamma p - (N_\gamma - p)q} \right]^{1-q/p_\gamma^*}, \tag{1.11}$$

*problem (1.9) with $\mu = 1$ has $k$ distinct pairs of nontrivial solutions.*

We conclude with the plan of the paper. In Sect. 2 we introduce some preliminary results: first, we show that the functionals involved in the proofs satisfy a suitable local Palais-Smale condition; next, we recall the construction of the sequence $\lambda_k$ of eigenvalues of $-\Delta_{\gamma,p}$ on $\Omega$ based on the cohomological index, which appear in our statements. In Sect. 3 we state the general multiplicity results that can be deduced for the Grushin equations from the abstract results in Perera [27]. Finally, in Sects. 4 and 5 we prove Theorems 1.1, 1.2, and Theorem 1.3, respectively.

## 2 Preliminaries

### 2.1 The Palais-Smale condition

The key compactness ingredient required in our proofs is the validity of the Palais-Smale condition under certain threshold for the involved variational functionals. We recall that, given a Banach space $W$, a functional $E \in C^1(W, \mathbb{R})$, satisfies the Palais-Smale condition at the level $c \in \mathbb{R}$, or the $(PS)_c$-condition for short, if every sequence $\{u_j\} \subset W$ satisfying $E(u_j) \to c$ and $E'(u_j) \to 0$, called a $(PS)_c$-sequence for $E$, has a strongly convergent subsequence.

Adapting the proofs for the ordinary $p$-Laplacian and $(p, q)$-Laplacian case, we can prove the following local Palais-Smale condition for the functionals corresponding to the considered problems, for the full range of the involved parameters.

**Proposition 2.1** *Let $\mu \geq 0$, $\lambda > 0$, $1 < q < p \leq r < p^*_\gamma$. Then, the energy functional*

$$E(u) = \frac{1}{p}\int_\Omega |\nabla_\gamma u|^p dz + \frac{\mu}{q}\int_\Omega |\nabla_\gamma u|^q dz - \frac{\lambda}{r}\int_\Omega |u|^r dz - \frac{1}{p^*_\gamma}\int_\Omega |u|^{p^*_\gamma} dz, \quad u \in \mathring{W}^{1,p}_\gamma(\Omega),$$

*satisfy the $(PS)_c$-condition for all*

$$c < \frac{1}{N_\gamma} S^{N_\gamma/p}_{\gamma,p}.$$

***Proof*** We first notice that for $\mu = 0$ and $r = p$, the above $(PS)$-condition for $E$ has been proved in Malanchini et al. [22] as a consequence of concentration-compactness arguments. Moreover, for $\mu = 0$ and $p < r < p^*_\gamma$, in Gandal et al. [11], it is proved that that every (PS)-sequence for $E$ under the indicated threshold has a weakly convergent subsequence to a non-zero critical point. Here we complete the proof, also considering the $(p, q)$-case.

Let $\{u_j\}$ be a (PS)-sequence for $E$ at level $c$, where $c < \frac{1}{N_\gamma} S^{N_\gamma/p}_{\gamma,p}$. By definition, it holds

$$E(u_j) = \frac{1}{p}\int_\Omega |\nabla_\gamma u_j|^p dz + \frac{\mu}{q}\int_\Omega |\nabla_\gamma u_j|^q dz - \frac{\lambda}{r}\int_\Omega |u_j|^r dz - \frac{1}{p^*_\gamma}\int_\Omega |u_j|^{p^*_\gamma} dz = c + o(1) \quad (2.1)$$

and

$$E'(u_j)v = \int_\Omega |\nabla_\gamma u_j|^{p-2}\nabla_\gamma u_j \cdot \nabla_\gamma v\, dz + \mu \int_\Omega |\nabla_\gamma u_j|^{q-2}\nabla_\gamma u_j \cdot \nabla_\gamma v\, dz - \lambda \int_\Omega |u_j|^{r-2}u_j v\, dz - \int_\Omega |u_j|^{p_\gamma^*-2}u_j v\, dz = o(\|v\|), \quad \forall v \in \mathring{W}^{1,p}_\gamma(\Omega). \tag{2.2}$$

We first show that $\{u_j\}$ is bounded. Taking $v = u_j$ in (2.2), we get

$$\int_\Omega |\nabla_\gamma u_j|^p\, dz + \mu \int_\Omega |\nabla_\gamma u_j|^q\, dz - \lambda \int_\Omega |u_j|^r\, dz - \int_\Omega |u_j|^{p_\gamma^*}\, dz = o(\|u_j\|). \tag{2.3}$$

If $p < r < p_\gamma^*$, subtracting (2.3) divided by $r$ from (2.1) gives

$$\left(\frac{1}{p} - \frac{1}{r}\right)\int_\Omega |\nabla_\gamma u_j|^p\, dz + \mu\left(\frac{1}{q} - \frac{1}{r}\right)\int_\Omega |\nabla_\gamma u_j|^q\, dz + \left(\frac{1}{r} - \frac{1}{p_\gamma^*}\right)\int_\Omega |u_j|^{p_\gamma^*}\, dz = c + o(1)$$

and this implies $\|u_j\|$ is bounded.

Analogously, if $r = p$, subtracting (2.3) divided by $p$ from (2.1) gives

$$\mu\left(\frac{1}{q} - \frac{1}{p}\right)\int_\Omega |\nabla_\gamma u_j|^q\, dz + \left(\frac{1}{p} - \frac{1}{p_\gamma^*}\right)\int_\Omega |u_j|^{p_\gamma^*}\, dz = c + o(1).$$

The latter together with (2.1) implies $\|u_j\|$ is bounded.

So, there exists $u \in \mathring{W}^{1,p}_\gamma(\Omega)$ such that, up to a subsequence,

$$u_j \rightharpoonup u \text{ in } \mathring{W}^{1,p}_\gamma(\Omega), \tag{2.4}$$

$$u_j \to u \text{ in } L^s(\Omega),\ 1 \le s < p_\gamma^*, \tag{2.5}$$

$$u_j \to u \text{ a.e. in } \Omega. \tag{2.6}$$

Let us show that $u_j \to u$ in $\mathring{W}^{1,p}_\gamma(\Omega)$. Define $\widetilde{u}_j := u_j - u$.

We will show that, if $\|\widetilde{u}_j\| \ge \varepsilon_0$ for some $\varepsilon_0 > 0$, then $c \ge \frac{1}{N_\gamma} S_{\gamma,p}^{N_\gamma/p}$, contrary to the assumption.

We start by noting that, from $u_j \to u$ in $L^r(\Omega)$, the relations (2.1) and (2.3) can be written as

$$\frac{1}{p}\int_\Omega |\nabla_\gamma u_j|^p dz + \frac{\mu}{q}\int_\Omega |\nabla_\gamma u_j|^q dz - \frac{\lambda}{r}\int_\Omega |u|^r dz - \frac{1}{p_\gamma^*}\int_\Omega |u_j|^{p_\gamma^*} dz = c + o(1) \tag{2.7}$$

and

$$\int_\Omega |\nabla_\gamma u_j|^p\, dz + \mu \int_\Omega |\nabla_\gamma u_j|^q\, dz - \lambda \int_\Omega |u|^r\, dz - \int_\Omega |u_j|^{p_\gamma^*}\, dz = o(1). \tag{2.8}$$

Moreover, taking $v = u$ in (2.2) and passing to the limit, we get

$$\int_\Omega |\nabla_\gamma u|^p \, dz + \mu \int_\Omega |\nabla_\gamma u|^q \, dz - \lambda \int_\Omega |u|^r \, dz - \int_\Omega |u|^{p_\gamma^*} \, dz = 0. \tag{2.9}$$

The Brezis-Lieb Lemma (see [5, Theorem 1]) gives

$$\begin{aligned} \int_\Omega |\nabla_\gamma u_j|^p \, dz - \int_\Omega |\nabla_\gamma u|^p \, dz &= \int_\Omega |\nabla_\gamma \widetilde{u}_j|^p \, dz + o(1), \\ \int_\Omega |\nabla_\gamma u_j|^q \, dz - \int_\Omega |\nabla_\gamma u|^q \, dz &= \int_\Omega |\nabla_\gamma \widetilde{u}_j|^q \, dz + o(1), \\ \int_\Omega |u_j|^{p_\gamma^*} \, dz - \int_\Omega |u|^{p_\gamma^*} \, dz &= \int_\Omega |\widetilde{u}_j|^{p_\gamma^*} \, dz + o(1). \end{aligned} \tag{2.10}$$

So, subtracting (2.9) from (2.8) gives

$$\int_\Omega |\nabla_\gamma \widetilde{u}_j|^p \, dz + \mu \int_\Omega |\nabla_\gamma \widetilde{u}_j|^q \, dz = \int_\Omega |\widetilde{u}_j|^{p_\gamma^*} \, dz + o(1). \tag{2.11}$$

Together with the Grushin Sobolev inequality, (2.11) gives

$$\int_\Omega |\nabla_\gamma \widetilde{u}_j|^p \, dz \le \left( S_{\gamma,p}^{-1} \int_\Omega |\nabla_\gamma \widetilde{u}_j|^p dz \right)^{p_\gamma^*/p} + o(1).$$

Since $\|\widetilde{u}_j\|$ is bounded away from zero, dividing this inequality by $\|\widetilde{u}_j\|^p$ gives

$$\int_\Omega |\nabla_\gamma \widetilde{u}_j|^p \, dz \ge S_{\gamma,p}^{N_\gamma/p} + o(1). \tag{2.12}$$

On the other hand, subtracting (2.8) divided by $p_\gamma^*$ from (2.7) gives

$$\frac{1}{N_\gamma} \int_\Omega |\nabla_\gamma u_j|^p dz + \mu \left( \frac{1}{q} - \frac{1}{p_\gamma^*} \right) \int_\Omega |\nabla_\gamma u_j|^q dz - \lambda \left( \frac{1}{r} - \frac{1}{p_\gamma^*} \right) \int_\Omega |u|^r \, dz = c + o(1)$$

and subtracting (2.9) divided by $N_\gamma$ from the above equation and using (2.10) gives

$$\begin{aligned} &\frac{1}{N_\gamma} \int_\Omega |\nabla_\gamma \widetilde{u}_j|^p dz + \mu \left( \frac{1}{q} - \frac{1}{p_\gamma^*} \right) \int_\Omega |\nabla_\gamma \widetilde{u}_j|^q dz + \mu \left( \frac{1}{q} - \frac{1}{p} \right) \int_\Omega |\nabla_\gamma u|^q dz \\ &\quad + \lambda \left( \frac{1}{p} - \frac{1}{r} \right) \int_\Omega |u|^r \, dz + \frac{1}{N_\gamma} \int_\Omega |u|^{p_\gamma^*} \, dz = c + o(1). \end{aligned}$$

Since $q < p \le r < p_\gamma^*$, this together with (2.12) gives $c \ge \frac{1}{N_\gamma} S_{\gamma,p}^{N_\gamma/p}$, that is, a contradiction. □

### 2.2 The eigenvalue problem

In this section we recall the construction of the eigenvalues based on the cohomological index, appearing in our statements, as a special case of the abstract construction in Perera [26] (eigenvalues based on the cohomological index for the *p*-Laplacian were first introduced in Perera [25]). We point out that such a sequence of eigenvalues for the *p*-Grushin operator has been also considered in Malanchini et al. [22] in order to obtain bifurcation results.

The Dirichlet spectrum of $-\Delta_{\gamma,p}$ in $\Omega$ consists of those $\lambda \in \mathbb{R}$ for which the problem

$$\begin{cases} -\Delta_{\gamma,p} u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.13}$$

has a nontrivial solution. For the properties of the above spectrum in the case $p = 2$ we refer e.g. to [17] and the reference therein. Concerning the case $p \neq 2$ we note that a complete description of the spectrum is still not known, even for the ordinary *p*-Laplacian case, that is, the case $\gamma = 0$ in (2.13). However, as in the ordinary *p*-Laplacian case, it is possible to construct an increasing and unbounded sequence of variational eigenvalues via a suitable minimax scheme, as we here outline.

Denoted by $W = \mathring{W}^{1,p}_{\gamma}(\Omega)$, with dual $W^*$ and dual pairing $(\cdot,\cdot)$, the above eigenvalue problem can be written as the operator equation

$$A_p u = \lambda B_p u \quad \text{in } W^*, \tag{2.14}$$

where $A_p, B_p \in C(W, W^*)$ are the potential operators given by

$$(A_p u, v) := \int_\Omega |\nabla_\gamma u|^{p-2} \nabla_\gamma u \cdot \nabla_\gamma v \, dz, \tag{2.15}$$

$$(B_p u, v) := \int_\Omega |u|^{p-2} uv \, dz. \tag{2.16}$$

(We call $f \in C(W, W^*)$ a potential operator if there exists a potential for *f*, that is, a functional $F \in C^1(W, \mathbb{R})$, such that $F' = f$). The functionals

$$I_p(u) = \frac{1}{p}\|u\|^p \quad \text{and} \quad J_p(u) = \frac{1}{p}\int_\Omega |u|^p \, dz \tag{2.17}$$

are potentials of $A_p$ and $B_p$ respectively, satisfying $I_p(0) = J_p(0) = 0$. Observe that the operator $A_p$ defined in (2.15) is $(p-1)$-homogeneous and odd, i.e.

$$A_p(tu) = |t|^{p-2} t A_p u \quad \forall u \in W, \ t \in \mathbb{R}$$

and

$$(A_p(u), v) \leq \|u\|^{p-1}\|v\|, \quad (A_p(u), u) = \|u\|^p \quad \forall u, v \in W.$$

Moreover, $B_p$ is a compact operator, since $\mathring{W}^{1,p}_{\gamma}(\Omega)$ is compactly embedded in $L^p(\Omega)$ (see [16] for Grushin embedding results).

In view of the above structural properties, the eigenvalue problem (2.14) falls in the abstract class considered by Perera in [26]. In particular, from [26, Prop. 3.1], it follows that eigenvalues of problem (2.14) coincide with critical values of the $C^1$-functional

$$\Psi(u) := \frac{1}{\int_\Omega |u|^p \, dz}, \qquad u \in S := \{u \in W \,|\, \|u\| = 1\}. \tag{2.18}$$

Now, we briefly recall the definition of the cohomological index. Let $W$ be a Banach space and let $\mathcal{A}$ be the class of symmetric subsets of $W \setminus \{0\}$. For $A \in \mathcal{A}$, let $\overline{A} = A/\mathbb{Z}_2$ be the quotient space of $A$ which identifies $u$ and $-u$, and let $f : \overline{A} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{A})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of $A$ is defined as

$$i(A) = \begin{cases} \sup\{m \geq 1 \,|\, f^*(\omega^{m-1}) \neq 0\} & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere $S^{m-1}$ in $\mathbb{R}^m$, $m \geq 1$ is the inclusion $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^\infty$, which induces isomorphisms on $H^q$ for $q \leq m-1$, so $i(S^{m-1}) = m$.

By means of the above index, a suitable sequence of eigenvalues can be constructed as follows (See [28, Theorem 4.6]).

**Theorem 2.2** *Let $\mathcal{F}$ denote the class of symmetric subsets of $S$. For $k \geq 1$, let*

$$\mathcal{F}_k = \{M \in \mathcal{F} \,|\, i(M) \geq k\}$$

*and define*

$$\lambda_k := \inf_{M \in \mathcal{F}_k} \sup_{u \in M} \Psi(u), \quad k \geq 1, \tag{2.19}$$

*where $\Psi$ has been defined in (2.18). Then*

$$\lambda_1 = \inf_{u \in S} \Psi(u) > 0$$

*is the first eigenvalue and $\lambda_1 \leq \lambda_2 \leq \ldots$ is an unbounded sequence of eigenvalues of problem (2.14).*

## 3 General multiplicity results

In this section we present some general multiplicity results that can be obtained by the abstract results in Perera [27] for equations involving the $p$-Grushin operator.

Let $W = \mathring{W}^{1,p}_{\gamma}(\Omega)$. Our first general result concerns equations of the form

$$A_p u + f(u) = \lambda g(u) + h(u) \quad \text{in } W^*, \tag{3.1}$$

where $A_p$ is defined as in (2.15), $f, g, h \in C(W, W^*)$ are odd potential operators and $\lambda > 0$ is a real parameter. Let $I_p$ and $J_p$ be the functionals defined in (2.17). We assume that the potentials $F$, $G$ and $H$, respectively of $f$, $g$ and $h$ such that $F(0) = G(0) = H(0) = 0$, satisfy the following hypotheses:

$(H_1)$ for some constants $\alpha > 0$ and $1 < q < p$,

$$0 \le F(u) \le \frac{\alpha}{q}(pI_p(u))^{q/p} \quad \forall u \in W,$$

$(H_2)$ for some constants $\beta > 0$ and $r > p$

$$G(u) \ge \frac{\beta}{r}(pJ_p(u))^{r/p} \quad \forall u \in W,$$

$(H_3)$ $G(u) = o(\|u\|^p)$ as $u \to 0$,

$(H_4)$ for some constants $\delta > 0$ and $p^*_\gamma > r$,

$$H(u) \ge \frac{\delta}{p^*_\gamma}(pJ_p(u))^{p^*_\gamma/p} \quad \forall u \in W,$$

$(H_5)$ $H(u) = o(\|u\|^p)$ as $u \to 0$,

$(H_6)$ there exists $c^* > 0$ such that the functional

$$E(u) = I_p(u) + F(u) - \lambda G(u) - H(u), \quad u \in W,$$

associated with (3.1) satisfies the $(PS)_c$ condition for all $c \in (0, c^*)$.

In what follows, let $\{\lambda_k\}$ be the sequence of eigenvalues defined by (2.19). A direct application of [27, Theorem 2.4] gives the following result.

**Theorem 3.1** *Assume that $(H_1) - (H_6)$ hold. Then equation (3.1) has $k$ distinct pairs of nontrivial solutions $\pm u^\lambda_1, \dots, \pm u^\lambda_k$ for all*

$$\lambda > \frac{r}{\beta} \sup_{\tau > 0} \left[ \frac{\lambda_k}{p\tau^{r-p}} + \frac{\alpha}{q\tau^{r-q}} \lambda_k^{q/p} - \frac{c^*}{\tau^r} - \frac{\delta \tau^{p^*_\gamma - r}}{p^*_\gamma} \right].$$

Our second multiplicity result concerns Grushin equations with $p$-linear perturbation, that is, equations of the form

$$A_p(u) + f(u) = \lambda B_p u + h(u) \quad \text{in } W^*, \tag{3.2}$$

where $B_p$ is defined as in (2.16), $f, h \in C(W, W^*)$ are odd potential operators and $\lambda > 0$ is a real parameter. We assume that the potentials $F$ and $H$, respectively of $f$ and $h$ with $F(0) = H(0) = 0$, satisfy the following hypotheses:

$(H_1')$ for some constants $\alpha_0, \alpha > 0$ and $1 < q < p$,

$$\frac{\alpha_0}{q}(pJ_p(u))^{q/p} \le F(u) \le \frac{\alpha}{q}(pI_p(u))^{q/p} \quad \forall u \in W,$$

$(H_2')$ for some constants $\delta > 0$,

$$H(u) \ge \frac{\delta}{p_\gamma^*}(pJ_p(u))^{p_\gamma^*/p} \quad \forall u \in W,$$

$(H_3')$ $H(u) = o(\|u\|^p)$ as $u \to 0$,

$(H_4')$ there exists $c^* > 0$ such that the functional

$$E(u) = I_p(u) + F(u) - \lambda J_p(u) - H(u), \quad u \in W$$

associated with (3.2) satisfies the $(PS)_c$ condition for all $c \in (0, c^*)$.

From [27, Theorem 2.5] the following result holds for equation (3.2).

**Theorem 3.2** *Assume that $(H_1') - (H_4')$ hold. Then equation (3.2) has k distinct pairs of nontrivial solutions $\pm u_1^\lambda, \dots, \pm u_k^\lambda$ for all*

$$\lambda > \lambda_k + p \sup_{\tau > 0} \left[ \frac{\alpha}{q\tau^{p-q}} \lambda_k^{q/p} - \frac{c^*}{\tau^p} - \frac{\delta \tau^{p_\gamma^* - p}}{p_\gamma^*} \right].$$

In the next section, we will apply the above theorems to obtain multiplicity of solutions for equation (1.1) and (1.7).

## 4 Proof of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2. To this aim, we apply Theorems 3.1 and 3.2 with $f, g, h \in C(\mathring{W}_\gamma^{1,p}(\Omega), (\mathring{W}_\gamma^{1,p}(\Omega))^*)$ given by

$$(f(u), v) = \int_\Omega |\nabla_\gamma u|^{q-2} \nabla_\gamma u \cdot \nabla_\gamma v \, dz, \tag{4.1}$$

$$(g(u), v) = \int_\Omega |u|^{r-2} uv \, dz, \tag{4.2}$$

$$(h(u), v) = \int_\Omega |u|^{p_\gamma^* - 2} uv \, dz, \tag{4.3}$$

for $u, v \in \mathring{W}_\gamma^{1,p}(\Omega)$. The corresponding potentials are

$$F(u)=\frac{1}{q}\int_\Omega |\nabla_\gamma u|^q\,dz,\quad G(u)=\frac{1}{r}\int_\Omega |u|^r\,dz,\quad H(u)=\frac{1}{p^*_\gamma}\int_\Omega |u|^{p^*_\gamma}\,dz, \tag{4.4}$$

respectively. Observe that, from Hölder's inequality

$$F(u)\le \frac{|\Omega|^{1-q/p}}{q}\left(\int_\Omega |\nabla_\gamma u|^p\,dz\right)^{q/p} \tag{4.5}$$

$$G(u)\ge \frac{1}{r|\Omega|^{r/p-1}}\left(\int_\Omega |u|^p\,dz\right)^{r/p} \tag{4.6}$$

$$H(u)\ge \frac{1}{p^*_\gamma|\Omega|^{p/(N_\gamma-p)}}\left(\int_\Omega |u|^p\,dz\right)^{p^*_\gamma/p}, \tag{4.7}$$

while the Sobolev inequality (1.5) ensures

$$G(u)=o(\|u\|^p)\quad \text{as } u\to 0,$$

being $r<p^*_\gamma$, and

$$H(u)=o(\|u\|^p)\quad \text{as } u\to 0.$$

Moreover, by Proposition 2.1, the functionals corresponding to problem (1.1) and (1.7), i.e.

$$E(u)=\frac{1}{p}\int_\Omega |\nabla_\gamma u|^p dz-\frac{\lambda}{r}\int_\Omega |u|^r dz-\frac{1}{p^*_\gamma}\int_\Omega |u|^{p^*_\gamma} dz,\quad u\in \mathring{W}^{1,p}_\gamma(\Omega)$$

and

$$E(u)=\frac{1}{p}\int_\Omega |\nabla_\gamma u|^p dz+\frac{1}{q}\int_\Omega |\nabla_\gamma u|^q dz-\frac{\lambda}{r}\int_\Omega |u|^r dz-\frac{1}{p^*_\gamma}\int_\Omega |u|^{p^*_\gamma} dz,\quad u\in \mathring{W}^{1,p}_\gamma(\Omega),$$

satisfy the $(PS)_c$-condition for all

$$c<c^*=\frac{1}{N_\gamma}S^{N_\gamma/p}_{\gamma,p}$$

when $p\le r<p^*_\gamma$. In view of the above observations, we can conclude as follows.

***Proof of Theorem 1.1*** We apply Theorem 3.1 with $f=0$ and $\alpha=0$. By (4.6) and (4.7), conditions $(H_2)$ and $(H_4)$ are verified with

$$\beta=\frac{1}{|\Omega|^{r/p-1}}\ \text{ and }\ \delta=\frac{1}{|\Omega|^{p/(N_\gamma-p)}},$$

respectively. The conclusion then follows. □

***Proof of Theorem 1.2*** (i) Let $p < r < p^*_\gamma$. We apply Theorem 3.1 with $f$ given by (4.1). By (4.5)–(4.7), conditions $(H_1)$, $(H_2)$ and $(H_4)$ hold with

$$\alpha = |\Omega|^{1-q/p},\ \beta = \frac{1}{|\Omega|^{r/p-1}},\ \delta = \frac{1}{|\Omega|^{p/(N_\gamma - p)}},$$

respectively. The conclusion then follows from Theorem 3.1.

(ii) Let $r = p \le q^*_\gamma$. We apply Theorem 3.2. Since $p \le q^*_\gamma$, the lower bound in $(H'_1)$ holds for some $\alpha_0 > 0$ by the Grushin Sobolev embedding. By (4.5) and (4.7), the upper bound in $(H'_1)$ and $(H'_2)$ hold with

$$\alpha = |\Omega|^{1-q/p}, \quad \delta = \frac{1}{|\Omega|^{p/(N_\gamma - p)}},$$

respectively. The conclusion then follows from Theorem 3.2. □

## 5 Proof of Theorem 1.3

This last section is devoted to the proof of Theorem 1.3. The proof follows the Euclidean outline in Candito-Livrea-Perera [7] and it is based on the abstract multiplicity theorem in Perera [27] which we preliminarily recall.

Let $W$ be a Banach space and let $E \in C^1(W, \mathbb{R})$ be an even functional, i.e. $E(-u) = E(u)$ for all $u \in E$. Assume that there exists $c^* > 0$ such that $E$ satisfies the $(PS)_c$-condition for all $c \in (0, c^*)$. The following result holds.

**Theorem 5.1** *(Perera [27, Theorem 2.1]) Let $C$ be a compact symmetric subset of $S = \{u \in W \,|\, \|u\| = 1\}$ with $i(C) = k \ge 1$. Assume that the origin is a strict local minimizer of $E$ with $E(0) = 0$ and there exists $R > 0$ such that*

$$\sup_{u \in A} E(u) \le 0, \quad \sup_{u \in X} E(u) < c^*, \tag{5.1}$$

*where*

$$A = \{Ru \,|\, u \in C\} \ \text{ and } \ X = \{tu \,|\, u \in C,\ 0 \le t \le R\}. \tag{5.2}$$

*Then, $E$ has $k$ distinct pairs of nontrivial critical points $\pm u_1, \dots, \pm u_k$, satisfying*

$$0 < E(u_j) \le \sup_{u \in X} E(u).$$

Now, let $W = \mathring{W}^{1,p}_\gamma(\Omega)$ and consider the functional corresponding to problem (1.9), i.e.

$$E(u) = \frac{1}{p}\int_\Omega |\nabla_\gamma u|^p dz + \frac{\mu}{q}\int_\Omega |\nabla_\gamma u|^q dz - \frac{\lambda}{p}\int_\Omega |u|^p dz - \frac{1}{p^*_\gamma}\int_\Omega |u|^{p^*_\gamma} dz, \quad u \in \mathring{W}^{1,p}_\gamma(\Omega).$$

The proof of Theorem 1.3 will follow by applying Theorem 5.1 to the above functional for a suitable choice of the subset $C$.

***Proof of Theorem 1.3*** In view of Proposition 2.1, we can assume $c^* = \frac{1}{N_\gamma} S_{\gamma,p}^{N_\gamma/p}$ as $(PS)$-threshold for $E$. Next, we observe that the origin is a strict local minimizer for $E$. Indeed, if $p \le q_\gamma^* = N_\gamma q/(N_\gamma - q)$, this follows from the Sobolev-type embeddings $\mathring{W}_\gamma^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$ and $\mathring{W}_\gamma^{1,p}(\Omega) \hookrightarrow L^{p_\gamma^*}(\Omega)$. If $p > q_\gamma^*$, by interpolation and the Grushin Sobolev inequality we get

$$\begin{aligned}\left(\int_\Omega |u|^p\,dz\right)^{1/p} &\le \left(\int_\Omega |u|^{q_\gamma^*}\,dz\right)^{\theta/q_\gamma^*} \left(\int_\Omega |u|^{p_\gamma^*}\,dz\right)^{(1-\theta)/p_\gamma^*} \\ &\le C\left(\int_\Omega |\nabla_\gamma u|^q\,dz\right)^{\theta/q} \left(\int_\Omega |u|^{p_\gamma^*}\,dz\right)^{(1-\theta)/p_\gamma^*},\end{aligned}$$

where $\theta = pq/N_\gamma(p-q) \in (0,1)$. Then, by Young's inequality we obtain

$$\int_\Omega |u|^p\,dz \le C_\varepsilon \left(\int_\Omega |\nabla_\gamma u|^q\,dz\right)^{p/q} + \varepsilon \left(\int_\Omega |u|^{p_\gamma^*}\,dz\right)^{p/p_\gamma^*}.$$

Therefore

$$\begin{aligned}E(u) \ge \frac{1}{p}\left(1 - \frac{\lambda\varepsilon}{S_{\gamma,p}}\right)\int_\Omega |\nabla_\gamma u|^p\,dz - \frac{1}{p_\gamma^* S_{\gamma,p}^{p_\gamma^*/p}}\left(\int_\Omega |\nabla_\gamma u|^p dz\right)^{p_\gamma^*/p} \\ + \frac{\mu}{q}\int_\Omega |\nabla_\gamma u|^q\,dz - \frac{\lambda C_\varepsilon}{p}\left(\int_\Omega |\nabla_\gamma u|^q\,dz\right)^{p/q}.\end{aligned}$$

The conclusion then follows from the arbitrariness of $\varepsilon > 0$ and the condition $q < p < p_\gamma^*$.

Now, by increasing $k$ if necessary, it may be assumed $\lambda_k < \lambda_{k+1}$. Consider the sublevel set

$$\Psi^{\lambda_k} = \{u \in S \,|\, \Psi(u) \le \lambda_k\},$$

where $S$ and $\Psi$ are defined as in (2.18). By Perera [26, Theorem 1.3], $\Psi^{\lambda_k}$ has a compact symmetric subset $C$ of index $k$. We now verify that the conditions in (5.1) hold for this choice of $C$.

Observe that, for $u \in C$,

$$\int_\Omega |\nabla_\gamma u|^p dz = 1, \quad \int_\Omega |u|^p dz \ge \frac{1}{\lambda_k} \ge \frac{1}{\lambda}. \tag{5.3}$$

So, for $u \in C$ and $t \ge 0$, by (5.3) and using Hölder's inequality we get

$$E(tu) = \frac{t^p}{p}\left(\int_\Omega |\nabla_\gamma u|^p dz - \lambda \int_\Omega |u|^p dz\right) + \frac{\mu t^q}{q}\int_\Omega |\nabla_\gamma u|^q dz - \frac{t^{p_\gamma^*}}{p_\gamma^*}\int_\Omega |u|^{p_\gamma^*} dz$$
$$\leq \frac{\mu |\Omega|^{1-q/p} t^q}{q} - \frac{t^{p_\gamma^*}}{p_\gamma^* |\Omega|^{p_\gamma^*/p-1} \lambda_k^{p_\gamma^*/p}}, \tag{5.4}$$

and taking the supremum over all $t \geq 0$ we have

$$\sup_{u\in C, t\geq 0} E(tu) \leq \left(\frac{1}{q} - \frac{1}{p_\gamma^*}\right) |\Omega| (\lambda_k^{q/p}\mu)^{p_\gamma^*/(p_\gamma^*-q)}. \tag{5.5}$$

Now, let $R > 0$ and define $A$ and $X$ as in (5.2). By (5.4) we have that

$$\sup_{u\in A} E(u) \leq \frac{\mu |\Omega|^{1-q/p} R^q}{q} - \frac{R^{p_\gamma^*}}{p_\gamma^* |\Omega|^{p_\gamma^*/p-1} \lambda_k^{p_\gamma^*/p}} \leq 0$$

for $R$ sufficiently large. Moreover, from (5.5) it follows

$$\sup_{u\in X} E(u) \leq \left(\frac{1}{q} - \frac{1}{p_\gamma^*}\right) |\Omega| (\lambda_k^{q/p}\mu)^{p_\gamma^*/(p_\gamma^*-q)} < \frac{1}{N_\gamma} S_{\gamma,p}^{N_\gamma/p}$$

by assumption (1.10).

Hence, by Theorem 5.1, it follows that there exist $k$ distinct pairs of nontrivial critical points $\pm u_j^\mu$, $j = 1, \ldots, k$, of the functional $E$, such that

$$0 < E(u_j^\mu) \leq \left(\frac{1}{q} - \frac{1}{p_\gamma^*}\right) |\Omega| (\lambda_k^{q/p}\mu)^{p_\gamma^*/(p_\gamma^*-q)}.$$

Finally, in order to prove that $u_j^\mu \to 0$ as $\mu \searrow 0$, first observe that, by the above estimate,

$$E(u_j^\mu) \to 0 \quad \text{as } \mu \searrow 0.$$

Moreover

$$E'(u_j^\mu)u_j^\mu = \int_\Omega |\nabla_\gamma u_j^\mu|^p dz + \mu \int_\Omega |\nabla_\gamma u_j^\mu|^q dz - \lambda \int_\Omega |u_j^\mu|^p dz - \int_\Omega |u_j^\mu|^{p_\gamma^*} dz = 0. \tag{5.6}$$

Hence

$$E(u_j^\mu) - \frac{1}{p}E'(u_j^\mu)u_j^\mu = \mu\left(\frac{1}{q} - \frac{1}{p}\right)\int_\Omega |\nabla_\gamma u_j^\mu|^q dz + \frac{1}{N_\gamma}\int_\Omega |u_j^\mu|^{p_\gamma^*} dz \to 0,$$

from which $\mu \int_\Omega |\nabla_\gamma u_j^\mu|^q dz \to 0$ and $\int_\Omega |u_j^\mu|^{p_\gamma^*} dz \to 0$. Then, by Hölder's inequality, $\int_\Omega |u_j^\mu|^p dz \to 0$, and so, from (5.6), $\|u_j^\mu\| \to 0$. □

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