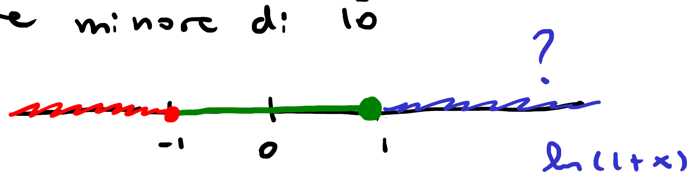
7 novembre 2024

Calcolo $\ln(1.2)$ con errore minore di 10^{-3}

Punto di partenza:



$$\forall x \in (-1,1] \qquad \ln(1+x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} x^n$$

$$\ln(1.2) = \ln(1 + \underbrace{\overset{=\frac{1}{5}}{0.2}}_{\in(-1,1]}) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}\frac{1}{5^n} \quad (*)$$

$\frac{1}{n}\frac{1}{5^n} =: b_n > 0$

$b_n \to 0$

$b_n \downarrow$

La serie $(*)$ soddisfa le ipotesi del crit. di Leibniz

$\Rightarrow$ $(*)$ converge (lo sapevo già)

$\forall n: \quad |S - S_n| \le b_{n+1}$

S: somma, S_n: somma parziale

Impongo

$$b_{n+1} < 10^{-3} \iff$$

$$\frac{1}{(n+1)5^{n+1}} < \frac{1}{10^3} \iff$$

$$(n+1)5^{n+1} > 1000$$

$n=2$: $\quad 3 \cdot 5^3 \quad$ no

$n=3$: $\quad 4 \cdot 5^4 = 4 \cdot 625 \quad$ sì!

$$\Rightarrow \underbrace{S \simeq S_3}_{\text{errore } < 10^{-3}} = \sum_{n=1}^{3} (-1)^{n-1} \frac{1}{n 5^n} = \frac{1}{5} - \frac{1}{2 \cdot 25} + \frac{1}{3 \cdot 125}$$

$$= \dots = \frac{137}{750} = 0.1826\dots$$

Excel: $0.182322\dots$

$\bullet\ \ln(0.7) = \ln(1 - 0.3) = \ln\Big(1 + \underbrace{\big(-\tfrac{3}{10}\big)}_{\in (-1,1]}\Big)$

$$= \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \left(-\frac{3}{10}\right)^n = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} (-1)^n \frac{3^n}{10^n}$$

$$= -\underbrace{\sum_{n=1}^{+\infty} \frac{3^n}{n \cdot 10^n}}_{?}$$

non è a termini alterni!

Oss: $\forall n \geq 1: \quad \dfrac{3^n}{n\, 10^n} \leq \dfrac{3^n}{10^n} = \left(\dfrac{3}{10}\right)^n$

$$\Rightarrow \underbrace{R_n \leq}\ R'_n = \frac{\left(\frac{3}{10}\right)^{n+1}}{1 - \frac{3}{10}} = \frac{10}{7} \cdot \left(\frac{3}{10}\right)^{n+1} = \underbrace{\frac{3}{7} \cdot \left(\frac{3}{10}\right)^n}$$

$\uparrow$ resto di $\sum_n \frac{3^n}{n 10^n}$ $\qquad$ $\uparrow$ resto di $\sum_n \left(\frac{3}{10}\right)^n$

Impongo $\quad \dfrac{3}{7} \cdot \left(\dfrac{3}{10}\right)^n < 10^{-3}$

$$\frac{7}{3} \left(\frac{10}{3}\right)^n > 1000$$

$$7 \cdot 10^n > 3^{n+1} \cdot 1000$$

$$7 \cdot 10^{n-3} > 3^{n+1}$$

$n = 4: \quad 7 \cdot 10 > 3^5 \qquad$ no

$n = 5: \quad 7 \cdot 10^2 > 3^6 = 9 \cdot 81 = 729$

$n = 6: \quad \underbrace{7 \cdot 10^3}_{7000} > 3^7 = \underbrace{3 \cdot 729}_{\simeq 2100} \qquad$ sì!

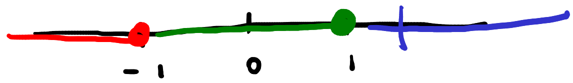
$$\Rightarrow \ln(0.7) \simeq -\sum_{n=1}^{6} \frac{3^n}{n \cdot 10^n} = \ldots = -\frac{427959}{12 \cdot 10^5}$$

$$= -0.35632\ldots$$

$$\text{Excel}: \quad -0.3566749\ldots$$

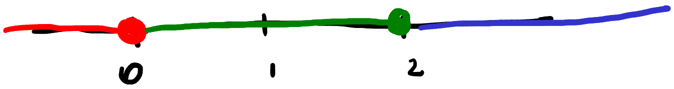
$\square$

- $\ln(2.5) = \ln(1 + 1.5)$



$\ln(1+x)$

$\ln(x)$



Oss: $\quad x > 1 \implies 0 < \frac{1}{x} < 1$

$$\ln(x) = -\ln\left(\frac{1}{x}\right)$$

$$\ln(2.5) = \ln\left(\frac{5}{2}\right) = -\ln\left(\frac{2}{5}\right) = -\ln\Big(1 + \underbrace{\left(-\tfrac{3}{5}\right)}_{\in\, (-1,1]}\Big)$$

$$= -\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}\left(-\frac{3}{5}\right)^n$$

$$= +\sum_{n=1}^{+\infty} \frac{1}{n}\left(\frac{3}{5}\right)^n$$

Proseguo come prima $\square$

"Trucco" analogo per valutare $\arctan(x)$ per $x \notin [-1,1]$

- $\forall x > 0: \arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$
- $\arctan(-x) = -\arctan(x)$

- Calcolo $\int_0^1 e^{-x^2}\, dx$ con errore $< 10^{-3}$

Parto da:

$$\forall x \in \mathbb{R}: \quad e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$\Rightarrow \quad \forall x \in \mathbb{R}: \qquad e^{-x^2} = \sum_{n=0}^{+\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{n!}$$

Teor. integr. termine a termine

$$\Rightarrow \quad \int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{+\infty} \int_0^1 (-1)^n \frac{x^{2n}}{n!}\,dx$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \left[\frac{x^{2n+1}}{2n+1} \right]_0^1$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \frac{1}{2n+1}$$

$b_n \to 0$ ....

Impongo

$$\frac{1}{(n+1)!\,(2n+3)} < 10^{-3}$$

....

- $\displaystyle\int_0^1 \frac{\sin(x)}{x}\,dx \qquad$ errore $< 10^{-4}$

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \qquad \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

$\uparrow$

Continua in $\mathbb{R}$ (e quindi in $[0,1]$)

Sviluppo f in serie di potenze di centro $x_0 = 0$.

Parto da:

$$\forall x \in \mathbb{R}: \qquad \sin(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

regola
del multiplo

$$\Rightarrow \qquad \forall x \in \mathbb{R}^* : \quad \frac{\sin(x)}{x} = \frac{1}{x} \cdot \sin(x)$$

$$= \sum_{n=0}^{+\infty} \frac{1}{x} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} \quad \odot$$

Osservo che la serie $\odot$ converge anche per $x=0$, quindi in tutto $\mathbb{R}$.

Se denoto con g la sua somma, ho mostrato che:

$$\forall x \in \mathbb{R}^* : \qquad f(x) = g(x)$$

Per $x=0$: f e g sono continue (f per costruzione, g perché somma di serie di potenze),

pertanto:

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x \to 0} g(x) = g(0)$$

Dunque: $\quad f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} \qquad \underline{\underline{\forall x \in \mathbb{R}}}$

Ora integro termine a termine:

$$\int_0^{+\cancel{3}\ \frac{1}{3}} f(x)\, dx = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} \int_0^{+\cancel{3}\ \frac{1}{3}} x^{2n}\, dx$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{+\cancel{3^{2n+1}}}{2n+1} \ \frac{1}{3^{2n+1}} \ldots$$

1. $\int_0^1 \frac{1-\cos(x^2)}{x^3}\,dx$ errore $< 10^{-4}$

$\cos(t) = 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \dots$

$$f(x) = \begin{cases} \frac{1-\cos(x^2)}{x^3} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\lim_{x \to 0} \frac{1-\cos(x^2)}{x^3} = \lim_{x \to 0} \frac{1-\left(1-\frac{x^4}{2}+\dots\right)}{x^3} = 0$

↑ continua in $\mathbb{R}$

$$\forall x \in \mathbb{R}: \quad \cos(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\Rightarrow \forall x \in \mathbb{R}: \quad \cos(x^2) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{4n}}{(2n)!} = 1 + \sum_{n=1}^{+\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

$$\Rightarrow \forall x \in \mathbb{R}: \quad 1-\cos(x^2) = -\sum_{n=1}^{+\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^{4n}}{(2n)!}$$

$$\Rightarrow \forall x \in \mathbb{R}^*: \quad \underbrace{\frac{1-\cos(x^2)}{x^3}}_{f(x)} = \underbrace{\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^{4n-3}}{(2n)!}}_{g(x)}$$

$f(0) = 0$ $\qquad$ $g(0) = 0$

$$\Rightarrow f(x) = g(x) \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \int_0^1 f(x)\,dx = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{(2n)!} \int_0^1 x^{4n+3}\,dx$$

$= \dots$

□

• Risolvo in serie l'equazione $y''+y=0$

$a_1(t) = 0 \quad \forall t \in \mathbb{R}$

$a_0(t) = 1 \quad \forall t \in \mathbb{R}$ analitiche in $t=0$ con rdc $=+\infty$

$\Rightarrow$ l'equazione ammette SFS costituito da funz. analitiche in $t=0$ con rdc $=+\infty$

Pongo $y(t) = \sum_{n=0}^{+\infty} \underbrace{c_n}_{\text{da determinare}} t^n \quad \forall t \in \mathbb{R}$

Derivo t. a t.:

$$\forall t \in \mathbb{R}: \quad y'(t) = \sum_{n=1}^{+\infty} n c_n t^{n-1}, \quad y''(t) = \sum_{n=2}^{+\infty} n(n-1) c_n t^{n-2}$$

Sostituisco nell'equazione:

$$\forall t \in \mathbb{R}: \quad \sum_{n=2}^{+\infty} n(n-1) c_n t^{n-2} + \sum_{n=0}^{+\infty} c_n t^n = 0$$

Faccio delle manipolazioni algebriche

Nota: sono tutte autorizzate perché le serie convergono assolutamente

$$\forall t \in \mathbb{R}: \quad \sum_{n=0}^{+\infty} (n+2)(n+1) c_{n+2} t^n + \sum_{n=0}^{+\infty} c_n t^n = 0$$

$$\forall t \in \mathbb{R}: \quad \sum_{n=0}^{+\infty} \left[(n+2)(n+1) c_{n+2} + c_n \right] t^n = 0 = \sum_{n=0}^{+\infty} 0 \cdot t^n$$

PISP

$\Rightarrow \quad \forall n: \quad (n+2)(n+1) c_{n+2} + c_n = 0$

$$\Rightarrow \quad \forall n: \quad c_{n+2} = -\frac{c_n}{(n+2)(n+1)}$$ relazione ricorsiva

Scelgo arbitrariamente $\underbrace{c_0}_{=y(0)}$ e $\underbrace{c_1}_{=y'(0)}$

e ricavo i coefficienti successivi:

$$c_2 = -\frac{c_0}{2\cdot 1} \qquad\qquad c_3 = -\frac{c_1}{3\cdot 2}$$

$$c_4 = -\frac{c_2}{4\cdot 3} = +\frac{c_0}{4\cdot 3\cdot 2\cdot 1} \qquad\qquad c_5 = -\frac{c_3}{5\cdot 4} = +\frac{c_1}{5\cdot 4\cdot 3\cdot 2}$$

$$c_6 = -\frac{c_4}{6\cdot 5} = -\frac{c_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} \qquad\qquad c_7 = -\frac{c_5}{7\cdot 6} = -\frac{c_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$c_{2n} = (-1)^n \frac{c_0}{(2n)!} \qquad\qquad c_{2n+1} = (-1)^n \frac{c_1}{(2n+1)!}$$

Scrivo la soluzione:

$$\forall t \in \mathbb{R} \quad y(t) = \sum_{n=0}^{+\infty} c_n t^n = \sum_{n=0}^{+\infty} c_{2n} t^{2n} + \sum_{n=0}^{+\infty} c_{2n+1} t^{2n+1}$$

$$= c_0 + \sum_{n=1}^{+\infty} (-1)^n \frac{c_0}{(2n)!} t^{2n} + c_1 t + \sum_{n=1}^{+\infty} (-1)^n \frac{c_1}{(2n+1)!} t^{2n+1}$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{c_0}{(2n)!} t^{2n} + \sum_{n=0}^{+\infty} (-1)^n \frac{c_1}{(2n+1)!} t^{2n+1}$$

$$= c_0 \underbrace{\sum_{n=0}^{+\infty} (-1)^n \frac{t^{2n}}{(2n)!}}_{= \cos(t)} + c_1 \underbrace{\sum_{n=0}^{+\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}}_{= \sin(t)}$$

$$= c_0 \cos(t) + c_1 \sin(t) \qquad \square$$

• $y'' - ty = 0$

$a_1(t) = 0 \quad \forall t$

$a_0(t) = -t \quad \forall t$ } analitiche in $t=0$ con rdc $= +\infty$

Come nell'es. precedente:

$$y(t) = \sum_{n=0}^{+\infty} c_n t^n \qquad \forall t \in \mathbb{R}$$

Derivo t.a t. e sostituisco:

$$\forall t \in \mathbb{R}: \quad \sum_{n=2}^{+\infty} n(n-1) c_n t^{n-2} - t \sum_{n=0}^{+\infty} c_n t^n = 0$$

$$\forall t \in \mathbb{R}: \quad \sum_{n=0}^{+\infty} (n+2)(n+1) c_{n+2} t^n - \sum_{n=0}^{+\infty} c_n t^{n+1} = 0$$

$$\forall t \in \mathbb{R}: \quad \sum_{n=0}^{+\infty} (n+2)(n+1) c_{n+2} t^n - \sum_{n=1}^{+\infty} c_{n-1} t^n = 0$$

$$\forall t \in \mathbb{R}: \quad 2c_2 + \sum_{n=1}^{+\infty} (n+2)(n+1) c_{n+2} t^n - \sum_{n=1}^{+\infty} c_{n-1} t^n = 0$$

$$\forall t \in \mathbb{R}: \quad 2c_2 + \sum_{n=1}^{+\infty} \left[(n+2)(n+1) c_{n+2} - c_{n-1} \right] t^n = 0 = 0 + \sum_{n=1}^{+\infty} 0\, t^n$$

PISP

$$\Rightarrow \begin{cases} 2c_2 = 0 \\ \forall n \geq 1: \quad (n+2)(n+1) c_{n+2} - c_{n-1} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} c_2 = 0 \\ \forall n \geq 1: \quad c_{n+2} = \dfrac{c_{n-1}}{(n+2)(n+1)} \end{cases}$$

Scelgo arbitrariamente c_0 e c_1; $c_2 = 0$

Calcolo i coefficienti successivi:

$$c_3 = \frac{c_0}{3 \cdot 2} \qquad c_4 = \frac{c_1}{4 \cdot 3} \qquad c_5 = \frac{c_2}{5 \cdot 4} = 0$$

$$c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2} \qquad c_8 = \frac{c_5}{\dots} = 0$$

$$\vdots$$

$$c_{11} = \dots = 0$$

$$c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}$$

$$c_9 = \frac{c_6}{9 \cdot 8} = \frac{c_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} \qquad c_{10} = \frac{c_7}{10 \cdot 9} = \frac{c_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}$$

$$c_{3n} = \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \ldots \cdot (3n-1)(3n)}$$

$$c_{3n+1} = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot \ldots \cdot (3n)(3n+1)}$$

$$c_{3n+2} = 0$$

Compongo la soluzione:

$\forall t \in \mathbb{R}$:

$$y(t) = \sum_{n=0}^{+\infty} c_n t^n$$

$$= \sum_{n=0}^{+\infty} c_{3n} t^{3n} + \sum_{n=0}^{+\infty} c_{3n+1} t^{3n+1} + \sum_{n=0}^{+\infty} \underbrace{c_{3n+2}}_{=0} t^{3n+2}$$

$$= c_0 + \sum_{n=1}^{+\infty} \frac{c_0}{2 \cdot 3 \cdots (3n-1)(3n)} t^{3n} + c_1 t + \sum_{n=1}^{+\infty} \frac{c_1}{3 \cdot 4 \cdots (3n)(3n+1)} t^{3n+1}$$

$\Rightarrow \quad \forall t \in \mathbb{R}$

$$y(t) = \underbrace{c_0 \left[1 + \sum_{n=1}^{+\infty} \frac{1}{2 \cdot 3 \cdots (3n-1)(3n)} t^{3n} \right]}_{y_1(t)} +$$

$$+ \underbrace{c_1 \left[t + \sum_{n=1}^{+\infty} \frac{1}{3 \cdot 4 \cdots (3n)(3n+1)} t^{3n+1} \right]}_{y_2(t)}$$