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Critical problems with Hardy potential on Stratified Lie Groups

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Dedicated to the memory of my dearest professor Enrico Jannelli

Abstract

We prove existence and nonexistence results for positive solutions to the subelliptic Brezis-Nirenberg type problem with Hardy potential

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{d^2} u = u^{2^*-1} + \lambda u \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \partial\Omega,$$

extending to the Stratified setting well-known Euclidean results by Jannelli [J. Diff. Equ. 156, 1999]. Here, $\Delta_{\mathbb{G}}$ is a sub-Laplacian on an arbitrary Carnot group $\mathbb{G}$, Ω is a bounded domain of $\mathbb{G}$, $0 \in \Omega$, d is the $\Delta_{\mathbb{G}}$-gauge, $\psi := |\nabla_{\mathbb{G}} d|$, where $\nabla_{\mathbb{G}}$ is the horizontal gradient associated to $\Delta_{\mathbb{G}}$, $0 \leq \mu < \overline{\mu}$, where $\overline{\mu} = \left(\frac{Q-2}{2}\right)^2$ is the best Hardy constant on $\mathbb{G}$ and $\lambda \in \mathbb{R}$. The main difficulty in this abstract framework is the lack of knowledge of the ground state solutions to the limit problem

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{d^2} u = u^{2^*-1} \text{ on } \mathbb{G},$$

whose explicit expression is not known, except for the case when $\mu = 0$ and $\mathbb{G}$ is a group of Iwasawa-type. So, a preliminary refined analysis of qualitative properties of solutions to the above problem on the whole space is required, which has independent interest. In particular, Lorentz regularity and a priori decay estimates are obtained.

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1 Introduction

In this paper we provide existence and qualitative properties of solutions for semilinear subelliptic equations with Hardy potential and critical nonlinearities in the setting of Carnot groups.

Let $\mathbb{G}$ be a Carnot group of homogeneous dimension $Q \geq 3$ and let $\Omega \subset \mathbb{G}$ be a smooth bounded domain, $0 \in \Omega$. We consider the following Brezis-Nirenberg type problem on $\mathbb{G}$

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu \dfrac{\psi^2}{d^2} u &= |u|^{2^*-2}u + \lambda u \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{\mathbb{G}}$ is a sub-Laplacian operator on $\mathbb{G}$, d is the natural gauge associated with the fundamental solution of $-\Delta_{\mathbb{G}}$ on $\mathbb{G}$, $\psi := |\nabla_{\mathbb{G}} d|$, where $\nabla_{\mathbb{G}}$ is the horizontal gradient associated to $\Delta_{\mathbb{G}}$, λ is a real parameter and $0 \leq \mu < \overline{\mu}$, where $\overline{\mu} = \left(\frac{Q-2}{2}\right)^2$ is the best constant in the Hardy inequality on Carnot groups

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi \geq \overline{\mu} \int_{\mathbb{G}} \psi^2 \frac{u^2}{d^2} \, \mathrm{d}\xi, \quad \forall u \in C_0^\infty(\mathbb{G}), \tag{1.2}$$

and it is not attained (see Section 2). We shall deal with weak solutions, i.e. solutions in the Sobolev-Stein space $\mathcal{S}_0^1(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{S}_0^1(\Omega)} := \left(\int_\Omega |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi \right)^{1/2}.$$

When dealing with $\Omega = \mathbb{G}$, we shall simply denote $\mathcal{S}^1(\mathbb{G}) = \mathcal{S}_0^1(\mathbb{G})$. We recall that $u \in \mathcal{S}_0^1(\Omega)$ is called a weak solution of (1.1) if it satisfies

$$\int_\Omega \langle \nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \varphi \rangle \, \mathrm{d}\xi - \mu \int_\Omega \psi^2 \frac{u\varphi}{d^2} \, \mathrm{d}\xi = \int_\Omega |u|^{2^*-1} \varphi \, \mathrm{d}\xi + \lambda \int_\Omega u\varphi \, \mathrm{d}\xi, \qquad \forall \varphi \in \mathcal{S}_0^1(\Omega).$$

Our aim in this paper is to study existence and nonexistence results of positive solutions to problem (1.1) depending on the parameters λ, μ, belonging to the appropriate ranges. This will require a preliminary refined qualitative analysis of the limit problem

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{d^2} u = |u|^{2^*-2} u \;\; \text{on } \mathbb{G}, \tag{1.3}$$

whose ground state solutions are not known, except for the case when $\mathbb{G}$ is a Iwasawa-type group and $\mu = 0$ (see [37], [23], [11], [31].) This is indeed the main difficulty in the abstract stratified setting and, in the spirit of [40] where the case $\mu = 0$ was treated, this leads to study sharp regularity and a priori decay estimates for solutions to (1.3).

We recall that in the ordinary Euclidean case, the critical problem with Hardy perturbation together with a great amount of generalizations, has been widely studied by many authors in the last decades (see [56], [32], [20], [19], [14], [33], [16], just to cite the ones that are most related to the present paper, but the list cannot be exhaustive). A starting point was Terracini's paper [56], where the symmetry and explicit form of the ground state solutions of the limit problem on $\mathbb{R}^n$ was obtained. Precisely, letting

$$\alpha = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu} \quad \text{and} \quad \alpha' = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}, \tag{1.4}$$

positive weak solutions $u \in H^1(\mathbb{R}^n)$ to the critical problem $-\Delta u - \mu\frac{u}{|x|^2} = u^{2^*-1}$ were proved to take the form

$$U_\varepsilon(x) = \frac{C_\varepsilon}{(\varepsilon|x|^{\frac{\alpha}{\sqrt{\overline{\mu}}}} + |x|^{\frac{\alpha'}{\sqrt{\overline{\mu}}}})^{\sqrt{\overline{\mu}}}}, \quad \varepsilon > 0, \tag{1.5}$$

where $C_\varepsilon = \left(\frac{4\varepsilon n(\overline{\mu}-\mu)}{n-2}\right)^{\sqrt{\overline{\mu}}/2}$. Subsequently, Jannelli in [32] studied the corresponding Brezis-Nirenberg-type problem on bounded domains of $\mathbb{R}^n$ (see the celebrated paper [9] for the case $\mu = 0$), that is the linearly perturbed critical problem of the type (1.1) for the ordinary Laplacian on $\mathbb{R}^n$. In particular, he proved that in the ordinary Laplacian case, when $0 < \mu \le \overline{\mu} - 1$, problem (1.1) has at least one positive weak solution for all $0 < \lambda < \lambda_1$, where λ_1 indicates the first eigenvalue of the operator $\Delta - \frac{\mu}{|x|^2} I$ with Dirichlet boundary conditions; instead, when $\overline{\mu} - 1 < \mu < \overline{\mu}$, problem (1.1) has a positive solution in a left neighborhood of λ_1 and it has no solutions in a right neighborhood of 0. These results enlightened the phenomenon of the so-called critical dimensions in the sense of Pucci-Serrin [50]. Moreover, in [32] the crucial rôle of the summability of the generalized fundamental solution of the involved operator in the existence and nonexistence thresholds was put into evidence.

Let us now focus on the analogous problem in the setting of stratified Lie groups. In this context, the study of critical growth problem is a topical issue and it has received a great renewed interest in the last recent years. For the case $\mu = 0$, we refer the reader to the classical references [25], [27], [38], [39], [57]. See also related results and developments in [6], [30], [40–45], [47], [48], [51], [52] and the references therein.

Concerning the case with Hardy potential, we recall that the study of the perturbed operator $\mathcal{L}_\mu = -\Delta_{\mathbb{G}} - \mu\dfrac{\psi^2}{d^2} I$ was started in the Heisenberg group $\mathbb{G} = \mathbb{H}^n$ by Garofalo and Lanconelli in the seminal paper [24], where the Hardy-type inequality on $\mathbb{H}^n$ was established and unique continuation results were obtained. Concerning problems of type (1.1) with a power-type nonlinearity, a first existence result on $\mathbb{H}^n$ in the case of a subcritical nonlinearity is due to Mokrani [46]. Recently, the author in [42], [44] studied the behavior at the singularity of solutions to critical problems with Hardy term respectively in the stratified and Grushin case. Bordoni-Filippucci-Pucci [7] studied the existence of solutions to critical problems with Hardy perturbation in general domains of $\mathbb{H}^n$. See also [2], [28], [49] for some further results involving the subelliptic Hardy potential.

Now, let us introduce our results. First of all, we consider the limit problem (1.3). First, we show that ground state solutions exist, straightforwardly following the Euclidean scheme. Then we focus on the asymptotic behavior at infinity of the ground states, which is, in fact, the crucial ingredient in the Brezis-Nirenberg [9] existence proof for problem (1.1). Therefore, we are lead to investigate the decay of nonnegative solutions to the limit problem (1.3), which is the Euler Lagrange equation of such ground states, up to a Lagrange multiplier.

In [42], by means of subelliptic Moser-type estimates, the author proved that weak positive solutions to (1.3) are singular at the origin and, in particular, they have the following behavior

$$u(\xi) \sim d(\xi)^{-\alpha} \quad \text{as } d(\xi) \to 0. \tag{1.6}$$

In the ordinary Euclidean case, this result immediately gives the decay of solutions at infinity by using the invariance of the equation under the appropriate Kelvin transform (see, for instance, [19],

[16], respectively for the local and the nonlocal case). In the stratified case, instead, a Kelvin-type transform with suitable conformal properties is only available in the subclass of Iwasawa-type groups (see [12]), so in the general case a more subtle and refined regularity analysis is needed in order to get the asymptotic decay at infinity. We shall obtain it by adapting the method developed by the author in [43] for the quasilinear autonomous case, which in turn borrows ideas from [58], [59] and [60].

We also observe that a useful tool in order to obtain the desired decay estimates for (1.3) is to use the transformation $v = d^{\alpha} u$, where α is defined in (1.4). We get that, if $u \in S^1(\mathbb{G})$ is a solution of (1.3), then $v \in S^1(\mathbb{G}, d^{-2\alpha}\, \mathrm{d}\xi)$ and satisfies the equation

$$-\mathrm{div}_{\mathbb{G}} \left(d^{-2\alpha} \nabla_{\mathbb{G}} v \right) = \frac{|v|^{2^*-2} v}{d^{2^*\alpha}} \quad \text{in}\, \mathbb{G}. \tag{1.7}$$

So we are lead to study the above subelliptic problem in the appropriate weighted horizontal Sobolev space. One of the advantage of working with the transformed problem (1.7), instead of the original one (1.3), is that its solutions are bounded at 0. Moreover, we observe that the equation (1.7) has its own interest, since it is the equation satisfied by the extremals of the weighted Sobolev inequality recalled in (2.6).

Our main result can be stated as follows. Here, with the notation $f \sim g$ as $d(\xi) \to \infty$, we mean that there exists a constant $C > 0$ such that $C^{-1} g(\xi) \le f(\xi) \le C g(\xi)$ for $d(\xi)$ large.

Theorem 1.1. *Let $u \in \mathcal{S}^1(\mathbb{G})$ be a nonnegative weak solution to the limit problem (1.3) on $\mathbb{G}$. Then, u satisfies*

$$u(\xi) \sim d(\xi)^{-\alpha'}, \quad \text{as } d(\xi) \to \infty, \tag{1.8}$$

where α' is defined in (1.4).

Following the original idea by Jannelli and Solimini [35], the first step in determining the above asymptotic behavior is to establish the sharp L^p-weak regularity of solutions. We point out that in the Euclidean cases, the radial decreasing symmetry of solutions together with the $L^{n/\alpha',\infty}$-regularity immediately gives the desired optimal decay estimate $|u(x)| \le C/|x|^{\alpha'}$, for $|x|$ large, as observed by Brasco et al in [8, Lemma 2.9]. In the stratified case, instead, the solutions are not radial and the proof requires many additional efforts.

Once we obtain the mentioned qualitative properties for the limit problem (1.3), we turn to study the Brezis-Nirenberg problem (1.1) on bounded domains of $\mathbb{G}$. We first prove a Pohozaev-type identity for our subelliptic non autonomous problem and we obtain the nonexistence of positive solutions, sufficiently regular up to the boundary, for $\lambda \le 0$, when Ω is a starshaped domain with respect to the dilations of the group. In what follows, Z denotes the infinitesimal generator of the dilations of the group (see formula (2.3) below).

Theorem 1.2. *If $\Omega \subset \mathbb{G}$ is a smooth connected bounded domain, δ_λ-starshaped about the origin, problem (1.1) has no nonnegative nontrivial solutions $u \in \mathcal{S}^1_0(\Omega) \cap C^1(\overline{\Omega} \setminus \{0\})$ such that $Zu/d \in L^2(\Omega)$, for $\lambda \le 0$.*

Concerning the question on how the regularity assumptions up to the boundary in the above theorem can be weakened, by assuming some additional geometric conditions on the boundary, we refer to the deep discussion by Garofalo-Vassilev [27] for the case $\mu = 0$ (see also Remark 4.4 below).

Then, we turn to the existence of positive solutions, confining our analysis to the admissible range $0 < \lambda < \lambda_1$, where $\lambda_1 = \lambda_1(\mu)$ denotes the first eigenvalue of the operator $-\Delta_{\mathbb{G}} - \mu \frac{\psi^2}{d^2} I$ with Dirichlet boundary condition. The following statement holds.

Theorem 1.3. *Let $\Omega \subset \mathbb{G}$ be a smooth bounded domain, $0 \in \Omega$.*
(i) If $0 < \mu \leq \overline{\mu} - 1$, problem (1.1) *has at least one positive solution $u \in \mathcal{S}^1_0(\Omega)$ for any $0 < \lambda < \lambda_1$.*
(ii) If $\overline{\mu} - 1 < \mu < \overline{\mu}$, there exists $\lambda_* \in (0, \lambda_1)$ such that problem (1.1) *has at least one positive solution $u \in \mathcal{S}^1_0(\Omega)$ when $\lambda_* < \lambda < \lambda_1$.*

In the Euclidean case, this analysis is completed by proving that when $\overline{\mu} - 1 < \mu < \overline{\mu}$ and Ω is a ball centered at 0, there exists $\lambda_{**} \in (0, \lambda_1)$ such that problem (1.1) has no solutions for $0 < \lambda < \lambda_{**}$, thus showing the *criticality* of the range of parameters $\overline{\mu} - 1 < \mu < \overline{\mu}$. This result is achieved in [32] by means of a subtle Pohozaev-type identity relying on the radial symmetry of the solutions. In this context, due to the lack of radiality, different methods are needed and this will constitute the next step of this research.

The paper is organized as follows. In Section 2, we introduce the main notation and definitions about the functional setting of Carnot groups. In Section 3, we study the limit problem on $\mathbb{G}$: we prove the existence of (ground state) solutions, we determine the sharp range of Lorentz regularity of general weak solutions, and then we get their asymptotic behavior at infinity by means of regularity tools such as reverse Hölder inequalities and Moser-type estimates on annuli involving L^p-weak norms. In Section 4, we prove a Pohozaev-type identity for our subelliptic problem and we deduce a non-existence result on bounded starshaped domains for $\lambda \leq 0$. Finally, in Section 5 we prove the existence results stated in Theorem 1.3.

2 The functional setting

We briefly recall the relevant definitions and notation related to the Carnot groups setting. For a complete overview, we refer to the monograph [5] and the classical papers [21], [22].

A *Carnot group* (or *Stratified group*) $(\mathbb{G}, \circ)$ is a connected, simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ admits a stratification, namely a decomposition $\mathfrak{g} = \bigoplus_{j=1}^r \mathfrak{G}_j$, such that $[\mathfrak{G}_1, \mathfrak{G}_j] = \mathfrak{G}_{j+1}$ for $1 \leq j < r$, and $[\mathfrak{G}_1, \mathfrak{G}_r] = \{0\}$. The number r is called the *step* of the group $\mathbb{G}$. The integer $Q = \sum_{i=1}^r i \dim(\mathfrak{G}_i)$ is called the *homogeneous dimension* of $\mathbb{G}$. We shall assume throughout that $Q \geq 3$.

By means of the natural identification of $\mathbb{G}$ with its Lie algebra via the exponential map (which we shall assume throughout), it is not restrictive to suppose that $\mathbb{G}$ is a homogeneous Lie group on $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \ldots \times \mathbb{R}^{N_r}$, with $N_i = \dim(\mathfrak{G}_i)$, equipped with a family of group-automorphisms (called *dilations*) δ_λ of the form

$$\delta_\lambda(\xi) = (\lambda \, \xi^{(1)}, \lambda^2 \, \xi^{(2)}, \cdots, \lambda^r \, \xi^{(r)}), \tag{2.1}$$

where $\xi^{(j)} \in \mathbb{R}^{N_j}$ for $j = 1, \ldots, r$. If we set $m := N_1$ and let $\widetilde{X}_1, \ldots, \widetilde{X}_m$ be the left invariant vector fields of $\mathfrak{G}_1$ that coincide at the origin with the first m partial derivatives, the second order differential operator $\mathcal{L}_{\mathbb{G}} = \sum_{i=1}^{m} \widetilde{X}_i^2$ is called the canonical sub-Laplacian on $\mathbb{G}$. If $X_1, \ldots, X_m$ is any basis of $span\{\widetilde{X}_1, \ldots, \widetilde{X}_m\}$ the operator

$$\Delta_{\mathbb{G}} = \sum_{i=1}^{m} X_i^2$$

is called a sub-Laplacian on $\mathbb{G}$. We shall denote by $\nabla_{\mathbb{G}} = (X_1, \ldots, X_m)$ the related horizontal gradient. Moreover, for any C^1 vector field $h = (h_1, h_2, \ldots, h_m)$, we shall indicate by

$$\mathrm{div}_{\mathbb{G}} h = \sum_{i=1}^{m} X_i h_i, \tag{2.2}$$

the divergence with respect to the vector fields X_j's.

Note that $\Delta_{\mathbb{G}}$ is left-translation invariant with respect to the group action and δ_λ-homogeneous of degree two. In other words, $\Delta_{\mathbb{G}}(u \circ \tau_\xi) = \Delta_{\mathbb{G}} u \circ \tau_\xi$, $\Delta_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda^2 \Delta_{\mathbb{G}} u \circ \delta_\lambda$. Moreover, due to the stratification condition, the Lie algebra generated by $X_1, \ldots, X_m$ is the whole $\mathfrak{g}$, and therefore it is everywhere of rank N; therefore, the operator $\Delta_{\mathbb{G}}$ satisfies the well-known Hörmander's hypoellipticity condition.

In connection with Pohozaev-type identities on groups, we shall also deal with the following vector field, that is the infinitesimal generator of the dilations (2.1), which has the form

$$Z = \sum_{i=1}^{r} \sum_{j=1}^{N_i} i\, \xi_j^{(i)} \frac{\partial}{\partial \xi_j^{(i)}}. \tag{2.3}$$

We recall that Z is characterized by the property that a function $u : \mathbb{G} \to \mathbb{R}$ is homogeneous of degree k with respect to δ_λ, i.e. $u(\delta_\lambda(\xi)) = \lambda^k u(\xi)$, if and only if $Zu = ku$.

The simplest example of Carnot group is the additive group $\mathbb{G} = (\mathbb{R}^N, +)$. In this case $Q = N$ and the sub-Laplacians are exactly the constant coefficients elliptic operators on $\mathbb{R}^N$. Moreover, if $\mathbb{G}$ is a Carnot group of homogeneous dimension $Q \leq 3$, then necessarily $\mathbb{G}$ is the ordinary Euclidean space. The simplest non-abelian Carnot group is the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$, which is a two-step Carnot group with homogeneous dimension $Q = 2n + 2$ and composition law given by $\xi \circ \xi' = (x + x', y + y', t + t' + 2(\langle x', y\rangle - \langle x, y'\rangle))$, for every $\xi = (x, y, t)$, $\xi' = (x', y', t') \in R^{2n+1}$, where $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

When $Q \geq 3$, sub-Laplacians possess the following property: there exists a suitable homogeneous symmetric norm d on $\mathbb{G}$, which we shall refer to as the $\Delta_{\mathbb{G}}$-*gauge*, such that

$$\Gamma(\xi) = \frac{C}{d(\xi)^{Q-2}} \tag{2.4}$$

is the fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at 0, for a suitable constant $C > 0$ (see [21]). By definition, a *homogeneous norm* on $\mathbb{G}$ is a continuous function $d : \mathbb{G} \to [0, +\infty)$, smooth away from the origin, such that $d(\delta_\lambda(\xi)) = \lambda\, d(\xi)$ for every $\lambda > 0$ and $d(\xi) = 0$ iff $\xi = 0$. We say that a homogeneous norm is symmetric if $d(\xi^{-1}) = d(\xi)$. Moreover, if we define $d(\xi, \eta) := d(\eta^{-1} \circ \xi)$, then d is a pseudo-distance on $\mathbb{G}$. We shall indicate by $B_r(\xi) = B_d(\xi, r)$ the d-ball with center at ξ and radius r. We also recall that any two homogeneous norms on a Carnot group are equivalent.

We denote by $\mathrm{d}\xi$ the Lebesgue measure on $\mathbb{R}^N$, which is a bi-invariant Haar measure on $\mathbb{G}$. A fundamental rôle in the functional analysis on Carnot groups is played by the following Sobolev-type inequality (see Folland [21]): there exists a positive constant $S = S(\mathbb{G})$ such that

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi \geq S \left(\int_{\mathbb{G}} |u|^{2^*} \,\mathrm{d}\xi \right)^{2/2^*}, \quad \forall\, u \in C_0^\infty(\mathbb{G}), \tag{2.5}$$

where $2^* = 2Q/(Q-2)$ is the critical exponent in this context. It is known that the best constant in (2.5) is attained (see [27], [57]), but the explicit form of the minimizers is known only for the class of Iwasawa-type groups (see [37], [23], [31], [11]). Qualitative properties of Sobolev minimizers on Carnot groups, such as their sharp summability in L^p-weak spaces and their asymptotic behavior at infinity, have been studied by the author in [40] for the pure Sobolev case, in [41] for the Hardy-Sobolev case, in [43] for the p-Sobolev inequality with exponent $1 < p < Q$.

In our calculation, we shall also use the following weighted version of inequality (2.5), which is a particular case of Caffarelli-Kohn-Nirenberg type inequalities on Carnot groups

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 d^{-2\alpha} \,\mathrm{d}\xi \geq S_\alpha \left(\int_{\mathbb{G}} |u|^{2^*} d^{-2^*\alpha} \,\mathrm{d}\xi \right)^{2/2^*}, \quad \forall\, u \in C_0^\infty(\mathbb{G}), \tag{2.6}$$

where d is the gauge associated with the fundamental solution of $\Delta_{\mathbb{G}}$ on $\mathbb{G}$ and $\alpha > \frac{2-Q}{2}$ (see [17], see also the Appendix in [42]). We shall use it for $\alpha = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}$, where $\overline{\mu} = (Q-2)/2$.

In the context of Carnot groups, the following Hardy-type inequality holds

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi \geq \left(\frac{Q-2}{2} \right)^2 \int_{\mathbb{G}} \psi^2 \frac{u^2}{d^2} \,\mathrm{d}\xi, \quad \forall u \in C_0^\infty(\mathbb{G}), \tag{2.7}$$

where $\psi = |\nabla_{\mathbb{G}} d|$. The above inequality was proved by Garofalo and Lanconelli in [24] for the Heisenberg group $\mathbb{G} = \mathbb{H}^n$. Then, it has been extended to all Carnot groups in [13]. The constant in the r.h.s. of formula (2.7) is sharp and it is never attained (see [13]). We recall that the weight function ψ appearing in the r.h.s. of (2.7) is δ_λ-homogeneous of degree 0 and it is constant if and only if $\mathbb{G}$ is the Euclidean additive group (see [5, Prop. 9.8.9]). See also [53] for a fractional version of such inequality in the Heisenberg group. See [54] for further related results.

By combining Sobolev and Hardy inequality (2.5) and (2.7), it follows that, for $\mu < \overline{\mu}$, there exists a constant $C > 0$, depending on Q and μ, such that

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi - \mu \int_{\mathbb{G}} \frac{\psi^2}{d^2} u^2 \geq C \left(\int_{\mathbb{G}} |u|^{2^*} \,\mathrm{d}\xi \right)^{2/2^*}, \quad \forall\, u \in C_0^\infty(\mathbb{G}). \tag{2.8}$$

Finally, we point out that a large variety of functional inequalities can be obtained on Carnot groups, and more generally on graded and homogeneous Lie groups. We quote [54, 55] and the references therein for recent developments on this topic.

Functional spaces. In view of (2.6), we shall use the weighted Sobolev space $S_0^1(\Omega, d^{-2\alpha}\mathrm{d}\xi)$ defined as the completion of $C_0^\infty(\Omega)$ with respect to the weighted norm

$$\|u\|_{\mathcal{S}_0^1(\Omega, d^{-2\alpha}\mathrm{d}\xi)} := \left(\int_\Omega |\nabla_{\mathbb{G}} u|^2 \, d^{-2\alpha} \mathrm{d}\xi \right)^{1/2}. \tag{2.9}$$

When $\Omega = \mathbb{G}$, we will simply denote the above space as $S^1(\mathbb{G}, d^{-2\alpha}\mathrm{d}\xi)$. For $\Omega \subset \mathbb{G}$, we shall denote by $\Gamma^{k,\beta}(\overline{\Omega})$, $0 < \beta < 1$, $k \in \mathbb{N} \cup \{0\}$, the Folland-Stein Hölder spaces (see [21]). Moreover, $L^{p,\infty}(\Omega)$, $p \geq 1$, will denote the classical weak-Lebesgue spaces. (see [29] for definitions and properties).

3 The limit problem

This section is devoted to the study of existence and qualitative properties of solutions to the limit problem on the entire space

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{d^2} u = |u|^{2^*-2} u, \qquad u \in \mathcal{S}^1(\mathbb{G}). \tag{3.1}$$

First of all, we prove the existence of ground state solutions for problem (3.1). Then, we investigate the optimal Lorentz regularity and the decay at infinity of such solutions, and in general of weak solutions to (3.1). We also recall that analogous qualitative analysis for Euclidean polyharmonic problems can be found in [14], [34], where however radial symmetry occurs.

3.1 Existence of ground states

Let $0 \leq \mu < \overline{\mu}$ and let

$$S_\mu := \inf_{u \in \mathcal{S}^1(\mathbb{G}), u \neq 0} \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi - \mu \int_{\mathbb{G}} \frac{\psi^2}{d^2} u^2 \, \mathrm{d}\xi}{\left(\int_{\mathbb{G}} |u|^{2^*} \, \mathrm{d}\xi\right)^{2/2^*}}. \tag{3.2}$$

Note that, when $\mu = 0$, then $S = S_0$ is the best constant in the Folland-Stein inequality (2.5).

It is clear that the infimum in (3.2) is positive, provided $\mu < \overline{\mu}$, due to (2.8). In addition, extremals for S_μ give rise to solutions to equation (3.1), which are usually referred to as ground state solutions.

Note that the ratio in (3.2) is invariant under the action of the following rescaling

$$u(\xi) \to C \lambda^{\frac{Q-2}{2}} u(\delta_\lambda \xi), \quad \lambda > 0, \, C \in \mathbb{R}.$$

In what follows, we prove that the best constant S_μ in (3.2) is achieved for any $0 \leq \mu < \overline{\mu}$. For the proof, we follow the Euclidean outline in [3].

Lemma 3.1. *Let $\mu < \overline{\mu}$. If $S_\mu < S$, then S_μ is achieved.*

Proof. Using the Ekeland's variational principle adapted to the present variational setting, we can choose a minimizing sequence $\{u_n\}$ for S_μ such that

$$\begin{aligned} \int_{\mathbb{G}} |\nabla_{\mathbb{G}} u_n|^2 \, \mathrm{d}\xi - \mu \int_{\mathbb{G}} \frac{\psi^2}{d^2} |u_n|^2 \, \mathrm{d}\xi &= \int_{\mathbb{G}} |u_n|^{2^*} \, \mathrm{d}\xi + o(1) \\ &= S_\mu^{Q/2} + o(1), \end{aligned} \tag{3.3}$$

and

$$-\Delta_{\mathbb{G}} u_n - \mu \frac{\psi^2}{d^2} u_n = |u_n|^{2^*-2} u_n + f_n, \tag{3.4}$$

where $f_n \to 0$ in the dual space of $\mathcal{S}^1(\mathbb{G})$. Up to a rescaling, we can assume that

$$\int_{B_2} |u_n|^{2^*} \, \mathrm{d}\xi = \frac{1}{2} S_\mu^{Q/2}. \tag{3.5}$$

We know that, since $\mu < \overline{\mu}$, u_n is a bounded sequence in $\mathcal{S}^1(\mathbb{G})$ by the Hardy inequality (1.2). Therefore, up to a subsequence, there exists $u \in \mathcal{S}^1(\mathbb{G})$ such that $u_n \rightharpoonup u$ weakly in $\mathcal{S}^1(\mathbb{G})$.

Now, we have to exclude that the weak limit vanishes. By contradiction, assume that $u_n \rightharpoonup 0$ in $\mathcal{S}^1(\mathbb{G})$. Adapting the arguments in [3, Prop. 4.1] to our context, we obtain that

$$\int_{B_1} |u_n|^{2^*} \, \mathrm{d}\xi = o(1). \tag{3.6}$$

Hence, from (3.5) and (3.6), we get that

$$\int_{B_2 \setminus B_1} |u_n|^{2^*} \, \mathrm{d}\xi = \frac{1}{2} S_\mu^{Q/2} + o(1). \tag{3.7}$$

Let us now test equation (3.4) with the function $\phi^2 u_n$, where ϕ is a cut-off function in $C_0^\infty(\mathbb{G})$ such that $\phi \equiv 1$ on $B_2 \setminus B_1$. By the Rellich-type compactness embedding in the stratified context (see [26]) and Hölder's inequality, arguing as in [3], we obtain that

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}}(\phi u_n)|^2 \, \mathrm{d}\xi - \mu \int_{\mathbb{G}} \frac{\psi^2}{d^2} |\phi u_n|^2 \, \mathrm{d}\xi \leq S_\mu \left(\int_{\mathbb{G}} |\phi u_n|^{2^*} \, \mathrm{d}\xi \right)^{\frac{Q-2}{Q}} + o(1). \tag{3.8}$$

Since ϕ has compact support in $\mathbb{G}$, using again the Rellich-type compactness theorem and the Sobolev inequality on $\mathbb{G}$, we get

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}}(\phi u_n)|^2 \, \mathrm{d}\xi - \mu \int_{\mathbb{G}} \frac{\psi^2}{d^2} |\phi u_n|^2 \, \mathrm{d}\xi = \int_{\mathbb{G}} |\nabla_{\mathbb{G}}(\phi u_n)|^2 \, \mathrm{d}\xi + o(1) \geq S \int_{\mathbb{G}} \left(|\phi u_n|^{2^*} \, \mathrm{d}\xi \right)^{2/2^*}. \tag{3.9}$$

So, by (3.8) and (3.9), we obtain that

$$S \left(\int_{\mathbb{G}} |\phi u_n|^{2^*} \, \mathrm{d}\xi \right)^{2/2^*} \leq S_\mu \left(\int_{\mathbb{G}} |\phi u_n|^{2^*} \, \mathrm{d}\xi \right)^{2/2^*},$$

which gives $\int_{\mathbb{G}} |\phi u_n|^{2^*} \, \mathrm{d}\xi = o(1)$. Therefore, we have that $\int_{B_2 \setminus B_1} |u_n|^{2^*} \, \mathrm{d}\xi = o(1)$ since $\phi \equiv 1$ on $B_2 \setminus B_1$. This contradicts (3.7). □

Theorem 3.2. *Let $\mu < \overline{\mu}$. The infimum S_μ is achieved if and only if $\mu \geq 0$.*

Proof. We first recall that the infimum $S = S_0$, i.e. the best Sobolev constant on $\mathbb{G}$, is achieved in $\mathcal{S}^1(\mathbb{G})$ (see [37] for $\mathbb{G} = \mathbb{H}^n$, [27], [57] for the general Carnot case). Now, adapting the arguments in [3, Theor. 1.3], let u be an arbitrarily chosen function in $C_0^\infty(\mathbb{G})$ and define the translated function $u_\eta(\xi) = u(\eta \circ \xi)$. It holds that

$$\begin{aligned} S_\mu &\le \lim_{d(\eta)\to\infty} \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u_\eta|^2 \, \mathrm{d}\xi - \mu \int_{\mathbb{G}} \frac{\psi^2}{d^2} u_\eta^2 \, \mathrm{d}\xi}{\left(\int_{\mathbb{G}} |u_\eta|^{2^*} \, \mathrm{d}\xi\right)^{2/2^*}} \\ &= \lim_{d(\eta)\to\infty} \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi - \mu \int_{\mathbb{G}} \frac{\psi^2(\eta^{-1}\circ\xi)}{d^2(\eta^{-1}\circ\xi)} u^2(\xi) \, \mathrm{d}\xi}{\left(\int_{\mathbb{G}} |u|^{2^*} \, \mathrm{d}\xi\right)^{2/2^*}} \\ &= \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi}{\left(\int_{\mathbb{G}} |u|^{2^*} \, \mathrm{d}\xi\right)^{2/2^*}}, \end{aligned}$$

where, in performing the limit as $d(\eta) \to \infty$, we have used that ψ is a bounded function, the property of the homogeneous norms $d(\eta^{-1} \circ \xi) \ge cd(\eta) - d(\xi)$, for all $\eta, \xi \in \mathbb{G}$ (see [5, Prop. 5.1.7]) and that u has compact support. Therefore, $S_\mu \le S$ for any $\mu \in \mathbb{R}$.

Now, in the case $\mu < 0$, it holds that $S_\mu \ge S$, that is, $S_\mu = S$. This easily implies that S_μ cannot be attained, since S is attained. If, instead, $0 \le \mu < \overline{\mu}$, let U be a minimizer for S. Then

$$S_\mu \le \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} U|^2 \, \mathrm{d}\xi - \mu \int_{\mathbb{G}} \frac{\psi^2}{d^2} U^2 \, \mathrm{d}\xi}{\left(\int_{\mathbb{G}} |U|^{2^*} \, \mathrm{d}\xi\right)^{2/2^*}} < \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} U|^2 \, \mathrm{d}\xi}{\left(\int_{\mathbb{G}} |U|^{2^*} \, \mathrm{d}\xi\right)^{2/2^*}} = S.$$

Hence, S_μ is attained by Lemma 3.1. □

3.2 L^p-weak regularity of solutions

In this section we study the sharp range of regularity, in the scale of Lorentz spaces, of solutions to (3.1). To study such qualitative properties, we shall use the transformation

$$v = d^\alpha u, \quad \text{where } \alpha = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}. \tag{3.10}$$

As verified by the author in [42], by means of the above transformation, the original problem (3.1) turns into the following weighted critical problem in divergence form

$$-\mathrm{div}_{\mathbb{G}}\left(d^{-2\alpha} \nabla_{\mathbb{G}} v\right) = \frac{|v|^{2^*-2} v}{d^{2^*\alpha}} \quad \text{in } \mathbb{G}, \tag{3.11}$$

where $\mathrm{div}_{\mathbb{G}}$ is the divergence with respect to the horizontal vector fields X_j's defined in (2.2).

We notice that the advantage of working with equation (3.11) instead of the original one is that the solutions of this second problem are bounded, as stated in Prop. 3.3 below.

For completeness, we recall here the definition of weak Lebesgue spaces. If X is a measure space and μ is a positive measure on X, for any $p \in (0, \infty)$, the weak L^p-space, also denoted by $L^{p,\infty}(X)$, is defined as the set of all μ-measurable functions $u : X \to \mathbb{R}$ such that

$$[u]_{L^{p,\infty}(X)} := \sup_{h>0} h \cdot \mu(\{|u| > h\})^{1/p} < \infty,$$

where $\mu(\{|u| > h\})$ denotes the measure of the set $\{\xi \in X : |u(\xi)| > h\}$. The map $[u]_{L^{p,\infty}(X)}$ is a quasi-norm on $L^{p,\infty}(X)$. For a complete treatment, including Hölder's and Young-type inequalities on such spaces, we refer to Grafakos [29]. We shall use these spaces on $X = \mathbb{G}$ endowed with suitable weighted Lebesgue measures.

The sharp range of summability for weak solutions to (3.11) is established below.

Proposition 3.3. *Let $v \in \mathcal{S}^1(\mathbb{G}, d^{-2\alpha}\,\mathrm{d}\xi)$ be a solution of equation (3.11). Then*

$$v \in L^{\frac{2^*}{2},\infty}(\mathbb{G}, d^{-2^*\alpha}\,\mathrm{d}\xi) \cap L^\infty(\mathbb{G}).$$

Proof. Step 1.(L^∞*-regularity.*) We recall that the local boundedness of solutions to (3.11) was proved by the author in [42] (see the proof of Theorem 1.1.) A suitable modification of that proof, using Moser's iteration on the whole space and starting from the global $L^{2^*}(d^{-2^*\alpha}\,\mathrm{d}\xi)$ regularity of v, which comes from (2.6), leads to $v \in L^\infty(\mathbb{G})$. We omit the details, referring for instance to the analogous proof in the fractional Euclidean case in [16, Prop. 4.5].

Step 2.(*Sharp Lorentz regularity.*) We adapt to the weighted subelliptic equation (3.11) the arguments by Vétois [59, Lemma 2.2], also used by the author in [43] for the quasilinear autonomous problem on Carnot groups.

In what follows, for simplicity of notation, we denote by μ_β the measure induced on $\mathbb{G}$ by the weight $d^{-\beta}$. For any $h > 0$, let $T_h(v) := \mathrm{sgn}(v)\min(|v|, h)$. Using $T_h v$ as a test function in equation (3.11), we have

$$\int_{\mathbb{G}} -\mathrm{div}_{\mathbb{G}}\left(d^{-2\alpha}\nabla_{\mathbb{G}} v\right) T_h(v)\,\mathrm{d}\xi = \int_{\mathbb{G}} \langle \nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} T_h(v)\rangle\,\mathrm{d}\mu_{2\alpha} = \int_{v\le h} |\nabla_{\mathbb{G}} v|^2\,\mathrm{d}\mu_{2\alpha}$$

and

$$\int_{\mathbb{G}} |v|^{2^*-1} T_h(v)\,\mathrm{d}\mu_{2^*\alpha} = \int_{v\le h} |v|^{2^*}\,\mathrm{d}\mu_{2^*\alpha} + h\int_{|v|>h} |v|^{2^*-1}\,\mathrm{d}\mu_{2^*\alpha}.$$

Hence

$$\int_{v\le h} |\nabla_{\mathbb{G}} v|^2\,\mathrm{d}\mu_{2\alpha} = \int_{v\le h} |v|^{2^*}\,\mathrm{d}\mu_{2^*\alpha} + h\int_{|v|>h} |v|^{2^*-1}\,\mathrm{d}\mu_{2^*\alpha}. \tag{3.12}$$

On the other hand

$$\int_{|v|\le h} |v|^{2^*}\,\mathrm{d}\mu_{2^*\alpha} = \int_{\mathbb{G}} |T_h(v)|^{2^*}\,\mathrm{d}\mu_{2^*\alpha} - h^{2^*}\mu_{2^*\alpha}(\{|v| > h\}) \tag{3.13}$$

and

$$\begin{aligned}\int_{|v|>h} |v|^{2^*-1}\,\mathrm{d}\mu_{2^*\alpha} =& (2^*-1)\int_0^{+\infty} s^{2^*-2}\mu_{2^*\alpha}(\{|v| > s \vee h\})\,\mathrm{d}s \\ =& h^{2^*-1}\mu_{2^*\alpha}(\{|v| > h\}) + (2^*-1)\int_h^{+\infty} s^{2^*-2}\mu_{2^*\alpha}(\{|v| > s\})\,\mathrm{d}s.\end{aligned} \tag{3.14}$$

From (3.12)-(3.14) it follows that

$$\int_{|v|\le h} |\nabla_{\mathbb{G}} v|^2\,\mathrm{d}\mu_{2\alpha} = \int_{\mathbb{G}} |T_h(v)|^{2^*}\,\mathrm{d}\mu_{2^*\alpha} + (2^*-1)h\int_h^{+\infty} s^{2^*-2}\mu_{2^*\alpha}(\{|v| > s\})\,\mathrm{d}s \tag{3.15}$$

Using the weighted Sobolev inequality (2.6), we get

$$\int_{\mathbb{G}} |T_h(v)|^{2^*} \,\mathrm{d}\mu_{2^*\alpha} \le C \left(\int_{\mathbb{G}} |\nabla_{\mathbb{G}} T_h(v)|^2 \,\mathrm{d}\mu_{2\alpha} \right)^{\frac{2^*}{2}} = C \left(\int_{|v|\le h} |\nabla_{\mathbb{G}} v|^2 \,\mathrm{d}\mu_{2\alpha} \right)^{\frac{2^*}{2}}. \tag{3.16}$$

From (3.13), (3.15), (3.16), being $\int_{\mathbb{G}} |T_h(v)|^{2^*} \,\mathrm{d}\mu_{2^*\alpha} = o(1)$ as $h \to 0$, we obtain

$$\begin{aligned} h^{2^*} \int_{|v|>h} \mathrm{d}\mu_{2^*\alpha} &\le \int_{\mathbb{G}} |T_h(v)|^{2^*} \,\mathrm{d}\mu_{2^*\alpha} \le \\ &\le C \left(h \int_h^{+\infty} s^{2^*-2} \mu_{2^*\alpha}(\{|v| > s\}) \,\mathrm{d}s \right)^{\frac{2^*}{2}} \quad (h \to 0). \end{aligned} \tag{3.17}$$

To simplify notation, let us introduce

$$g(s) := s^{2^*-2} \mu_{2^*\alpha}(\{|v| > s\}), \quad G(h) := \left(\int_h^{+\infty} g(s) \,\mathrm{d}s \right)^{1-\frac{2^*}{2}}. \tag{3.18}$$

It holds that

$$G'(h) = \left(\frac{2^*}{2} - 1 \right) \left(\int_h^{+\infty} g(s) \,\mathrm{d}s \right)^{-\frac{2^*}{2}} g(h). \tag{3.19}$$

Rewriting (3.17) in terms of g and G, we get

$$h^2 g(h) \le C \left(h \int_h^{+\infty} g(s) \,\mathrm{d}s \right)^{\frac{2^*}{2}}, \tag{3.20}$$

from which

$$\left(\int_h^{+\infty} g(s) \,\mathrm{d}s \right)^{-\frac{2^*}{2}} g(h) \le C h^{\frac{2^*}{2}-2} = C h^{\frac{4-Q}{Q-2}}.$$

Hence, taking into account (3.19), we have

$$G'(h) \le C h^{\frac{4-Q}{Q-2}} \quad (h \to 0). \tag{3.21}$$

By integrating, we obtain

$$G(h) - G(0) \le C h^{\frac{2}{Q-2}} \tag{3.22}$$

for sufficiently small value of h, where $G(0) := \lim_{h\to 0} G(h)$.

Let us rewrite, now, (3.14) in terms of G. We have

$$\int_{|v|>h} |v|^{2^*-1} \,\mathrm{d}\mu_{2^*\alpha} = h^{2^*-1} \mu_{2^*\alpha}(\{|v| > h\} + (2^* - 1) G(h)^{\frac{2}{2-2^*}}. \tag{3.23}$$

So, being $\frac{2}{2-2^*} = \frac{2-Q}{2}$, from (3.23) we get

$$h G(h)^{\frac{2-Q}{2}} \le \frac{1}{2^*-1} h \int_{|v|>h} |v|^{2^*-1} \,\mathrm{d}\mu_{2^*\alpha} = o(1) \tag{3.24}$$

as $h \to 0$, due to $v \in L^{2^*}(\mathbb{G}, \mathrm{d}\mu_{2^*\alpha})$. It follows from (3.22) and (3.24) that $G(0) > 0$, that is $\int_0^\infty g(s)\,\mathrm{d}s < +\infty$. From (3.20) and taking into account that $G(h)$ is a non decreasing function, we get

$$h^{\frac{2^*}{2}}\mu_{2^*\alpha}(\{v > h\}) \le C\left(\int_h^{+\infty} g(s)\,\mathrm{d}s\right)^{\frac{2^*}{2}} = CG(h)^{\frac{\frac{2^*}{2}}{1-\frac{2^*}{2}}} = CG(h)^{-\frac{Q}{2}} \le CG(0)^{-\frac{Q}{2}}\ (h \to 0),$$

that implies, together with $v \in L^\infty$, that $v \in L^{\frac{2^*}{2},\infty}(\mathbb{G}, \mathrm{d}\mu_{2^*\alpha})$. □

3.3 Asymptotic behavior of solutions

We begin with some preliminary steps in order to prove the upper bound estimate for solutions to the transformed problem (3.11), which is stated in Theorem 3.6 below. The first step of the proof consists in a preliminary reverse-Hölder inequality for solutions to our problem on annuli in an exterior domain (see [60], [43] for analogous results in unweighted contexts).

In what follows, denoted by B_R the d-ball with center at 0 and radius R, we let

$$A_R := B_{5R} \setminus \overline{B}_{2R} \quad \text{and} \quad \widetilde{A}_R := B_{6R} \setminus \overline{B}_R, \quad R > 0. \tag{3.25}$$

The following uniform estimate with respect to R holds. We prove it for a general weighted Schrödinger-type equation modelled on our problem.

Lemma 3.4. *Let $V \in L^{Q/2}(\mathbb{G}, d^{-2^*\alpha}\,\mathrm{d}\xi)$ and let $v \in \mathcal{S}^1(\mathbb{G}, d^{-2\alpha}\,\mathrm{d}\xi)$ be a nonnegative solution to*

$$-\mathrm{div}_{\mathbb{G}}\left(d^{-2\alpha}\nabla_{\mathbb{G}} v\right) \le d^{-2^*\alpha} V v \quad \text{in } \mathbb{G}. \tag{3.26}$$

Let $t > 2^*$. Then, there exists $R_0 > 0$ depending on t such that for any $R \ge R_0$, it holds

$$\left(\fint_{A_R} v^t\,\mathrm{d}\mu_{2^*\alpha}\right)^{1/t} \le C\left(\fint_{\widetilde{A}_R} v^{2^*}\,\mathrm{d}\mu_{2^*\alpha}\right)^{1/2^*}, \tag{3.27}$$

where $\fint_{A_R} v^t\,\mathrm{d}\mu_{2^*\alpha} = \frac{1}{\mu_{2^*\alpha}(A_R)}\int_{A_R} v^t\,\mathrm{d}\mu_{2^*\alpha}$ and C is a positive constant depending on t, but not on R.

Proof. For any $R > 0$, we consider the linear transformation

$$w(\xi) := v(\delta_R \xi), \quad \xi \in \mathbb{G},$$

where we are omitting, for ease of notation, the dependence on R.

By (3.26) and the homogeneity properties of the involved operator, w satisfies

$$-\mathrm{div}_{\mathbb{G}}\left(d^{-2\alpha}\nabla_{\mathbb{G}} w\right) \le d^{-2^*\alpha} V_R w \quad \text{in } \mathbb{G}, \tag{3.28}$$

where

$$V_R(\xi) = R^{2-(2^*-2)\alpha} V(\delta_R \xi).$$

We shall prove estimate (3.27) for w on the annulus $\widetilde{A}_1$.

Set $w_m := \min(w, m)$, for $m \geq 1$. For any $\eta \in C_0^\infty(\widetilde{A}_1)$, $\eta \geq 0$ and $s \geq 1$, the test function $\varphi = \eta^2 w_m^{2(s-1)} w$ into (3.28) gives

$$\int_{\widetilde{A}_1} \langle \nabla_{\mathbb{G}} w, \nabla_{\mathbb{G}} \varphi \rangle \, \mathrm{d}\mu_{2\alpha} \leq \int_{\widetilde{A}_1} V_R w \, \varphi \, \mathrm{d}\mu_{2^*\alpha}. \tag{3.29}$$

Concerning the left hand side of (3.29), it is easy to see that for any sufficiently small $\delta > 0$, there exists $C_\delta > 0$ such that

$$\begin{aligned} \int_{\widetilde{A}_1} \langle \nabla_{\mathbb{G}} w, \nabla_{\mathbb{G}} \varphi \rangle \, \mathrm{d}\mu_{2\alpha} \geq & (1-\delta) \frac{2(s-1)+1}{s^2} \int_{\widetilde{A}_1} |\nabla_{\mathbb{G}}(\eta w_m^{s-1} w)|^2 \, \mathrm{d}\mu_{2\alpha} \\ & - C_\delta \int_{\widetilde{A}_1} |\nabla_{\mathbb{G}} \eta|^2 w_m^{2(s-1)} w^2 \, \mathrm{d}\mu_{2\alpha}. \end{aligned} \tag{3.30}$$

So, by choosing $\delta = 1/2$ in (3.30) and using the weighted Sobolev inequality (2.6), we obtain

$$\int_{\widetilde{A}_1} \langle \nabla_{\mathbb{G}} w, \nabla_{\mathbb{G}} \varphi \rangle \, \mathrm{d}\mu_{2\alpha} \geq C_1 \left(\int_{\widetilde{A}_1} |\eta w_m^{s-1} w|^{2\chi} \, \mathrm{d}\mu_{2^*\alpha} \right)^{1/\chi} - C_2 \int_{\widetilde{A}_1} |\nabla_{\mathbb{G}} \eta|^2 w_m^{2(s-1)} w^2 \, \mathrm{d}\mu_{2\alpha}, \tag{3.31}$$

for some constants $C_1, C_2 > 0$ depending on Q, s, where $\chi = 2^*/2$.

On the other hand, for the right hand side of (3.29), by Hölder's inequality we get

$$\begin{aligned} \int_{\widetilde{A}_1} V_R w \, \varphi \, \mathrm{d}\mu_{2^*\alpha} & \leq \|V_R\|_{L^{\frac{Q}{2}}(\widetilde{A}_1, r\mathrm{d}\mu_{2^*\alpha})} \left(\int_{\widetilde{A}_1} |\eta w_m^{s-1} w|^{2\chi} \, \mathrm{d}\mu_{2^*\alpha} \right)^{1/\chi} \\ & = \|V\|_{L^{\frac{Q}{2}}(\widetilde{A}_R, \mathrm{d}\mu_{2^*\alpha})} \left(\int_{\widetilde{A}_1} |\eta w_m^{s-1} w|^{2\chi} \, \mathrm{d}\mu_{2^*\alpha} \right)^{1/\chi}. \end{aligned} \tag{3.32}$$

So, by (3.29), (3.31) and (3.32), we get

$$\begin{aligned} \left(\int_{\widetilde{A}_1} |\eta w_m^{s-1} w|^{2\chi} \, \mathrm{d}\mu_{2^*\alpha} \right)^{1/\chi} \leq & C_3 \int_{\widetilde{A}_1} |\nabla_{\mathbb{G}} \eta|^2 w_m^{2(s-1)} w^2 \, \mathrm{d}\mu_{2\alpha} \\ & + C_3 \|V\|_{L^{\frac{Q}{2}}(\widetilde{A}_R, \mathrm{d}\mu_{2^*\alpha})} \left(\int_{\widetilde{A}_1} |\eta w_m^{s-1} w|^{2\chi} \, \mathrm{d}\mu_{2^*\alpha} \right)^{1/\chi} \end{aligned} \tag{3.33}$$

for some constant $C_3 = C_3(Q, s) > 0$.

Now, fix $t > 2^*$ and let $k \in \mathbb{N}$ such that $2\chi^k \leq t \leq 2\chi^{k+1}$. Then, there exists a positive constant $C_3 = C_3(Q, t)$ such that (3.33) holds for all $1 \leq s \leq \chi^k$.

Since $V \in L^{Q/2}(\mathbb{G}, \mathrm{d}\mu_{2^*\alpha})$, there exists $R_0 > 0$ such that

$$C_3 \|V\|_{L^{\frac{Q}{2}}(\widetilde{A}_R, \mathrm{d}\mu_{2^*\alpha})} \leq 1/2, \qquad \text{for any } R \geq R_0. \tag{3.34}$$

Therefore, for all $R \geq R_0$, it holds

$$\left(\int_{\widetilde{A}_1} |\eta w_m^{s-1} w|^{2\chi} \, \mathrm{d}\mu_{2^*\alpha} \right)^{1/\chi} \leq C \int_{\widetilde{A}_1} |\nabla_{\mathbb{G}} \eta|^2 w_m^{2(s-1)} w^2 \, \mathrm{d}\mu_{2\alpha},$$

for all $1 \leq s \leq \chi^k$, where $C > 0$ depends only on Q, t.

At this point, by choosing an appropriate cut-off function η and applying Moser's iteration technique, after finitely many iterations we have

$$\left(\int_{A_1} w^t \, \mathrm{d}\mu_{2^*\alpha}\right)^{1/t} \leq C \left(\int_{\widetilde{A}_1} w^{2^*} \, \mathrm{d}\mu_{2^*\alpha}\right)^{1/2^*} \tag{3.35}$$

for $R \geq R_0$, where C does not depend on R. Finally, by a simple change of variable, (3.27) follows from (3.35). □

Now, we prove the crucial estimate on the L^∞-norm of the solutions to (3.11) on annuli in terms of the sharp L^p-weak norm, which gives the sharp decay of solutions at ∞. In what follows, we indicate by

$$\widehat{A}_R = B_{4R} \setminus \overline{B}_{3R}, \quad R > 0, \tag{3.36}$$

and A_R will denote, as before, the larger annulus $A_R = B_{5R} \setminus \overline{B}_{2R}$.

Theorem 3.5. *Let $v \in \mathcal{S}^1(\mathbb{G},\, d^{-2\alpha}\,\mathrm{d}\xi)$ be a solution to* (3.11)*. Then, there exist constants $R_0, C > 0$, such that for any $R \geq R_0$*

$$\sup_{\widehat{A}_R} |v| \leq \frac{C}{\mu_{2^*\alpha}(A_R)^{2/2^*}} \, [v]_{L^{2^*/2,\infty}(A_R, \mathrm{d}\mu_{2^*\alpha})}, \tag{3.37}$$

where C does not depend on R.

Proof. We begin by observing that, if $v \in \mathcal{S}^1(\mathbb{G},\, d^{-2\alpha}\,\mathrm{d}\xi)$ is a solution to (3.11), by Kato's inequality adapted to the stratified context (see [15] for related results), $|v|$ satisfies the differential inequality

$$-\mathrm{div}_{\mathbb{G}}\left(d^{-2\alpha}\nabla_{\mathbb{G}}|v|\right) \leq d^{-2^*\alpha} V|v| \quad \text{in } \mathbb{G}, \qquad \text{with } V = |v|^{2^*-2}. \tag{3.38}$$

Reasoning as before, we set

$$w(\xi) := |v(\delta_R \xi)|, \qquad R > 0,\ \xi \in \mathbb{G}.$$

Then, in particular, w satisfies the inequality

$$-\mathrm{div}_{\mathbb{G}}\left(d^{-2\alpha}\nabla_{\mathbb{G}} w\right) \leq d^{-2^*\alpha} V_R w \quad \text{in } A_1, \tag{3.39}$$

where

$$V_R(\xi) = R^{2-(2^*-2)\alpha}\, V(\delta_R \xi)$$

and V is defined in (3.38).

Let, now, $t > 2^*$ be fixed. Observe that $V_R \in L^{t_0}(A_1, \mathrm{d}\mu_{2^*\alpha})$, for $t_0 = \frac{t}{2^*-2} > \frac{Q}{2}$ since $v \in L^t(\mathbb{G}, \mathrm{d}\mu_{2^*\alpha})$, as proved in Prop. 3.3. Then, by performing subelliptic Moser-type estimates as in Capogna et al. in [10] (see [10, Theorem 3.4 and Lemma 3.29]) with the use of the weighted Sobolev inequality (2.6), we get that, for any $q > 0$,

$$\sup_B w \leq C \left(\fint_{2B} w^q \, \mathrm{d}\mu_{2^*\alpha}\right)^{1/q}, \tag{3.40}$$

for any ball $B = B(\xi, r)$ such that $2B = B(\xi, 2r) \subset A_1$, where C is a positive constant depending on Q, q and linearly on $\|V_R\|_{L^{t_0}(A_1,\, d\mu_{2^*\alpha})}$. In fact, the constant C can be made independent on R, for sufficiently large R, since the following estimate holds:

$$\|V_R\|_{L^{t_0}(A_1,\, d\mu_{2^*\alpha})} \leq C\|v\|^{2^*-2}_{L^{2^*}(\mathbb{G}, d\mu_{2^*\alpha})} \quad \forall R \geq R_0, \tag{3.41}$$

where C only depends on Q, t_0 and R_0 is choosen so that (3.34) holds for the potential $V = |v|^{2^*-2}$. Indeed, we get that, for any $R \geq R_0$

$$\begin{aligned} \|V_R\|_{L^{t_0}(A_1,\, d\mu_{2^*\alpha})} &= R^{2-(2^*-2)\alpha}\|V \circ \delta_R\|_{L^{t_0}(A_1,\, d\mu_{2^*\alpha})} \\ &= R^{2-(2^*-2)\alpha - \frac{1}{t_0}(Q-2^*\alpha)}\|V\|_{L^{t_0}(A_R, d\mu_{2^*\alpha})} \\ &= R^{2-(2^*-2)\alpha - \frac{1}{t_0}(Q-2^*\alpha)}\|v\|^{2^*-2}_{L^{t}(A_R, d\mu_{2^*\alpha})} \\ &\leq CR^{2-(2^*-2)\alpha - \frac{1}{t_0}(Q-2^*\alpha)} R^{(Q-2^*\alpha)(\frac{1}{2^*}-\frac{1}{t})(2^*-2)}\|v\|^{2^*-2}_{L^{2^*}(\widetilde{A}_R, d\mu_{2^*\alpha})} \\ &\leq C\|v\|^{2^*-2}_{L^{2^*}(\mathbb{G}, d\mu_{2^*\alpha})}, \end{aligned} \tag{3.42}$$

with $C > 0$ not depending on R, where we have used Lemma 3.4 applied to (3.38), the fact that

$$\mu_{2^*\alpha}(A_R) = \int_{A_R} d^{-2^*\alpha}\, d\xi = C\int_{2R}^{5R} \rho^{-2^*\alpha}\rho^{Q-1}\, d\rho = CR^{Q-2^*\alpha}, \tag{3.43}$$

and that $2 - (2^* - 2)\alpha - \frac{1}{t_0}(Q - 2^*\alpha) + (Q - 2^*\alpha)(\frac{1}{2^*} - \frac{1}{t})(2^* - 2) = 0$. Therefore, the constant C in (3.40) does not depend on R, for $R \geq R_0$.

Finally, a covering argument on the inner annulus $\widehat{A}_1 \subset\subset A_1$ gives from (3.40) that

$$\sup_{\widehat{A}_1} w \leq C\left(\fint_{A_1} w^q\, d\mu_{2^*\alpha}\right)^{1/q},$$

that is, by rescaling,

$$\sup_{\widehat{A}_R} |v| \leq C\left(\fint_{A_R} |v|^q d\mu_{2^*\alpha}\right)^{1/q}, \tag{3.44}$$

for $R \geq R_0$, where C depends on q, but not on R.

Finally, we choose q in (3.44) so that $0 < q < 2^*/2$. By Hölder's inequality for weak Lebesgue norms (see Grafakos [29], Ex. 1.1.11) we have

$$\left(\fint_{A_R} |v|^q d\mu_{2^*\alpha}\right)^{1/q} \leq C_q\left(\mu_{2^*\alpha}(A_R)\right)^{-2/2^*}[v]_{L^{2^*/2,\infty}(A_R, d\mu_{2^*\alpha})}. \tag{3.45}$$

Thus, by (3.44) and (3.45), estimate (3.37) follows. □

From Theorem 3.5, the estimate from above for solutions of the transformed problem (3.11) easily follows.

Theorem 3.6. *Let* $v \in \mathcal{S}^1(\mathbb{G}, d^{-2\alpha}\,\mathrm{d}\xi)$ *be a solution to* (3.11). *Then, there exist* $C, R_0 > 0$ *such that*

$$|v(\xi)| \leq \frac{C}{d(\xi)^{Q-2-2\alpha}}, \qquad for\ d(\xi) > R_0.$$

Proof. The thesis follows from estimate (3.37), taking into account Prop. 3.3 and the fact that $\mu_{2^*\alpha}(A_R) = CR^{Q-2^*\alpha}$ as verified in (3.43). Hence, for $R = d(\xi)$ sufficiently large, we get the desired estimate. □

We are now able to conclude, turning to our original problem (3.1).

Proof of Theorem 1.1. The estimate from above immediately follows from Theorem 3.6 taking into account the definition of v (see (3.10)).

To prove the estimate from below, we first observe that if u is a nonnegative solution to (3.1), then u is strictly positive. Indeed, observe that u is a smooth function out of the origin by the subelliptic regularity theory in Folland [21]. Moreover, if $u \geq 0$, then u also satisfies the inequality

$$-\Delta_{\mathbb{G}} u \geq 0, \tag{3.46}$$

that is, u is $\Delta_{\mathbb{G}}$-superharmonic. Henceforth, by Bony's strong maximum principle (see, for instance, [5, Theorem 5.13.8]), it follows that $u > 0$ in $\mathbb{G} \setminus \{0\}$. Now, let us set, as before, $v = d^\alpha u$, and define

$$g := C_1 d^{-(Q-2-2\alpha)}, \quad \text{where } \ C_1 = \min_{\partial B_d(0,1)} v > 0.$$

We have that

$$v - g \geq 0 \quad \text{on } \partial B_d(0,1).$$

Moreover, by equation (3.11), and being $\mathrm{div}_{\mathbb{G}}\left(d^{-2\alpha}\nabla_{\mathbb{G}} g\right) = 0$ in $\mathbb{G} \setminus \{0\}$, as it can be easily verified by direct computation, it follows that

$$-\mathrm{div}_{\mathbb{G}}\left(d^{-2\alpha}\nabla_{\mathbb{G}}(v-g)\right) \geq 0 \quad \text{in } B_d(0,1)^C. \tag{3.47}$$

Then, by adapting the weak maximum principle for sub-Laplacians on unbounded domains in [5, Coroll. 5.13.6] to the above weighted operator away from the singularity, we get that $v - g \geq 0$ on $B_d(0,1)^C$, that is equivalent to the desired estimate from below for u. □

4 A Pohozaev-type identity and nonexistence results

In this section, we turn to the linearly perturbed critical problem (1.1) on bounded domains of $\mathbb{G}$. First, we establish a Pohozaev-type identity for our problem and consequently we derive some non-existence results on δ_λ-starshaped domains.

In the subelliptic setting, Pohozaev-type identities have been introduced in the Heisenberg group model case by Garofalo-Lanconelli [25], and by Garofalo-Vassilev [27] in the general Carnot setting, and they present several additional difficulties with respect to the Euclidean case (see also [38], [39], [6], [41], [4] for further related results).

Here we provide a Pohozaev-type identity for weak solutions, sufficiently regular up to the boundary, to our non-autonomous problem (1.1). An analogous identity for the Grushin operator with Hardy term was obtained by the author in [44]. In addition to the usual technical difficulties due to the stratification, we have to manage here the lack of regularity at the origin.

We stress that, as already observed in the case $\mu = 0$ in [27], the a priori C^1-regularity assumption up to the boundary required on the solutions in order to implement Pohozaev-type identities represents a strong assumption in the subelliptic context due to the possible loss of regularity of the solutions near the characteristic set of the boundary, while, instead, the interior $C^\infty(\Omega \setminus \{0\})$-regularity of weak solutions is ensured by the local subelliptic regularity theory in Folland [21]. In Remark 4.4 at the end of the section, we recall how the boundary regularity assumptions can be somehow weakened, in the context of step-two Carnot groups, following the arguments in [27].

In what follows, $\mathrm{d}\sigma$ denotes the $(N-1)$-dimensional Hausdorff measure on $\partial\Omega$.

Theorem 4.1. *Let $\Omega \subset \mathbb{G}$ be a smooth bounded domain, $0 \in \Omega$, and let $u \in \mathcal{S}^1_0(\Omega) \cap C^1(\overline{\Omega} \setminus \{0\})$ be a solution of (1.1) such that $Zu/d \in L^2(\Omega)$. Then, the following identity holds*

$$\lambda \int_\Omega u^2 \,\mathrm{d}\xi = \frac{1}{2} \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 \langle Z, \nu\rangle \,\mathrm{d}\sigma, \tag{4.1}$$

where Z is the infinitesimal generator of the dilations δ_λ defined in (2.3) and $\nu = (\nu_1, \dots, \nu_N)$ is the outward normal to $\partial\Omega$.

Remark 4.2. We emphasize that in the Euclidean case, where d is the Euclidean distance and $Z = x \cdot \nabla$, the assumption $Zu/d \in L^2(\Omega)$ is easily implied by the assumption $|\nabla u| \in L^2(\Omega)$. In the present context, the assumption $|\nabla_{\mathbb{G}} u| \in L^2(\Omega)$ is not sufficient to ensure the required L^2-summability of the radial derivative Zu/d, due to the fact that the infinitesimal generator Z of the dilations δ_λ involves commutators of the horizontal fields X_j up to maximum length. We also emphasize that the above assumption is used in the estimate (4.14) below and an attempt to avoid this kind of hypothesis, in the case $\mu = 0$ and in presence of a Hardy-Sobolev nonlinearity, was made by the author in [41], in the context of step-two Carnot groups, by performing a preliminary a priori pointwise estimate of Zu near the origin. We refer the interested reader to [41, Section 3].

Proof of Theorem 4.1. Due to the lack of regularity of solutions at the origin recalled in (1.6), we begin by considering approximating domains $\Omega \setminus B_{r_n}$, for an appropriate sequence of radii $r_n \to 0$, as in [41]. To this aim, observe that, from Federer's coarea formula (see [18]), if $B_R = B_d(0, R)$ is a d-ball centered at 0 contained in Ω, then

$$\begin{aligned}\int_0^R \mathrm{d}s \int_{\partial B_s} \left(\psi^2 \frac{u^2}{d^2} + |u|^{2^*} + u^2 + |\nabla_{\mathbb{G}} u|^2 + \frac{|Zu|^2}{d^2} \right) \frac{1}{|\nabla d|} \,\mathrm{d}\sigma \\ = \int_{B_R} \left(\psi^2 \frac{u^2}{d^2} + |u|^{2^*} + u^2 + |\nabla_{\mathbb{G}} u|^2 + \frac{|Zu|^2}{d^2} \right) \mathrm{d}\xi.\end{aligned} \tag{4.2}$$

From $u \in \mathcal{S}^1_0(\Omega)$ and $Zu/d \in L^2(\Omega)$, the integral in the r.h.s. of (4.2) is finite. Then, there exists a sequence $r_n \to 0$ such that

$$r_n \int_{\partial B_{r_n}} \left(\psi^2 \frac{u^2}{d^2} + |u|^{2^*} + u^2 + |\nabla_{\mathbb{G}} u|^2 + \frac{|Zu|^2}{d^2} \right) \frac{1}{|\nabla d|} \,\mathrm{d}\sigma \longrightarrow 0, \quad \text{as } n \to \infty. \tag{4.3}$$

Set $\Omega_{r_n} := \Omega \setminus B_{r_n}$. Multiplying equation (1.1) by Zu and integrating over Ω_{r_n}, we get

$$\int_{\Omega_{r_n}} -\Delta_{\mathbb{G}} u Z u \,\mathrm{d}\xi = \int_{\Omega_{r_n}} f(\xi, u)\, Zu \,\mathrm{d}\xi, \tag{4.4}$$

where we have set

$$f(\xi, u) := \mu\psi^2 \frac{u}{d^2} + |u|^{2^*-2}u + \lambda u.$$

Since $u \in C^2(\Omega_{r_n}) \cap C^1(\overline{\Omega}_{r_n})$, the following Rellich-type identity holds for u on Ω_{r_n} (see [25], [27]):

$$\begin{aligned}\int_{\Omega_{r_n}} -\Delta_{\mathbb{G}} u Z u \,\mathrm{d}\xi = &-\frac{Q-2}{2}\int_{\Omega_{r_n}} |\nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi \\ &+ \frac{1}{2}\int_{\partial\Omega_{r_n}} |\nabla_{\mathbb{G}} u|^2 \langle Z, \nu\rangle \,\mathrm{d}\sigma - \int_{\partial\Omega_{r_n}} \langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}}\rangle Zu \,\mathrm{d}\sigma,\end{aligned} \tag{4.5}$$

where we denote by $\nu_{\mathbb{G}} = (\nu_{\mathbb{G}}^{(1)}, \dots, \nu_{\mathbb{G}}^{(N)})$ the vector field with components $\nu_{\mathbb{G}}^{(i)} = \langle X_i, \nu\rangle$.

On the other hand, concerning the right hand side of (4.4), if we let $F(\xi, u) := \int_0^u f(\xi, t)\,\mathrm{d}t$, it holds

$$\begin{aligned}\int_{\Omega_{r_n}} f(\xi, u) Zu \,\mathrm{d}\xi &= \int_{\Omega_{r_n}} Z(F(\xi, u))\,\mathrm{d}\xi - \int_{\Omega_{r_n}} \langle Z, \nabla_\xi F(\xi, u)\rangle \,\mathrm{d}\xi \\ &= -\int_{\Omega_{r_n}} \mathrm{div} Z\, F(\xi, u)\,\mathrm{d}\xi + \int_{\partial\Omega_{r_n}} F(\xi, u)\langle Z, \nu\rangle \,\mathrm{d}\sigma \\ &\quad - \int_{\Omega_{r_n}} \langle Z, \nabla_\xi F(\xi, u)\rangle \,\mathrm{d}\xi.\end{aligned} \tag{4.6}$$

Hence, by (4.4), (4.5) and (4.6), and taking into account that $divZ = Q$, we obtain

$$\begin{aligned}&Q\int_{\Omega_{r_n}} F(\xi, u)\,\mathrm{d}\xi + \int_{\Omega_{r_n}} \langle Z, \nabla_\xi F(\xi, u)\rangle \,\mathrm{d}\xi - \frac{Q-2}{2}\int_{\Omega_{r_n}} |\nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi \\ &\quad = \int_{\partial\Omega_{r_n}} \langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}}\rangle Zu \,\mathrm{d}\sigma - \frac{1}{2}\int_{\partial\Omega_{r_n}} |\nabla_{\mathbb{G}} u|^2 < Z, \nu > \mathrm{d}\sigma \\ &\quad + \int_{\partial\Omega_{r_n}} F(\xi, u) < Z, \nu > \mathrm{d}\sigma.\end{aligned} \tag{4.7}$$

Now, let $r_n \to 0$ in the above identity. Taking into account that

$$F(\xi, u) = \frac{\mu}{2}\psi^2 \frac{u^2}{d^2} + \frac{1}{2^*}|u|^{2^*} + \frac{\lambda}{2}u^2 \tag{4.8}$$

and due to the integrability of the functions $F(\xi, u)$ and $|\nabla_{\mathbb{G}} u|^2$, we get

$$Q\int_{\Omega_{r_n}} F(\xi, u)\,\mathrm{d}\xi - \frac{Q-2}{2}\int_{\Omega_{r_n}} |\nabla_{\mathbb{G}} u|^2 \mathrm{d}\xi \longrightarrow Q\int_{\Omega} F(\xi, u)\,\mathrm{d}\xi - \frac{Q-2}{2}\int_{\Omega} |\nabla_{\mathbb{G}} u|^2 \mathrm{d}\xi \tag{4.9}$$

as $r_n \to 0$. Moreover, computing the second integral in (4.7), we get

$$\begin{aligned}\int_{\Omega_{r_n}} \langle Z, \nabla_\xi F(\xi,u)\rangle \,\mathrm{d}\xi &= \int_{\Omega_{r_n}} \psi Z\psi \frac{u^2}{d^2}\,\mathrm{d}\xi - \mu \int_{\Omega_{r_n}} \psi^2 u^2 d^{-3} Zd \,\mathrm{d}\xi \\ &= -\mu \int_{\Omega_{r_n}} \psi^2 \frac{u^2}{d^2}\,\mathrm{d}\xi,\end{aligned} \tag{4.10}$$

where we have used that $Z\psi = 0$ and $Zd = d$, since they are δ_λ-homogeneous functions, respectively of degree zero and one. Therefore, again from the integrability of the function $\psi^2 \frac{u^2}{d^2}$, we have

$$\int_{\Omega_{r_n}} \langle Z, \nabla_\xi F(\xi,u)\rangle \,\mathrm{d}\xi \longrightarrow \int_{\Omega} \langle Z, \nabla_\xi F(\xi,u)\rangle \,\mathrm{d}\xi, \qquad \text{as } r_n \to 0. \tag{4.11}$$

Now we verify that the integrals on ∂B_{r_n} in (4.7) vanish as $r_n \to 0$. Indeed, since $\nu = -\frac{\nabla d}{|\nabla d|}$ on ∂B_{r_n}, then

$$\langle Z, \nu\rangle = -\frac{Zd}{|\nabla d|} = -\frac{d}{|\nabla d|} \quad \text{on } \partial B_{r_n}.$$

From this, and using (4.3), we have

$$\begin{aligned}&\int_{\partial B_{r_n}} \left(F(\xi,u) - \frac{1}{2}|\nabla_{\mathbb{G}} u|^2\right) |\langle Z,\nu\rangle| \,\mathrm{d}\sigma \\ &\quad = r_n \int_{\partial B_{r_n}} \left(\frac{\mu}{2}\psi^2 \frac{u^2}{d^2} + \frac{1}{2^*}|u|^{2^*} + \frac{\lambda}{2}u^2 - \frac{1}{2}|\nabla_{\mathbb{G}} u|^2\right) \frac{1}{|\nabla d|}\,\mathrm{d}\sigma \longrightarrow 0, \qquad \text{as } r_n \to 0.\end{aligned} \tag{4.12}$$

For the first integral in the r.h.s. of (4.7), observe that

$$|\langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}}\rangle| = |\langle \nabla_{\mathbb{G}} u, \frac{\nabla_{\mathbb{G}} d}{|\nabla d|}\rangle| \le \psi \frac{|\nabla_{\mathbb{G}} u|}{|\nabla d|} \le c \frac{|\nabla_{\mathbb{G}} u|}{|\nabla d|} \quad \text{on } \partial B_{r_n}. \tag{4.13}$$

Hence, by (4.13) and using the assumption $Zu/d \in L^2$, we get

$$\begin{aligned}\left|\int_{\partial B_{r_n}} \langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}}\rangle Zu \,\mathrm{d}\sigma\right| &\le c \left(\int_{\partial B_{r_n}} \frac{|\nabla_{\mathbb{G}} u||Zu|}{|\nabla d|}\,\mathrm{d}\sigma\right) \\ &\le c r_n \left(\int_{\partial B_{r_n}} \frac{|\nabla_{\mathbb{G}} u|^2}{|\nabla d|}\,\mathrm{d}\sigma\right)^{1/2} \left(\int_{\partial B_{r_n}} \frac{|Zu|^2}{d^2|\nabla d|}\,\mathrm{d}\sigma\right)^{1/2} \\ &= o(1), \quad \text{as } r_n \to 0.\end{aligned} \tag{4.14}$$

So, letting $r_n \to 0$ in (4.7), by (4.9), (4.11), (4.12), (4.14), and remembering that, since $u = 0$ on $\partial\Omega$, then $F(\xi,u) = 0$ on $\partial\Omega$ and $\int_{\partial\Omega} \langle \nabla_{\mathbb{G}} u, \nu_{\mathbb{G}}\rangle Zu \,\mathrm{d}\sigma = \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 <Z,\nu> \mathrm{d}\sigma$, we get the following identity on the whole Ω, where we have substituted the explicit expressions of each term (see (4.8) and (4.10)):

$$\begin{aligned}\frac{Q}{2^*}\int_\Omega\left(\mu\psi^2\frac{u^2}{d^2}+|u|^{2^*}+\lambda u^2\right)\mathrm{d}\xi+\lambda\int_\Omega u^2\,\mathrm{d}\xi-\frac{Q-2}{2}\int_\Omega|\nabla_{\mathbb{G}}u|^2\,\mathrm{d}\xi\\ =\frac{1}{2}\int_{\partial\Omega}|\nabla_{\mathbb{G}}u|^2\langle Z,\nu\rangle\,\mathrm{d}\sigma.\end{aligned}\tag{4.15}$$

On the other hand, using u as a test function in (1.1), we have

$$\int_\Omega|\nabla_{\mathbb{G}}u|^2\,\mathrm{d}\xi=\int_\Omega\left(\mu\psi^2\frac{u^2}{d^2}+|u|^{2^*}+\lambda u^2\right)\mathrm{d}\xi.\tag{4.16}$$

Hence, substituting (4.16) in (4.15), and taking into account that $\frac{Q}{2^*}-\frac{Q-2}{2}=0$, the thesis follows. □

We now recall the definition of δ_λ-starshaped domains.

Definition 4.3. *Let Ω be a C^1 connected open set, $0\in\Omega$. We say that Ω is a δ_λ-starshaped domain with respect to the origin if*

$$\langle Z,\nu\rangle(\xi)\geq 0,\ \forall\xi\in\partial\Omega.$$

The integral identity provided by Theorem 4.1 allows us to prove the nonexistence of nonnegative solutions on bounded starshaped domains stated in Theorem 1.2 as follows.

Proof of Theorem 1.2. If $\lambda<0$, from (4.1) it immediately follows that $u\equiv 0$ in Ω. If $\lambda=0$, from (4.1) we get

$$\int_{\partial\Omega}|\nabla_{\mathbb{G}}u|^2\langle Z,\nu\rangle\,\mathrm{d}\sigma=0.\tag{4.17}$$

Then, by a subelliptic unique continuation argument (see Corollary A.1 in [25] and Corollary 10.7 in [27]), whose application requires u to be nonnegative, we conclude that $u\equiv 0$ in Ω. We refer for the details to the analogous proofs in [25], [27]. □

Remark 4.4. It is not our purpose here to treat the very delicate problem of the boundary regularity of weak solutions to subelliptic problems due to the presence of characteristic points, for which we refer to the deep related discussion in Garofalo-Vassilev [27], Sections 3-4. We only recall that the phenomenon of the possible loss of regularity near the characteristic points of the boundary was discovered by D. Jerison [36], who constructed an explicit example of a smooth domain Ω in the Heisenberg group $\mathbb{H}^n$ and a $\Delta_{\mathbb{H}^n}$-harmonic function in Ω, vanishing on $\partial\Omega$, which is at most in a Hölder class $\Gamma^{0,\alpha}$ near a characteristic point of $\partial\Omega$ (see [36]). Hence, in this context, assuming strong a priori regularity of solutions on the boundary can be a serious obstacle to overcome. However, in [27], Pohozaev identities and related non-existence results for Yamabe-type equations are obtained, in the context of step-two Carnot groups, by assuming $\Gamma^{0,\alpha}$-Hölder regularity of solutions up to the boundary, and boundedness of $\nabla_{\mathbb{G}}u$ and Zu near the characteristic set; moreover, suitable geometric conditions on $\partial\Omega$ ensuring such regularity of solutions are provided. Such conditions are satisfied, in particular, by the gauge balls in the Heisenberg-type groups. Henceforth, significant examples of domains which do not support weak solutions of critical equations, other than the trivial one, are given. The arguments in [27] were adapted in [41, Sect. 4] to the Hardy-Sobolev case and remain valid in the present singular case, since the solutions, away from 0, fulfill the necessary interior regularity. We also emphasize that in [27], the validity of Schauder-type estimates at the non-characteristic set of the boundary was assumed, in absence of a full result for Carnot groups, except for the Heisenberg case. In this regard, we quote that the most recent developments in this direction can be found in Baldi-Citti-Cupini [1], where Schauder-type estimates at the non-characteristic boundary are proved for a prototype class of Carnot groups.

5 Existence results on bounded domains

In this section, exploiting the qualitative analysis for solutions to the limit problem performed in Section 3, we prove Theorem 1.3.

Let $U > 0$ be a fixed minimizer for S_μ and consider, for $\varepsilon > 0$, the family of rescaled functions

$$U_\varepsilon(\xi) = \varepsilon^{\frac{2-Q}{2}} U(\delta_{\frac{1}{\varepsilon}}(\xi)).$$

The functions U_ε are solutions, up to multiplicative constant, of the limit equation

$$-\Delta_{\mathbb{G}} U_\varepsilon - \mu \frac{\psi^2}{d^2} U_\varepsilon = U_\varepsilon^{2^*-1} \text{ in } \mathbb{G}. \tag{5.1}$$

Moreover, they satisfy

$$\int_{\mathbb{G}} \left(|\nabla_{\mathbb{G}} U_\varepsilon|^2 - \mu \frac{\psi^2}{d^2} U_\varepsilon^2 \right) \mathrm{d}\xi = \int_{\mathbb{G}} U_\varepsilon^{2^*} \mathrm{d}\xi = S_\mu^{\frac{Q}{2}}, \quad \text{for all } \varepsilon > 0. \tag{5.2}$$

Let $R > 0$ be such that $B_d(0, R) \subset \Omega$ and let $\varphi \in C_0^\infty(B_d(0, R))$, $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_d(0, R/2)$ and define

$$u_\varepsilon(\xi) := \varphi(\xi) U_\varepsilon(\xi). \tag{5.3}$$

The following asymptotic expansions hold.

Lemma 5.1. *Let $0 \le \mu \le \overline{\mu} - 1$. The functions u_ε satisfy the following estimates, as $\varepsilon \to 0$:*

$$\int_\Omega \left(|\nabla_{\mathbb{G}} u_\varepsilon|^2 - \mu \frac{\psi^2}{d^2} u_\varepsilon^2 \right) \mathrm{d}\xi = S_\mu^{\frac{Q}{2}} + O(\varepsilon^{2\sqrt{\overline{\mu}-\mu}}) \tag{5.4}$$

$$\int_\Omega u_\varepsilon^{2^*} \mathrm{d}\xi = S_\mu^{\frac{Q}{2}} + O(\varepsilon^{2^*\sqrt{\overline{\mu}-\mu}}) \tag{5.5}$$

$$\int_\Omega u_\varepsilon^2 \mathrm{d}\xi \ge \begin{cases} C\,\varepsilon^2 + O(\varepsilon^{2\sqrt{\overline{\mu}-\mu}}) & \text{if } 0 \le \mu < \overline{\mu} - 1 \\ C\,\varepsilon^2 |\log \varepsilon| + O(\varepsilon^2) & \text{if } \mu = \overline{\mu} - 1. \end{cases} \tag{5.6}$$

Proof. For the first estimate, using the equation (5.1) with test function $\varphi^2 U_\varepsilon$, we get

$$\begin{aligned} \int_\Omega \left(|\nabla_{\mathbb{G}} u_\varepsilon|^2 - \mu \frac{\psi^2}{d^2} u_\varepsilon^2 \right) \mathrm{d}\xi &= \int_\Omega \langle \nabla_{\mathbb{G}} U_\varepsilon, \nabla_{\mathbb{G}}(\varphi^2 U_\varepsilon) \rangle \,\mathrm{d}\xi + \int_\Omega |\nabla_{\mathbb{G}} \varphi|^2 U_\varepsilon^2 \,\mathrm{d}\xi - \mu \frac{\psi^2}{d^2} \varphi^2 U_\varepsilon^2 \,\mathrm{d}\xi \\ &= \int_\Omega \varphi^2 U_\varepsilon^{2^*} \mathrm{d}\xi + \int_\Omega |\nabla_{\mathbb{G}} \varphi|^2 U_\varepsilon^2 \,\mathrm{d}\xi \\ &= \int_{\mathbb{G}} U_\varepsilon^{2^*} \mathrm{d}\xi + \int_\Omega |\nabla_{\mathbb{G}} \varphi|^2 U_\varepsilon^2 \,\mathrm{d}\xi + \sigma(\varphi, \varepsilon), \end{aligned} \tag{5.7}$$

where

$$\sigma(\varphi, \varepsilon) = -\int_\Omega U_\varepsilon^{2^*} + \int_\Omega (\varphi^2 - 1) U_\varepsilon^{2^*}.$$

Now, using the estimate $U \le C d^{-\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}$ for d large, proved in Theorem 1.1, and using the polar coordinates formula for integrals of radial functions on homogeneous groups, we get

$$\begin{aligned}\int_\Omega |\nabla_{\mathbb{G}}\varphi|^2 U_\varepsilon^2 \,\mathrm{d}\xi &\le C \int_{B_R \setminus B_{R/2}} U_\varepsilon^2 \,\mathrm{d}\xi = C \int_{B_R \setminus B_{R/2}} \varepsilon^{2-Q}\, U^2(\delta_{\frac{1}{\varepsilon}}\xi)\,\mathrm{d}\xi \\ &= C\varepsilon^2 \int_{B_{R/\varepsilon}\setminus B_{R/2\varepsilon}} U^2 \mathrm{d}\xi \\ &\le C\varepsilon^2 \int_{B_{R/\varepsilon}\setminus B_{R/2\varepsilon}} \frac{1}{d^{2(\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu})}}\,\mathrm{d}\xi \\ &= C\varepsilon^2 \int_{R/2\varepsilon}^{R/\varepsilon} \frac{\rho^{Q-1}}{\rho^{Q-2+2\sqrt{\bar\mu-\mu}}}\,\mathrm{d}\rho \\ &= C\varepsilon^{2\sqrt{\bar\mu-\mu}}.\end{aligned} \tag{5.8}$$

Moreover,

$$\begin{aligned} 0 &\le \int_\Omega \left(1-\varphi^2\right) U_\varepsilon^{2^*}\,\mathrm{d}\xi \le \int_{B_R^C} U_\varepsilon^{2^*}\,\mathrm{d}\xi = \int_{B_{R/\varepsilon}^C} U^{2^*}\,\mathrm{d}\xi \\ &\le C \int_{B_{R/\varepsilon}^C} \frac{1}{d^{2^*(\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu})}}\,\mathrm{d}\xi \\ &= C \int_{R/\varepsilon}^{+\infty} \frac{\rho^{Q-1}}{\rho^{Q+2^*\sqrt{\bar\mu-\mu}}}\,\mathrm{d}\rho \\ &= C\varepsilon^{2^*\sqrt{\bar\mu-\mu}},\end{aligned} \tag{5.9}$$

and an analogous estimate holds for the remaining term in $\sigma(\varphi,\varepsilon)$. So, from (5.7), taking into account (5.2), (5.8) and (5.9), the estimate (5.4) follows. Next, we have

$$\begin{aligned}\int_\Omega u_\varepsilon^{2^*}\,\mathrm{d}\xi &= \int_\Omega U_\varepsilon^{2^*}\,\mathrm{d}\xi + \int_\Omega (\varphi^{2^*}-1)\, U_\varepsilon^{2^*}\,\mathrm{d}\xi \\ &= \int_{\mathbb{G}} U_\varepsilon^{2^*}\,\mathrm{d}\xi - \int_{\Omega^C} U_\varepsilon^{2^*}\,\mathrm{d}\xi + \int_\Omega (\varphi^{2^*}-1)\, U_\varepsilon^{2^*}\,\mathrm{d}\xi \\ &= S_\mu^{\frac{Q}{2}} + O(\varepsilon^{2^*\sqrt{\bar\mu-\mu}}),\end{aligned}$$

that is, estimate (5.5). Finally, we compute

$$
\begin{aligned}
\int_\Omega u_\varepsilon^2 \,\mathrm{d}\xi &= \int_\Omega \varphi^2 U_\varepsilon^2 \,\mathrm{d}\xi \;\geq \int_{B_{R/2}} U_\varepsilon^2 \,\mathrm{d}\xi = \varepsilon^2 \int_{B_{R/2\varepsilon}} U^2 \,\mathrm{d}\xi \\
&= \varepsilon^2 \left(\int_{B_1} U^2 \mathrm{d}\xi \;+ \int_{B_{R/2\varepsilon}\setminus B_1} U^2 \mathrm{d}\xi \right) \\
&\geq C\varepsilon^2 \left(1 \;+ \int_{B_{R/2\varepsilon}\setminus B_1} \frac{1}{d^{2(\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu})}} \,\mathrm{d}\xi \right) \\
&= C\varepsilon^2 \left(1 \;+ \int_1^{R/2\varepsilon} \frac{\rho^{Q-1}}{\rho^{Q-2+2\sqrt{\overline{\mu}-\mu}}} \,\mathrm{d}\rho \right) \\
&= \begin{cases} C\,\varepsilon^2 \;+\; O(\varepsilon^{2\sqrt{\overline{\mu}-\mu}}) & \text{if } 0 \leq \mu < \overline{\mu}-1 \\ C\,\varepsilon^2 |\log \varepsilon| \;+ O(\varepsilon^2) & \text{if } \mu = \overline{\mu}-1, \end{cases}
\end{aligned}
$$

as $\varepsilon \to 0$, where we have used the estimate from below on U provided by Theorem 1.1.

The proof is therefore complete. □

Proof of Theorem 1.3. *Part (i)* Reasoning as in [9], we know that a sufficient condition for the existence of a positive solution to (1.1) when $0 < \lambda < \lambda_1$ is that

$$
S_{\mu,\lambda} := \inf_{u \in S_0^1(\Omega)} Q_{\mu,\lambda}(u) = \inf_{u \in S_0^1(\Omega)} \frac{\int_\Omega |\nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi - \mu \int_\Omega \dfrac{\psi^2}{d^2} u^2 \,\mathrm{d}\xi - \lambda \int_\Omega u^2 \,\mathrm{d}\xi}{\left(\int_\Omega |u|^{2^*} \,\mathrm{d}\xi\right)^{2/2^*}} < S_\mu, \tag{5.10}
$$

since this ensures that $S_{\mu,\lambda}$ is achieved.

In order to verify (5.10), we compute the ratio $Q_{\mu,\lambda}(u)$ on the family of Sobolev concentrating functions u_ε introduced in (5.3). From the preceding lemma, if $0 \leq \mu < \overline{\mu} - 1$, we get

$$
Q_{\mu,\lambda}(u_\varepsilon) \leq \frac{\left(S_\mu^{\frac{Q}{2}} \;- C\lambda\varepsilon^2 + \,O(\varepsilon^{2\sqrt{\overline{\mu}-\mu}}) \right)}{\left(S_\mu^{\frac{Q}{2}} \;+ O\left(\varepsilon^{2^*\sqrt{\overline{\mu}-\mu}}\right) \right)^{2/2^*}} = S_\mu - C\lambda\varepsilon^2 \;+\; O(\varepsilon^{2\sqrt{\overline{\mu}-\mu}}) < S_\mu,
$$

for any $\lambda > 0$ and $\varepsilon > 0$ sufficiently small.

Similarly, if $\mu = \overline{\mu} - 1$ we have

$$
Q_{\mu,\lambda}(u_\varepsilon) \leq \frac{\left(S_\mu^{\frac{Q}{2}} \;- C\lambda\varepsilon^2 |\log \varepsilon| + \,O(\varepsilon^2) \right)}{\left(S_\mu^{\frac{Q}{2}} \;+ O(\varepsilon^{2^*}) \right)^{2/2^*}} = S_\mu - C\lambda\varepsilon^2 |\log \varepsilon| \;+\; O(\varepsilon^2) < S_\mu,
$$

for any $\lambda > 0$ and $\varepsilon > 0$ sufficiently small. This concludes the proof. □

Part (ii) We use a standard bifurcation argument, as in [14]. Let $\varphi_1 \neq 0$ be a solution of the eigenvalue problem

$$\begin{cases} -\Delta_{\mathbb{G}}\varphi_1 - \mu\dfrac{\psi^2}{d^2}\varphi_1 = \lambda_1\varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.11}$$

Then, $Q_{\mu,\lambda_1}(\varphi_1) = 0$; by continuity, there exists $\lambda_* < \lambda_1$ such that $Q_{\mu,\lambda}(\varphi_1) < S$ for $\lambda_* < \lambda < \lambda_1$. □

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