Existence results for critical problems involving p-sub-Laplacians on Carnot groups

Annunziata Loiudice

Abstract We provide existence results for the quasilinear subelliptic Dirichlet problem

$$-\Delta_{p,\mathbb{G}} u = |u|^{p^*-2}u + g(\xi, u) \quad \text{in } \Omega, \quad u \in S_0^{1,p}(\Omega),$$

where $\Delta_{p,\mathbb{G}}$ is the p-sub-Laplacian on a Carnot group $\mathbb{G}$, $p^* = pQ/(Q-p)$ is the critical Sobolev exponent in this context, Ω is a bounded domain of $\mathbb{G}$ and $g(\xi, u)$ is a subcritical perturbation. By means of standard variational methods adapted to the stratified context, we prove the existence of solutions both in the mountain pass and in the linking case. A crucial ingredient in this abstract framework is the knowledge of the exact rate of decay of the p-Sobolev extremals on Carnot groups.

Keywords p-sub-Laplacian · Carnot groups · Critical exponents · Existence results

1 Introduction

In this paper we provide existence results for the following quasilinear subelliptic problem with critical Sobolev exponent

$$\begin{cases} -\Delta_{p,\mathbb{G}} u = |u|^{p^*-2}u + g(\xi, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_g}$$

under suitable subcritical assumptions on the lower order perturbation $g(\xi, u)$. Here, $\Delta_{p,\mathbb{G}}$ is the p-sub-Laplacian operator on a Carnot group $\mathbb{G}$ of homogeneous dimension Q, where $1 < p < Q$, the exponent $p^* = pQ/(Q-p)$ is the critical Sobolev exponent in this context, Ω is a bounded domain of $\mathbb{G}$ and

Annunziata Loiudice
Department of Mathematics, University of Bari, Via Orabona 4, Bari (Italy), e-mail: annunziata.loiudice@uniba.it

the lower order term satisfies subcritical growth assumptions. In particular, we obtain existence results for the case $g(\xi, u) = \lambda |u|^{p-2}u$, with $\lambda \in \mathbb{R}$.

The present results extend to the p-sub-Laplacian case the existence results obtained by the author in [30] for the semilinear Carnot case $p = 2$, subsequently generalized in [31] to the semilinear case with a Hardy-Sobolev nonlinearity and in [35] to the case with a Hardy-type perturbation.

We recall that a great deal of interest has been paid in the literature to the topic of subelliptic equations with critical Sobolev exponent or general power-type nonlinearities on stratified Lie groups. See e.g. [4], [6], [12], [22], [23], [25], [29], [30–35], [37], [39], [41], [44], [45], [48] and the references therein. In particular, in the recent papers [44], [45], interesting generalizations of variational-type results are obtained for Rockland operators on general graded Lie groups (see [15] for an overview on this functional setting). Concerning the quasilinear case, we recall that Vassilev in [48] studies the main aspects of the critical equation

$$-\Delta_{p,\mathbb{G}} u = |u|^{p^*-2}u, \qquad u \in S_0^{1,p}(\Omega),$$

where Ω is an arbitrary open subset of $\mathbb{G}$. Precisely, he proves global boundedness and interior regularity of solutions, discusses the problem of the regularity near the characteristic set of the boundary and, in the case $\Omega = \mathbb{G}$, obtains the existence of ground state solutions.

In [33], the author establishes the decay of positive entire solutions $u \in S^{1,p}(\mathbb{G})$ of the critical equation

$$-\Delta_{p,\mathbb{G}} u = u^{p^*-1} \quad \text{in } \mathbb{G},$$

obtaining that they have the following asymptotic behavior at infinity

$$u(\xi) \sim \frac{1}{d(\xi)^{(Q-p)/(p-1)}} \quad \text{as } d(\xi) \to \infty, \tag{1}$$

where d is any fixed homogeneous norm on $\mathbb{G}$. This result applies, in particular, to the extremals of the p-Sobolev inequality on Carnot groups (4) and it turns out to be a useful tool in adapting the well-known Brezis-Nirenberg type techniques [8] to problems of the type (P_g), in absence of the explicit knowledge of Sobolev minimizers. In fact, such minimizers are only known when $\mathbb{G}$ is a Iwasawa-type group and $p = 2$.

Further recent results for quasilinear equations and systems on Carnot groups can be found e.g. in [7], [13], [14], [19], [39], [41], [43], [46], [47]. In particular, Pucci-Temperini in [41] obtain existence of entire solutions to the problem

$$-\Delta_{p,H} u = k(\xi)|u|^{p^*-2}u + \lambda w(\xi)|u|^{q-2}u, \quad u \in S^1(\mathbb{H}^n),$$

in the model case of the Heisenberg group $\mathbb{G} = \mathbb{H}^n$, where $p \leq q < p^*$, under appropriate hypotheses on k and w.

Let us, now, introduce our existence results on bounded domains for problem (P_g). Let $\mathbb{G}$ be a Carnot group of homogeneous dimension Q and, for $1 < p < Q$, let

$$\Delta_{p,\mathbb{G}} u = \sum_{i=1}^{m} X_i(|Xu|^{p-2} X_i u)$$

be the p-sub-Laplacian operator on $\mathbb{G}$ (see Section 2 for the definition). We denote by $S_0^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{S_0^{1,p}(\Omega)} := (\int_\Omega |Xu|^p \,\mathrm{d}\xi)^{1/p}.$$

We shall deal with weak solutions of problem (P_g), i.e. functions $u \in S_0^{1,p}(\Omega)$ such that

$$\int_\Omega |Xu|^{p-2} < Xu, X\phi > \mathrm{d}\xi = \int_\Omega |u|^{p^*-2} u\phi \,\mathrm{d}\xi + \int_\Omega g(\xi, u)\phi \,\mathrm{d}\xi, \;\; \forall \phi \in \mathcal{C}_0^\infty(\Omega).$$

Let the functional $J : S_0^{1,p}(\Omega) \to \mathbb{R}$ be defined as

$$J(u) = \frac{1}{p} \int_\Omega |Xu|^p \,\mathrm{d}\xi - \frac{1}{p^*} \int_\Omega |u|^{p^*} \,\mathrm{d}\xi - \int_\Omega G(\xi, u) \,\mathrm{d}\xi,$$

where $G(\xi, s) = \int_0^s g(\xi, t) dt$. If g is continuous, then $J \in C^1(S_0^{1,p}(\Omega), \mathbb{R})$ and the critical points of J corresponds to weak solutions of equation (P_g).

Concerning the lower order term, following [1], [24], we assume that g is subcritical in the following sense

$$\begin{cases} g : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function satisfying:} \\ \qquad \forall \varepsilon > 0, \exists\, a_\varepsilon \in L^{\frac{pQ}{Q(p-1)+p}} \text{ such that} \\ |g(\xi, s)| \leq a_\varepsilon(\xi) + \varepsilon |s|^{\frac{Q(p-1)+p}{Q-p}} \text{ for a.e.} \, \xi \in \Omega, \forall s \in \mathbb{R}. \end{cases} \tag{2}$$

Moreover, other assumptions will be required on the primitive $G(\xi, s) = \int_0^s g(\xi, t) \,\mathrm{d}t$. In particular, we assume that

$$G(\xi, s) \geq 0 \quad \text{for a.e.} \, \xi \in \Omega, \quad \forall s \in \mathbb{R}, \tag{3}$$

while $g(\xi, s)$ is allowed to change sign. Further assumptions on G will be required, according to the different cases to be considered, namely the case when J has a mountain pass geometry or the case where J has a linking structure, with or without resonance. Roughly speaking, these three cases correspond to

$$0 \leq \lim_{s \to 0^+} \frac{G(\xi, s)}{s^p} < \frac{\lambda_1}{p}, \quad \lim_{s \to 0^+} \frac{G(\xi, s)}{s^p} = \frac{\lambda_1}{p}, \quad \frac{\lambda_1}{p} < \lim_{s \to 0^+} \frac{G(\xi, s)}{s^p},$$

where λ_1 denotes the first eigenvalue of $-\Delta_{p,\mathbb{G}}$ with Dirichlet boundary condition, that is,

$$\lambda_1 = \min_{u \in S_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|Xu\|_p^p}{\|u\|_p^p}.$$

As in the Euclidean case, in view of Lemma 1 below, the existence results in the different three cases will be obtained by constructing suitable Palais-Smale sequences (in short PS-sequences) for J at a level $c \in (0, \frac{S^{Q/p}}{Q})$, where S denotes the best constant in the Sobolev inequality (4). To this aim, as in the semilinear cases in [30], [31], [35] the behavior at infinity of the extremals for the Sobolev inequality on groups recalled in (1) will be used in a crucial way. We finally quote that analogous considerations have been used in the Euclidean setting to treat the quasilinear nonlocal case (see [38]).

The paper is organized as follows. In Section 2, we introduce the functional framework of Carnot groups; in Section 3 we treat the case when the functional J has a mountain pass geometry: we state the existence results in Theorems 1, 2 and 3, introducing the appropriate additional hypotheses on G, and we give a sketch of the proofs; in Section 4, we consider the case when J has a linking geometry, treating both the resonance and the non resonance case; the related existence results are contained in Theorem 4 and 5.

2 The functional setting

Let us briefly introduce the functional setting of Carnot groups. For a complete overview, we refer the reader to the monographs [5], [15] and the classical papers [16], [17].

A Carnot group $(\mathbb{G}, \circ)$ (or Stratified Lie group) is a connected, simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ admits a stratification, namely a decomposition $\mathfrak{g} = \bigoplus_{j=1}^r V_j$, such that $[V_1, V_j] = V_{j+1}$ for $1 \leq j < r$, and $[V_1, V_r] = \{0\}$. The number r is called the *step* of the group $\mathbb{G}$ and the integer $Q = \sum_{i=1}^r i \,\mathrm{dim} V_i$ is the *homogeneous dimension* of $\mathbb{G}$. Note that, if $Q \leq 3$, then $\mathbb{G}$ is necessarily the ordinary Euclidean space $\mathbb{G} = (\mathbb{R}^N, +)$.

The simplest non-abelian Carnot group is the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$, which is a two-step Carnot group with homogeneous dimension $Q = 2n + 2$ and composition law given by $\xi \circ \xi' = (x + x', y + y', t + t' + 2(\langle x', y\rangle - \langle x, y'\rangle))$, for every $\xi = (x, y, t)$, $\xi' = (x', y', t') \in R^{2n+1}$, where $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

By means of the natural identification of $\mathbb{G}$ with its Lie algebra via the exponential map, which we shall assume throughout, it is not restrictive to suppose that $\mathbb{G}$ is a homogeneous Carnot group, according to the definition in [5], i.e. $\mathbb{G} = \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \ldots \times \mathbb{R}^{N_r}$, where $N_i = \mathrm{dim} V_i$, endowed with dilations δ_λ of the form

$$\delta_\lambda(\xi) = (\lambda\,\xi^{(1)}, \lambda^2\,\xi^{(2)}, \cdots, \lambda^r\,\xi^{(r)}),$$

where $\xi^{(j)} \in \mathbb{R}^{N_j}$ for $j = 1, \ldots, r$. Let $m := N_1$ and let $X_1, \ldots, X_m$ be the set of left invariant vector fields of V_1 that coincide at the origin with the first m partial derivatives. It holds that $rank(Lie\{X_1, \ldots, X_m\}) = N$ at any point of $\mathbb{G}$. We shall denote by $X = (X_1, \ldots, X_m)$ such system of vector fields. Then, the differential operator

$$\Delta_{p,\mathbb{G}} u := \sum_{i=1}^{m} X_i(|Xu|^{p-2} X_i u)$$

is called the canonical *p-sub-Laplacian* on $\mathbb{G}$. Note that for any $c > 0$, one has $\Delta_{p,\mathbb{G}}(cu) = c^{p-1}\Delta_{p,\mathbb{G}} u$ and furthermore, since the X_j's are homogeneous of degree one with respect to the dilations δ_λ, the operator $\Delta_{p,\mathbb{G}}$ is homogeneous of degree p with respect to δ_λ, namely

$$\Delta_{p,\mathbb{G}}(u \circ \delta_\lambda) = \lambda^p \Delta_{p,\mathbb{G}} u \circ \delta_\lambda.$$

By definition, a homogeneous norm on $\mathbb{G}$ is a continuous function $d : \mathbb{G} \to [0, +\infty)$, smooth away from the origin, such that $d(\delta_\lambda(\xi)) = \lambda\, d(\xi)$, for every $\lambda > 0$ and $\xi \in \mathbb{G}$,$d(\xi) = 0$ if and only if $\xi = 0$. We recall that any two homogeneous norms on a Carnot group $\mathbb{G}$ are equivalent, as observed in [5, Prop. 5.1.4]. If we let $d(\xi, \eta) := d(\eta^{-1} \circ \xi)$, d is a pseudo-distance on $\mathbb{G}$. Throughout the paper, d will indicate a fixed homogeneous norm on $\mathbb{G}$; we shall denote by $B(\xi, r)$ the d-ball with center at ξ and radius r, i.e.

$$B(\xi, r) = \{\eta \in \mathbb{G} \,|\, d(\xi^{-1} \circ \eta) < r\},$$

and we will simply denote by B_r the d-ball centered at 0 with radius r.

The starting point of the variational formulation of our problem is the validity of the following Sobolev-type inequality on $\mathbb{G}$ (see Folland [16]): for any $p \in (1, Q)$, there exists $S > 0$, depending on p and $\mathbb{G}$, such that

$$\int_{\mathbb{G}} |Xu|^p \,\mathrm{d}\xi \geq S \left(\int_{\mathbb{G}} |u|^{p^*} \,\mathrm{d}\xi \right)^{p/p^*}, \quad \forall\, u \in C_0^\infty(\mathbb{G}). \tag{4}$$

It is know that the best constant in (4) is achieved (see [23], [48]); however, the explicit form of the extremal functions is not known, except for the case when $p = 2$ and $\mathbb{G}$ is a group of Iwasawa type (see Jerison-Lee [28], Frank-Lieb [18] for the Heisenberg case, Ivanov-Minchev-Vassilev [27] and Christ-Liu-Zhang [11] for the remaining cases). Nevertheless, relevant qualitative properties of such extremals in the general case have been obtained by the author in [33].

Concerning the main regularity tools for quasilinear subelliptic equations, such as Moser-type estimates and Harnack-type inequality, we refer to Capogna-Danielli-Garofalo [9]. Moreover, we indicate the paper [2] for an overview on the main aspects of nonlinear potential theory on Carnot groups. We also quote [3] for a strong maximum principle for quasilinear equations involving Hörmander vector fields.

3 The mountain pass case

In this section, we treat the case when J has a mountain pass geometry. We introduce the additional needed assumptions on G and we state the related existence results in Theorems 1, 2 and 3 below. Finally, we sketch the proof of the theorems.

Before introducing the additional assumptions which ensure the mountain pass geometry, we state a compactness result which is valid under the only assumption (2). Recall here that a sequence $\{u_n\} \subset S_0^1(\Omega)$ is called a Palais-Smale sequence (PS sequence in short) for J at level c if $J(u_n) \to c$ and $J'(u_n) \to 0$ in $S^{-1,p'}(\Omega)$.

Lemma 1 *Assume that* (2) *holds. If* $\{u_n\} \subset S_0^1(\Omega)$ *is a PS sequence for* J *at level* $c \in (0, \frac{S^{Q/p}}{Q})$, *there exists* $u \in S_0^1(\Omega) \setminus \{0\}$ *such that* $u_n \rightharpoonup u$ *up to a subsequence and* $J'(u) = 0$.

The proof is standard and it will be omitted, referring to the Euclidean counterpart (see, for instance, Lemma 1 in [1]). In view of the above result, the solutions to problem (P_g), both in the mountain pass and in the linking case, will be obtained by constructing a PS sequence at a level $c \in (0, \frac{S^{Q/p}}{Q})$.

Assume, now, that there exist an open subset $\Omega_0 \subset \Omega$ and some constants $\sigma, \delta, \mu > 0$ and $a, b > 0$, $a < b$, such that

$$G(\xi, s) \leq \frac{1}{p}(\lambda_1 - \sigma)|s|^p \qquad \text{for a.e. } \xi \in \Omega, \quad \forall |s| \leq \delta \tag{5}$$

and

$$G(\xi, s) \geq \mu \qquad \text{for a.e. } \xi \in \Omega_0, \forall s \in [a, b]. \tag{6}$$

Under these assumptions, the following existence results hold.

Theorem 1 *Assume that* (2), (3), (5), (6) *hold.*
If $1 < p^2 < Q$, *then problem* (P_g) *admits a positive solution.*
If $Q = p^2$ *and* μ *in* (6) *is large enough, then problem* (P_g) *admits a positive solution.*

From the above theorem, we obtain, for the particular case $g(\xi, u) = \lambda |u|^{p-2} u$, the following result, which was proved in the Euclidean context by Garcìa Azorero and Peral [20] (see also [26] for related regularity and nonexistence results).

Corollary 1 *Let $1 < p^2 \leq Q$. Then, problem*

$$\begin{cases} -\Delta_{p,\mathbb{G}} u = |u|^{p^*-2}u + \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P_λ}$$

admits a positive solution for any $\lambda \in (0, \lambda_1)$.

If, instead, $p < Q < p^2$, in the ordinary Euclidean setting we are in the case of *critical dimensions* in the sense of Pucci-Serrin [40]. Therefore, in order to get existence of solutions for problem (P_g), the assumption (6) is no longer sufficient; we require that there exists a nonempty open set $\Omega_0 \subset \Omega$ such that

$$\lim_{s\to+\infty} \frac{G(\xi,s)}{s^\beta} \quad \text{uniformly in } \Omega_0, \tag{7}$$

where $\beta = \frac{p(Qp+p-2Q)}{(p-1)(Q-p)}$. Under this additional assumption, the following result holds.

Theorem 2 *Let $1 < p < Q < p^2$. Assume that conditions* (2)*,* (3)*,* (5) *and* (7) *hold. Then problem (P_g) admits a positive solution.*

We notice that, in the Euclidean setting, Theorems 1 and 2 were generalization of results proved in [21].

Finally, in the same range of dimensions considered in Theorem 2, we can also prove the following result about problem (P_λ), which provides existence of solutions in a left neighborhood of λ_1. For the semilinear subelliptic counterpart, see [30, Theorem 1.2].

Theorem 3 *Let $\Lambda = S|\Omega|^{-p/Q}$ and assume that $1 < p < Q < p^2$ and $\lambda \in (\lambda_1 - \Lambda, \lambda_1)$. Then, problem (P_λ) admits a positive solution.*

In what follows, we prove the existence results stated above. The idea of the proofs is to find a nonnegative function $v \in S_0^{1,p}(\Omega)$ such that $\max_{t\geq 0} J(tv) < \frac{S^{Q/p}}{Q}$. Indeed, noting that there exists $t_v > 0$ such that $J(t_v v) < 0$, consider the set

$$\Gamma_v = \{\gamma \in C([0,1]) | \gamma(0) = 0, \gamma(1) = t_v v\}$$

and the inf-max value

$$c := \inf_{\gamma\in\Gamma_v} \max_{t\in[0,1]} J(\gamma(t)).$$

By standard variational arguments (see, for instance, [42]), if such v exists, we obtain a PS sequence at level $c \in (0, \frac{S^{Q/p}}{Q})$. In the proofs of Theorems 1, 2 and 3, a different choice of v will be done.

Proof of Theorem 1. Let $U > 0$ be a fixed extremal function for (4). We can assume, up to a normalization, that $\|XU\|_p^p = \|U\|_{p^*}^{p^*} = S^{Q/p}$. For $\epsilon > 0$, define

$$U_\epsilon(\xi) = \epsilon^{-(Q-p)/p} U(\delta_{1/\epsilon}(\xi)), \quad \xi \in \mathbb{G}. \tag{8}$$

Of course, U_ϵ are also minimizers and verify $\|XU_\epsilon\|_p^p = \|U_\epsilon\|_{p^*}^{p^*} = S^{Q/p}$. Now, observe that it is not restrictive to suppose $0 \in \Omega$. Let $R > 0$ be such that $B_R \subset \Omega$ and let $\varphi \in C_0^\infty(B_R)$ be a cut-off function, $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_{R/2}$. Define

$$u_\epsilon(\xi) = \varphi(\xi) U_\epsilon(\xi). \tag{9}$$

Reasoning as in [30], by exploiting the asymptotic estimate (1) proved in [33], we are able to prove that

$$\|Xu_\epsilon\|_p^p \le S^{Q/p} + C\epsilon^{(Q-p)/(p-1)}, \quad \|u_\epsilon\|_{p^*}^{p^*} \ge S^{Q/p} - C\epsilon^{Q/(p-1)}. \tag{10}$$

We claim that, for ϵ sufficiently small, it holds

$$\max_{t\ge 0} J(tu_\epsilon) < \frac{1}{Q} S^{Q/p}. \tag{11}$$

Indeed, assume by contradiction that for all $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$J(t_\epsilon u_\epsilon) \ge \frac{1}{Q} S^{Q/p}. \tag{12}$$

It is easily seen that, as $\epsilon \to 0$, the sequence $\{t_\epsilon\}$ is upper and lower bounded by two positive constants; moreover, by the expansions (10), as $\epsilon \to 0$ we have

$$\begin{aligned} \frac{\|X(t_\epsilon u_\epsilon)\|_p^p}{p} - \frac{\|t_\epsilon u_\epsilon\|_{p^*}^{p^*}}{p^*} &\le \frac{S^{Q/p}}{Q} + \left(t_\epsilon^p - 1 - \frac{Q-p}{Q}(t_\epsilon^{p^*} - 1)\right) \frac{S^{Q/p}}{p} \\ &\quad + O(\epsilon^{(Q-p)/(p-1)}) \\ &\le \frac{S^{Q/p}}{Q} + O(\epsilon^{(Q-p)/(p-1)}). \end{aligned} \tag{13}$$

It can be also verified that there exists $c_1, c_2 > 0$ such that, for ϵ sufficiently small

$$c_1\epsilon^{1/p} < d(\xi) < c_2\epsilon^{1/p} \ \text{ implies } \ a < t_\epsilon u_\epsilon(\xi) < b,$$

where a, b are as in (6). Hence, since $B_\epsilon \subset \Omega_0$ for small ϵ, by (3) and (6) we have

$$\int_\Omega G(\xi, t_\epsilon u_\epsilon) \ge c\mu \int_{c_1\epsilon^{1/p}}^{c_2\epsilon^{1/p}} \rho^{Q-1} \mathrm{d}\rho \ge c\mu\epsilon^{Q/p}, \tag{14}$$

where we used the appropriate polar coordinates formula. So, if $Q > p^2$, we get that there exists a function $\zeta = \zeta(\epsilon)$ such that $\lim_{\epsilon\to 0} \zeta(\epsilon) = +\infty$ such that, for ϵ small,

$$\int_\Omega G(\xi, t_\epsilon u_\epsilon) \ge \zeta(\epsilon) \cdot \epsilon^{(Q-p)/(p-1)}.$$

Hence, from (13) and the latter estimate, we get that, for ϵ small enough,

$$J(t_\epsilon u_\epsilon) < \frac{S^{Q/p}}{Q}.$$

Analogously, if $Q = p^2$, from (13) and (14), we get

$$J(t_\epsilon u_\epsilon) \le \frac{S^{Q/p}}{Q} + O(\epsilon^p) - c\mu\epsilon^p < \frac{S^{Q/p}}{Q},$$

for suitable small ϵ and μ large enough. Hence,(12) cannot hold. So, from (11), we obtain a Palais-Smale sequence for J, belonging to the cone of nonnegative functions in $S_0^{1,p}(\Omega)$, at level $c \in (0, \frac{S^{Q/p}}{Q})$: its weak limit is nonnegative, nontrivial and it solves (P_g).

Finally, such solution turns out to be strictly positive by the nonlinear strong maximum principle in [3]. □

Proof of Theorem 2. The proof follows the scheme of the proof of Theorem 1, except for estimate (14), which is replaced by the following considerations. From (7), there exists an increasing function ζ such that $\lim_{t\to+\infty}\zeta(t) = +\infty$ such that $G(\xi, s) \ge \zeta(s)\cdot s^\beta$ for a.e. $\xi \in \Omega_0$ and all $s > 0$. Therefore,

$$\begin{aligned}\int_\Omega G(\xi, t_\epsilon u_\epsilon) &\ge \zeta\left(c\epsilon^{(p-Q)/p}\right)\epsilon^{\beta(p-Q)/p}\int_0^\epsilon \rho^{Q-1}\mathrm{d}\rho \\ &\ge \zeta\left(c\epsilon^{(p-Q)/p}\right)\epsilon^{(Q-p)/(p-1)},\end{aligned} \tag{15}$$

where it is used that $\min_{d(\xi)\le\epsilon} t_\epsilon u_\epsilon \ge c\epsilon^{(p-Q)/p}$. □

Proof of Theorem 3. Also in this case, we prove that the PS sequence obtained by the mountain pass argument is at level below the compactness threshold. Following the idea in [10], let e_1 be the first positive eigenfunction of $-\Delta_{p,\mathbb{G}}$ in Ω and let us estimate $J(te_1)$, where $t > 0$. We have

$$\begin{aligned}J(te_1) &= \frac{\lambda_1-\lambda}{p}t^p\|e_1\|_p^p - \frac{Q-p}{Qp}t^{p^*}\|e_1\|_{p^*}^{p^*} \\ &\le \frac{\lambda_1-\lambda}{p}|\Omega|^{p/Q}t^p\|e_1\|_{p^*}^p - \frac{Q-p}{Qp}t^{p^*}\|e_1\|_{p^*}^{p^*} \\ &\le \frac{(\lambda_1-\lambda)^{Q/p}}{Q}|\Omega|,\end{aligned} \tag{16}$$

where in the last inequality we have maximized with respect to $t \ge 0$. So, if $\lambda \in (\lambda_1 - \Lambda, \lambda_1)$, then

$$\max_{t\ge0} J(te_1) < \frac{1}{Q}S^{Q/p},$$

and the existence of a solution follows as in the preceding proofs. □

4 The case with linking geometry

This section is devoted to the case with linking geometry. We introduce the necessary notation and, under the appropriate hypotheses on G, we state the related existence results (see Theorems 4 and 5 below). Finally, after some preliminary lemmas, we sketch the proofs.

4.1 Statement of the results

Let us introduce some further notation. Let $w \in S^{-1,p'}(\Omega)$, the dual space of $S_0^{1,p}(\Omega)$. We denote by $E_w^\perp$ the subspace of $S_0^{1,p}(\Omega)$ orthogonal to w, i.e.

$$E_w^\perp = \{u \in S_0^{1,p}(\Omega) \mid \langle w, u \rangle = 0\},$$

where $\langle , \rangle$ is the duality product between $S^{-1,p'}(\Omega)$ and $S_0^{1,p}(\Omega)$; denote by $B^1 = \{u \in S_0^{1,p}(\Omega) \mid \|u\|_p = 1\}$. Let

$$\overline{\lambda} = \sup_{w \in S^{-1,p'}(\Omega)} \inf_{u \in E_w^\perp \cap B^1} \|Xu\|_p^p .$$

It is possible to verify that $\overline{\lambda} \le \lambda_2$, where λ_2 is the second eigenvalue of $-\Delta_{p,\mathbb{G}}$. If $p = 2$, then $\overline{\lambda} = \lambda_2$; if $p \neq 2$, it is not clear whether the equality holds or not. However, it holds that $\overline{\lambda} > \lambda_1$ (see Lemma 2 below).

The non-resonance case corresponds to requiring the following assumptions on $G(\xi, u)$:

$$\begin{aligned} \frac{1}{p}(\lambda_1 + \sigma)|s|^p \le G(\xi, s) \le \frac{1}{p}(\overline{\lambda} - \sigma)|s|^p \quad & \text{for a.e.}\, \xi \in \Omega, \forall |s| \le \delta \\ G(\xi, s) \ge \frac{1}{p}(\lambda_1 + \sigma)|s|^p - \frac{1}{p^*}|s|^{p^*} \quad & \text{for a.e.}\, \xi \in \Omega, \forall s \neq 0. \end{aligned} \tag{17}$$

We observe that (5) and (17) imply $g(\xi, 0) = 0$ a. e. in Ω and $u = 0$ is a solution of (P_g). In this case, we prove the following result.

Theorem 4 *If $1 < p^2 \le Q$, assume that* (2), (3) *and* (17) *hold; if $1 < p < Q < p^2$, assume that* (2), (3), (7), (17) *hold. Then, problem (P_g) admits a nontrivial solution.*

We conclude with the case of resonance near the origin. We assume that there exists $\delta > 0$ and $\sigma \in (0, 1/p^*)$ such that

$$\begin{aligned} \frac{1}{p}\lambda_1 |s|^p \le G(\xi, s) \le \frac{1}{p}(\overline{\lambda} - \sigma)|s|^p \quad & \text{for a.e.}\, \xi \in \Omega, \forall |s| \le \delta \\ G(\xi, s) \ge \frac{1}{p}\lambda_1 |s|^p - (\frac{1}{p^*} - \sigma)|s|^{p^*} \quad & \text{for a.e.}\, \xi \in \Omega, \forall s \in \mathbb{R}. \end{aligned} \tag{18}$$

We also need the following condition on $G(\xi, s)$ at infinity: there exists an open nonempty set $\Omega_0 \subset \Omega$ such that

$$\lim_{s \to +\infty} \frac{G(\xi, s)}{s^\gamma} = +\infty \quad \text{uniformly in} \Omega_0, \tag{19}$$

where $\gamma = \frac{Qp(Qp+2p-2Q)}{(Q-p)(Qp+p-Q)}$. The following result holds.

Theorem 5 *Let $1 < p < Q$ and assume that* (2), (3), (18), (19) *hold. Then, problem (P_g) admits a nontrivial solution.*

We observe that $\gamma < p^*$ for any $1 < p < Q$ and $\gamma > 0$ for $p > \frac{2Q}{Q+2}$. From Theorem 5, we deduce the following result for problem (P_λ).

Corollary 2 *Let $p > 1$ and Q such that $\frac{Q^2}{Q+1} > p^2$. Then, problem (P_λ) admits a nontrivial solution for $\lambda = \lambda_1$.*

4.2 Proof of the results

In this subsection, we sketch the proof of Theorems 4 and 5. We begin with some preliminary lemmas. We first need a lemma which provides a sufficient condition for the linking geometry to hold.

Lemma 2 *For any $w \in S^{-1,p'}(\Omega)$ such that $\langle w, e_1 \rangle \neq 0$ there exists a constant $c_w > 0$ depending on w such that if $u \in E_w^\perp$, then $\|Xu\|_p^p - \lambda_1 \|u\|_p^p \geq c_w \|Xu\|_p^p$; therefore, $\overline{\lambda} > \lambda_1$.*

Proof The proof follows the Euclidean outline in [1], so we omit it. □

Now, let e_1 denote the positive eigenfunction relative to λ_1 and such that $\|e_1\|_p = 1$ and let Ω_0 be as in (7) or (19). Without restriction, we can assume that $0 \in \Omega_0 \subset \Omega$. Denote by B_r the d-ball centered at 0 with radius r. For $m \in \mathbb{N}$ sufficiently large so that $B_{2/m} \subset \Omega_0$, define the functions $\phi_m : \Omega \to \mathbb{R}$ as follows

$$\phi_m(\xi) := \begin{cases} 0 & \text{if } \xi \in B_{1/m} \\ m\, d(\xi) - 1 & \text{if } \xi \in B_{2/m} \setminus B_{1/m} \\ 1 & \text{if } \xi \in \Omega \setminus B_{2/m}. \end{cases} \tag{20}$$

Let $e_1^m := \phi_m e_1$ and let $E^m := \operatorname{span}\{e_1^m\}$. We prove the following approximation result.

Lemma 3 *As $m \to \infty$, there holds*

$$e_1^m \to e_1 \quad \text{in } S_0^{1,p}(\Omega) \quad \text{and} \quad \|X(e_1^m)\|_p^p \leq \lambda_1 + \nu m^{p-Q}, \tag{21}$$

for a suitable $\nu > 0$.

Proof Let us compute

$$\begin{aligned}\|X(e_1^m - e_1)\|_p &= \|e_1 X\phi_m + (\phi_m - 1)Xe_1\|_p \\ &\le \|e_1 X\phi_m\|_p + \|(\phi_m - 1)Xe_1\|_p \\ &\le c(m^{p-Q} + m^{-Q}) \to 0,\end{aligned} \tag{22}$$

where we have used that

$$\int_\Omega |X\phi_m|^p = m^p \int_{B_{2/m}\setminus B_{1/m}} |Xd|^p \le Cm^p |B_{2/m}\setminus B_{1/m}| = Cm^{p-Q},$$

due to the boundedness of $\psi = |Xd|$. Hence, $e_1^m \to e_1$ and by the definition of e_1^m, the second estimate in (21) follows. □

Let now observe that, for all $\delta > 0$, there exists $w \in S^{-1,p'}(\Omega)$ such that $\min_{u\in E_w^\perp \cap B^1} \|Xu\|_p \ge \overline{\lambda} - \delta$. Let, for such w, $E_\delta^\perp := E_w^\perp$.

Lemma 4 *Assume that* (2), (3) *and either* (17) *or* (18) *hold. Then, there exists* $\alpha, \delta, \rho > 0$ *such that*

$$J(u) \ge \alpha \quad \forall u \in \partial B_\rho \cap E_\delta^\perp.$$

Now, consider the family of Sobolev minimizers U_ϵ defined in (8) and, for $m \in \mathbb{N}$, take a cut-off function $\eta_m \in C_0^\infty(B_{1/m})$, $0 \le \eta \le 1$, such that $\eta_m \equiv 1$ in $B_{1/2m}$ and $\|X\eta_m\|_\infty \le 3m$. Then, for $\epsilon > 0$, define

$$u_\epsilon^m(\xi) = \eta_m(\xi) U_\epsilon(\xi), \quad \xi \in \mathbb{G}. \tag{23}$$

Then, as $\epsilon m \to 0$, analogous estimates as in (10) hold

$$\|Xu_\epsilon^m\|_p^p \le S^{Q/p} + C(\epsilon m)^{(Q-p)/(p-1)}, \quad \|u_\epsilon^m\|_{p^*}^{p^*} \ge S^{Q/p} - C(\epsilon m)^{Q/(p-1)}. \tag{24}$$

Note that, by construction, for all $\epsilon > 0$ and $m \in \mathbb{N}$ we get

$$\mathrm{supp}(u_\epsilon^m) \cap \mathrm{supp}(e_1^m) = \emptyset. \tag{25}$$

Now, define

$$Q_m^\epsilon = \{u \in S_0^{1,p}(\Omega) \,|\, u = ae_1^m + bu_\epsilon^m, |a| \le R, 0 \le b \le R\}.$$

It can be verified that ∂Q_m^ϵ and $\partial B_\rho \cap E^\perp$ link (see [42]) if $R > \rho$, where ρ is as in Lemma 4. Moreover, by (25), if R and m are large enough, then $J(u) \le 0$ for all $u \in \partial Q_m^\epsilon$. By these choices on R and m, if we let

$$\Gamma = \{h \in C(Q_m^\epsilon, S_0^{1,p}(\Omega) \,|\, h(u) = u, \forall u \in \partial Q_m^\epsilon\},$$

by standard arguments, we obtain a PS sequence for J at level

$$c = \inf_{h \in \Gamma} \max_{u \in Q_m^\epsilon} J(h(u)).$$

Then, the conclusions of Theorems 4 and 5 will follow by showing that, for ϵ sufficiently small, $c < \frac{S^{Q/p}}{Q}$.

Proof of Theorem 4. Let $1 < p^2 \leq Q$. Choose m large enough so that $\nu m^{p-Q} < \sigma$, where ν is as in Lemma 3 and σ is as in (17). It follows that

$$\forall w \in E^m \quad J(w) \leq 0. \tag{26}$$

We prove that there exists $\epsilon > 0$ such that

$$\max_{u \in Q_m^\epsilon} J(u) < \frac{1}{Q} S^{Q/p}. \tag{27}$$

Arguing by contradiction, assume that

$$\forall \epsilon > 0, \quad \max_{u \in Q_m^\epsilon} J(u) \geq \frac{1}{Q} S^{Q/p}.$$

By the compactness of the set $\{u \in Q_m^\epsilon | J(u) \geq 0\}$, for all $\epsilon > 0$ there exist $w_\epsilon \in E^m$ and $t_\epsilon \geq 0$ such that, letting $v_\epsilon = w_\epsilon + t_\epsilon u_\epsilon^m$, it holds

$$J(v_\epsilon) = \max_{u \in Q_m^\epsilon} J(u) \geq \frac{1}{Q} S^{Q/p},$$

that is

$$\frac{1}{p} \|X v_\epsilon\|_p^p - \int_\Omega G(\xi, v_\epsilon) - \frac{1}{p^*} \|v_\epsilon\|_{p^*}^{p^*} \geq \frac{1}{Q} S^{Q/p}, \quad \forall \epsilon > 0. \tag{28}$$

As in Theorem 1, it follows that t_ϵ is bounded between two positive constants. We now estimate the term $\int_\Omega G(\xi, t_\epsilon u_\epsilon)$. We claim that there exists a function $\zeta = \zeta(\epsilon)$ satisfying $\lim_{\epsilon \to 0} \zeta(\epsilon) = +\infty$, such that for ϵ sufficiently small, it holds

$$\int_\Omega G(\xi, t_\epsilon u_\epsilon) \geq \zeta(\epsilon) \cdot \epsilon^{(Q-p)/(p-1)}. \tag{29}$$

The above estimate can be seen as follows. For ϵ sufficiently small, there exists a constant $c_1 > 0$ such that $t_\epsilon U_\epsilon(\xi) \in (0, \delta)$ for all ξ such that $d(\xi) \geq c_1 \epsilon^{(p-1)/p^2}$; we also observe that, if $\xi \in B_{1/2m}$, then $u_\epsilon^m = U_\epsilon$. Therefore, by (17) and (1), we get

$$\int_{\Omega} G(\xi, t_\epsilon u_\epsilon) \geq c \int_{c_1\epsilon^{1/p}}^{1/2m} U_\epsilon^p(\xi)$$
$$\geq c\epsilon^{(Q-p)/(p-1)} \int_{c_1\epsilon^{1/p}}^{1/2m} \rho^{(p^2-Q-p+1)/(p-1)} d\rho \qquad (30)$$
$$= c\epsilon^{(Q-p)/(p-1)} \begin{cases} \epsilon^{(p^2-Q)/(p-1)} & \text{if } Q > p^2 \\ |\log \epsilon| & \text{if } Q = p^2. \end{cases}$$

Hence, the function $\zeta(\epsilon)$ in (29) is obtained.

So, from (13), (25), (26) and (30), we have

$$J(v_\epsilon) \leq J(t_\epsilon u_\epsilon) \leq \frac{S^{Q/p}}{Q} + (c - \zeta(\epsilon))\epsilon^{(Q-p)/(p-1)},$$

and choosing ϵ small, we get a contradiction with (28).

The case $p < Q < p^2$ can be treated analogously, taking into account estimate (15). □

Proof of Theorem 5. The proof follows the scheme of that of Theorem 4: as before, we can show that, (27) holds for sufficiently large $m \in \mathbb{N}$, under the asymptotic assumption (19) on G. We omit the details, referring to the Euclidean outline in [1]. □

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