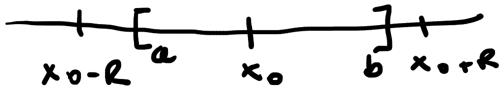
31 ottobre 2024

$$\sum_{n=0}^{+\infty} \underbrace{c_n (x-x_0)^n}_{f_n(x)} =: f(x) \qquad R>0$$

$$\sum_{n=1}^{+\infty} \underbrace{n\, c_n (x-x_0)^{n-1}}_{f_n'(x)} = f'(x)$$



$$\sum_{n=2}^{+\infty} \underbrace{n(n-1)\, c_n (x-x_0)^{n-2}}_{(f_n')'(x)} = f''(x)$$

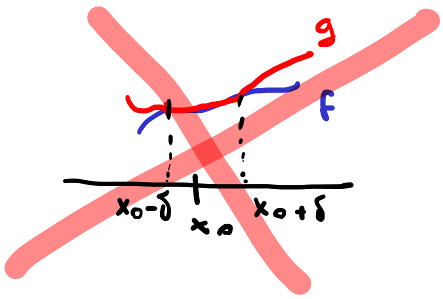
Motivo il PISP

$$f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n \qquad x \in (x_0-R_a,\ x_0+R_a)$$

$$g(x) = \sum_{n=0}^{+\infty} b_n (x-x_0)^n \qquad x \in (x_0-R_b,\ x_0+R_b)$$

Ipotesi: $\exists\, \delta>0$ t.c. $\forall x \in (x_0-\delta, x_0+\delta)$: $f(x) = g(x)$

$\Rightarrow \forall k$: $f^{(k)}(x_0) = g^{(k)}(x_0)$

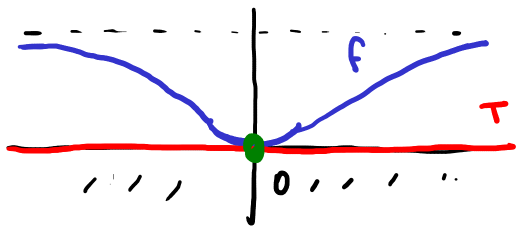


$\Rightarrow \forall k$: $\dfrac{f^{(k)}(x_0)}{k!} = \dfrac{g^{(k)}(x_0)}{k!}$

$\overset{(*)}{\Leftrightarrow} \forall k$: $a_k = b_k$ $\square$

Es. (di funzione non svilupp. in serie di Taylor)

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



Si verifica che $\forall k: \; f^{(k)}(0) = 0$

$$\Rightarrow \quad \forall k: \; \frac{f^{(k)}(0)}{k!} = 0$$

$$\Rightarrow \quad \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n = T(x) \equiv 0$$

f e T coincidono <u>soltanto</u> in $x=0$.

Oss:

$$f(x) = \overbrace{3}^{c_2}x^2 + \overbrace{5}^{c_1}x \overbrace{-1}^{c_0} \qquad \text{funz. polinomiale di centro } 0$$

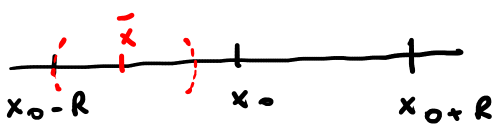
Posso "ricentrare" f a piacere. Per esempio, scelgo $\bar{x} = 2$

$$f(x) = 3(x-2+2)^2 + 5(x-2+2) - 1$$

$$= 3\left[(x-2)^2 + 4 + 4(x-2)\right] + 5(x-2) + 10 - 1$$

$$= \underbrace{3}_{\tilde{c}_2}(x-2)^2 + \underbrace{17}_{\tilde{c}_1}(x-2) + \underbrace{21}_{\tilde{c}_0} \qquad \text{funz. polinomiale di centro } 2$$

$$f(x) = \sum_{n=0}^{+\infty} c_n (x-x_0)^n$$



$$= \sum_{n=0}^{+\infty} c_n (x - \bar{x} + \bar{x} - x_0)^n$$

$$= \sum_{n=0}^{+\infty} c_n \sum_{k=0}^{n} \binom{n}{k} (x-\bar{x})^k (\bar{x}-x_0)^{n-k}$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left(c_n \binom{n}{k} (x-\bar{x})^k (\bar{x}-x_0)^{n-k} \right)$$

è lecita?

sì! perché la serie conv. assol.

$$= \sum_{k=0}^{+\infty} \sum_{n=k}^{+\infty} c_n \binom{n}{k} \underbrace{(x-\bar{x})^k}_{\text{non dipende da } n} (\bar{x}-x_0)^{n-k}$$

$$= \sum_{k=0}^{+\infty} \underbrace{\left(\sum_{n=k}^{+\infty} c_n \binom{n}{k} (\bar{x}-x_0)^{n-k} \right)}_{=: \tilde{c}_k} (x-\bar{x})^k$$

centro $\bar{x}$

Dimostro la cond. suff. per l'analiticità.

Ipotesi: $\exists M, \delta > 0$ t.c.

$$|f^{(k)}(x)| \leq M \quad \forall x \in (x_0-\delta, x_0+\delta) \quad \forall k \in \mathbb{N}.$$

Tesi:

$$\forall x \in (x_0-\delta, x_0+\delta): \quad f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

Pongo $T_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$ pol. di Taylor di centro x_0 e ordine n

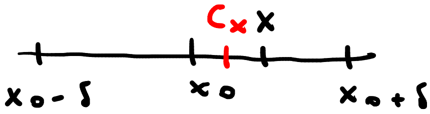
Tesi:

$$\forall x \in (x_0-\delta, x_0+\delta): \quad f(x) = \lim_{n \to +\infty} T_n(x) \quad (=)$$

$$\forall x \in (x_0-\delta, x_0+\delta): \quad \lim_{n \to +\infty} |\underbrace{f(x) - T_n(x)}_{\text{resto di Taylor di centro } x_0 \text{ e ordine } n}| = 0$$

Fisso $x \in (x_0-\delta, x_0+\delta)$

Dall' Analisi I ricordo che



esiste c_x compreso tra x_0 e x (e quindi tra $x_0-\delta$ e $x_0+\delta$) t.c.

$$f(x) - T_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$$

Quindi:

$$0 \le |f(x) - T_n(x)| = \frac{\overbrace{|f^{(n+1)}(c_x)|}^{\le M}}{(n+1)!}|x-x_0|^{n+1} \le \underbrace{M \underbrace{\frac{|x-x_0|^{n+1}}{(n+1)!}}_{\xrightarrow[n\to+\infty]{} 0}}_{\downarrow\, 0}$$

$$\frac{a^{n+1}}{(n+1)!} \sim \frac{a^n}{n!} \xrightarrow[n\to+\infty]{} 0$$

TCO

$$\Rightarrow \quad |f(x) - T_n(x)| \xrightarrow[n\to+\infty]{} 0 \qquad \square$$

Esempi:

$$f(x) = \sin(x) \qquad |\underset{\pm\sin\ \ \pm\cos}{f^{(n)}(x)}| \le 1 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \forall \delta > 0 : \quad |f^{(n)}(x)| \le 1 \quad \forall x \in (x_0-\delta, x_0+\delta)$$

$$\Rightarrow f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad \forall x \in (x_0-\delta, x_0+\delta)$$

δ è arbitrario $\Rightarrow$

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad \forall x \in \mathbb{R}$$

$$(\forall x_0 \in \mathbb{R})$$

In particolare, con $x_0 = 0$:

$$\forall x \in \mathbb{R}: \quad \sin(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n \underset{\text{AM I}}{=} \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Analogamente:

$$\forall x \in \mathbb{R}: \quad \cos(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

- $f(x) = e^x$

$\forall n: \quad f^{(n)}(x) = e^x$

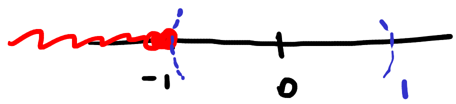
$\forall \delta > 0: \quad |f^{(n)}(x)| = e^x \leq e^\delta \qquad \forall x \in (x_0 - \delta, x_0 + \delta)$
$\forall n$

$$\Rightarrow \forall \delta > 0: \quad e^x = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \qquad \forall x \in (-\delta, \delta)$$

$$\Rightarrow \quad e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \qquad \forall x \in \mathbb{R}$$

- $f(x) = \ln(1+x)$

$f'(x) = \dfrac{1}{1+x} = (1+x)^{-1}$



$f''(x) = -(1+x)^{-2}$

$\vdots$

Oss: f e le sue derivate non sono limitate in intorni di $x = -1$

$\Rightarrow (f^{(n)})$ non è equi-limitata

$\Rightarrow$ la cond. suff. non è applicabile.

Verifico che $f(x) = \ln(1+x)$ è analitica in $x_0 = 0$ (con r.d.c. $R = 1$)

definita in $(-1, +\infty)$

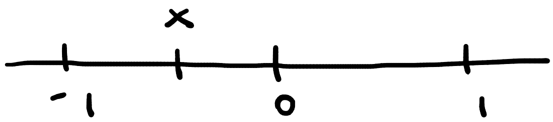
Punto di partenza:

$$\forall x \in (-1,1): \quad \frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$$

noto da AM I

$$\Rightarrow \forall x \in (-1,1): \quad \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{+\infty} (-x)^n = \sum_{n=0}^{+\infty} (-1)^n x^n$$

$\in (-1,1)$



$$\Rightarrow \forall x \in (-1,1): \quad \int_0^x \frac{1}{1+t}\,dt = \sum_{n=0}^{+\infty} \int_0^x (-1)^n t^n\,dt$$

teor di integr. t.a t.

$$\Rightarrow \forall x \in (-1,1): \quad \Big[\ln(1+t)\Big]_0^x = \sum_{n=0}^{+\infty} (-1)^n \left[\frac{t^{n+1}}{n+1}\right]_0^x$$

$$\Rightarrow \forall x \in (-1,1): \quad \ln(1+x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n} =: g(x)$$

$x = -1$: $\ln(1+x)$ non ha significato

$$\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{(-1)^n}{n} = -\sum_{n=1}^{+\infty} \frac{1}{n} \quad \text{non conv.}$$

$x = 1$: $\ln(1+x) = \ln(2)$ ✓

$\sum_{n=1}^{+\infty} (-1)^n \frac{1^n}{n}$ converge

Pongo $g(x) := \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}$

La serie conv. punt. in $(-1,1]$

$\Rightarrow$ g è continua in $(-1,1]$, dunque anche per $x = 1$

$\Rightarrow g(1) = \lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \ln(1+x) = \ln(2)$

Quindi: l'uguaglianza

$$\ln(1+x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}$$

vale anche per $x = 1$, ossia in $(-1,1]$ $\square$

Verifico che $f(x) = \arctan(x)$ ($\mathrm{dom}(f) = \mathbb{R}$) è svilupp. in serie di Taylor di centro $x_0 = 0$.

Riparto da:

$\forall x \in (-1,1)$: $\frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n$

$\Rightarrow \forall x \in (-1,1)$: $\frac{1}{1+\underbrace{x^2}_{\in [0,1) \subset (-1,1)}} = \sum_{n=0}^{+\infty} (-1)^n x^{2n}$

$\Rightarrow \forall x \in (-1,1)$: $\int_0^x \frac{1}{1+t^2}\, dt = \sum_{n=0}^{+\infty} (-1)^n \int_0^x t^{2n}\, dt$

$\Rightarrow \forall x \in (-1,1)$:

$$\left[\arctan(t)\right]_0^x = \sum_{n=0}^{+\infty} (-1)^n \left[\frac{t^{2n+1}}{2n+1}\right]_0^x$$

$$\Rightarrow \forall x \in (-1,1): \quad \arctan(x) = \underbrace{\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}_{=:\, g(x)}$$

In $x = \pm 1$: arctan è continua

$$\sum_{n=0}^{+\infty} (-1)^n \frac{(\pm 1)^{2n+1}}{2n+1} = \pm \sum_{n=0}^{+\infty} (-1)^n \frac{1}{2n+1} \quad \text{converge}$$

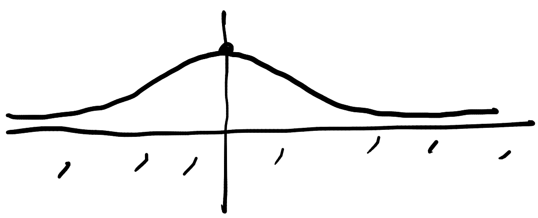
$\Rightarrow g$ è continua anche in ± 1

Come nell'es. precedente, per continuità:

$$\arctan(x) = \underbrace{\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}_{\text{ha r.d.c.} = 1} \quad \forall x \in [-1,1]$$

$$\sqrt[n]{\left|\frac{(-1)^n}{2n+1}\right|} = \frac{1}{\sqrt[n]{2n+1}} \xrightarrow[n\to+\infty]{} 1$$

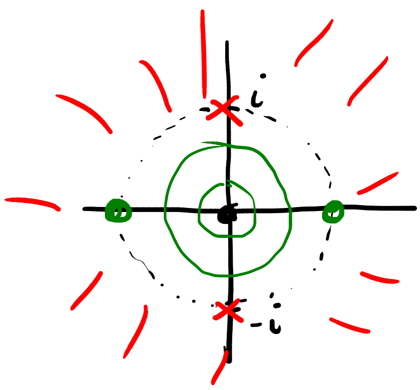
$$f'(x) = \frac{1}{1+x^2}$$



$x \in \mathbb{R}$

$$\varphi(z) = \frac{1}{1+z^2}, \quad z \in \mathbb{C}$$

$$1 + z^2 = 0 \iff z^2 = -1 \qquad z = \pm i$$



Verifico che sinh e cosh sono analitiche in $x_0 = 0$

$$\cosh(x) := \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \forall x \in \mathbb{R}$$

Punto di partenza:

$$\forall x \in \mathbb{R}: \quad e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$\Rightarrow \forall x \in \mathbb{R}: \quad e^{-x} = \sum_{n=0}^{+\infty} (-1)^n \frac{x^n}{n!}$$

regola della somma $\Rightarrow$

$$\forall x \in \mathbb{R}: \quad e^x \pm e^{-x} = \sum_{n=0}^{+\infty} \left(\frac{x^n}{n!} \pm (-1)^n \frac{x^n}{n!} \right)$$

regola del multiplo

$$\Rightarrow \forall x \in \mathbb{R}: \quad \frac{e^x \pm e^{-x}}{2} = \sum_{n=0}^{+\infty} \frac{1}{2} \left(\frac{x^n}{n!} \pm (-1)^n \frac{x^n}{n!} \right)$$

$$\Rightarrow \forall x \in \mathbb{R}: \quad \cosh(x) \ (\sinh) = \sum_{n=0}^{+\infty} \frac{1 \pm (-1)^n}{2} \frac{x^n}{n!}$$

n pari: 1 (0) — n disp.: 0 (1)

$$= \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}$$

$$\Rightarrow \forall x \in \mathbb{R}: \quad \sinh(x) = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$\square$

Calcolo $\cos(0.5)$ con errore $< 10^{-3}$

Parto da: $\forall x \in \mathbb{R}$: $\cos(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$$\Rightarrow \cos(0.5) = \cos\left(\tfrac{1}{2}\right) = \sum_{n=0}^{+\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{1}{4^n (2n)!} \quad (*)$$

Problema: calcolare la somma della serie con errore $< 10^{-3}$

$\Leftrightarrow$ determinare una somma parziale S_N t.c.

$$|S - S_N| < 10^{-3}$$

$$\Leftrightarrow \quad |R_N| < 10^{-3}$$

Osservo che $b_n := \frac{1}{4^n (2n)!}$:

$b_n > 0$, $b_n \to 0$, $b_n \downarrow$

$\Rightarrow$ posso applicare il crit. di Leibniz

$\Rightarrow$ la serie $(*)$ converge   vabbè!

$\quad |R_n| \le b_{n+1} = \frac{1}{4^{n+1}(2n+2)!}$

Quindi: se voglio che $|R_n| < 10^{-3}$, mi basta imporre che

$$\frac{1}{4^{n+1}(2n+2)!} < 10^{-3} \quad \odot$$

$\odot \iff 4^{n+1}(2n+2)! > 1000$

$n=1: \quad 4^2 \cdot 4! = 16 \cdot 24 \quad$ no

$n=2: \quad 4^3 \cdot 6! = 64 \cdot 720 \quad \underline{\text{sì}}$

$$\Rightarrow \underset{\cos(0.5)}{\overset{\shortparallel}{S}} \approx S_2 = \sum_{k=0}^{2} (-1)^k \frac{1}{4^k (2k)!}$$

errore $< 10^{-3}$

$$= 1 - \frac{1}{4 \cdot 2} + \frac{1}{16 \cdot 24} = \frac{\cdots}{\cdots} = \frac{337}{384}$$

$$= \underbrace{0.877}60\ldots$$

Excel: $\underbrace{0.877}582$