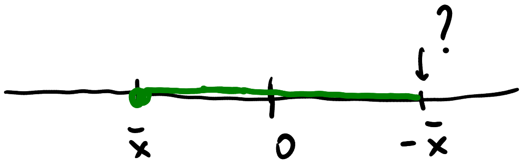
24 ottobre 2024



Dimostro il lemma.

Suppongo che $\sum_n c_n \bar{x}^n$ converga.

Per la cond. necessaria per la convergenza: $(c_n \bar{x}^n)$ è una succ. infinitesima, pertanto è limitata:

$$\exists M > 0 \quad \text{t.c.} \quad |c_n \bar{x}^n| \le M \quad \forall n$$

Fisso $x \in (-|\bar{x}|, |\bar{x}|)$ (cioè: $|x| < |\bar{x}|$)

Voglio provare che $\sum_n c_n x^n$ conv. assolutamente

Valuto:

$$|c_n x^n| \overset{\bar{x} \neq 0}{=} \left| c_n \bar{x}^n \frac{x^n}{\bar{x}^n} \right| = \underbrace{|c_n \bar{x}^n|}_{\le M} \underbrace{\left| \frac{x^n}{\bar{x}^n} \right|}_{\ge 0}$$

$$\le M \left| \left(\frac{x}{\bar{x}}\right)^n \right| = M \left| \frac{x}{\bar{x}} \right|^n$$

$$= M \left(\frac{|x|}{|\bar{x}|} \right)^n$$

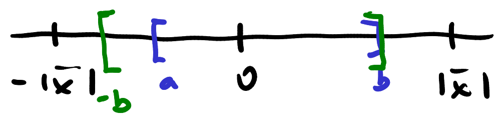
$|x| < |\bar{x}| \Rightarrow 0 \le \frac{|x|}{|\bar{x}|} < 1 \Rightarrow \sum_n \left(\frac{|x|}{|\bar{x}|} \right)^n$ serie geom. convergente

$\overset{\text{multiplo}}{\Rightarrow} \quad \sum_n M \left(\frac{|x|}{|\bar{x}|} \right)^n$ conv.

$\overset{\text{crit. confr.}}{\Rightarrow} \quad \sum_n |c_n x^n|$ conv. $\iff \sum_n c_n x^n$ è assol. conv.

Dimostro che la serie converge totalmente in un intervallo $[-a, a]$ con $0 < a < |\bar{x}|$

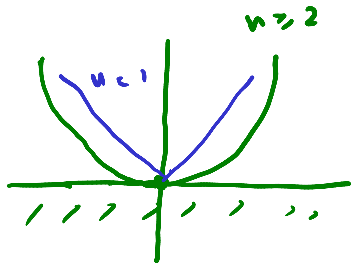
(Da questo segue la conv. totale in qualsiasi interv. compatto contenuto in $(-|\bar{x}|, |\bar{x}|)$



)

Fisso $a \in (0, |\bar{x}|)$

$$\sup_{x\in[-a,a]} |c_n x^n| = \sup_{x\in[-a,a]} |c_n|\,|x|^n$$



$$= |c_n|\, a^n = |c_n a^n|$$

$\sum_n |c_n a^n|$ converge perché $a \in (-|\bar{x}|, |\bar{x}|)$

$\Rightarrow \sum_n \sup_{x\in[-a,a]} |c_n x^n|$ converge

$\Leftrightarrow \sum_n c_n x^n$ conv. totalmente in $[-a, a]$,

$\forall a \in (0, |\bar{x}|)$. $\square$

Determino il r.d.c. di:

- $\sum_n (-1)^n \frac{x^n}{n}$ $\qquad c_0 = 0 \qquad c_n = \frac{(-1)^n}{n} \quad \forall n \geq 1$

$\Rightarrow \forall n \geq 1 : |c_n| = \frac{1}{n} \Rightarrow \forall n \geq 1 : \sqrt[n]{|c_n|} = \frac{1}{\sqrt[n]{n}}$ succ. regolare (noto!)

$\Rightarrow$ "$\lim_n$" $\sqrt[n]{|c_n|} = \lim_n \frac{1}{\sqrt[n]{n}} = 1 =: \alpha$

$\Rightarrow R = 1.$

- $\sum_n x^n$ $\quad \forall n: \quad c_n = 1$

(Convenzione: $x^0 \equiv 1$)

$$\Rightarrow \sqrt[n]{|c_n|} = 1 \Rightarrow \exists \lim_n \sqrt[n]{|c_n|} = 1 =: \alpha$$

$$\Rightarrow R = 1$$

- $\sum_n x^{2n}$ $\quad c_{2n} = 1, \quad c_{2n-1} = 0 \quad \forall n \geq 1$

$$\Leftrightarrow \quad c_k = \begin{cases} 0 & k \text{ dispari} \\ 1 & k \text{ pari} \end{cases}$$

$$\Rightarrow \sqrt[k]{|c_k|} = \begin{cases} 0 & k \text{ disp.} \\ 1 & k \text{ pari} \end{cases}$$

$$\Rightarrow \nexists \lim_{k \to +\infty} \sqrt[k]{|c_k|} \quad \text{però: } \exists \lim_{k \to +\infty}{}'' \sqrt[k]{|c_k|} = 1$$

$$\Rightarrow R = 1$$

- $\sum_n \frac{x^n}{n!}$ $\quad \forall n: \quad c_n = \frac{1}{n!} \quad \Rightarrow \cancel{\sqrt[n]{|c_n|} = \sqrt[n]{\frac{1}{n!}}}$??

$$\cancel{?} \lim_{n \to +\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n \to +\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n \to +\infty} \frac{n!}{(n+1)!}$$

$$= \lim_{n \to +\infty} \frac{1}{n+1} = 0 =: \alpha$$

$$\Rightarrow R = +\infty$$

$$\left[\Rightarrow \lim_{n \to +\infty} \sqrt[n]{\frac{1}{n!}} = 0 \quad \Rightarrow \lim_{n \to +\infty} \sqrt[n]{n!} = +\infty \right]$$

- $\sum_n n!\, x^n$ $\quad \forall n: \quad c_n = n!$

$$\ldots \quad \alpha = +\infty \quad \Rightarrow \quad R = 0.$$

Dimostro il teorema.

Voglio applicare il criterio della radice alla serie di termine $c_n x^n$, quindi devo valutare

$$\lim_{n\to+\infty} \sqrt[n]{|c_n x^n|} = \lim_{n\to+\infty} \sqrt[n]{|c_n||x|^n} = \lim_{n\to+\infty} \left(\sqrt[n]{|c_n|} \cdot \underbrace{|x|}_{\text{succ. costante}} \right)$$

$$= \Big(\underbrace{\lim_{n\to+\infty} \sqrt[n]{|c_n|}}_{=:\alpha} \Big) |x| = \alpha |x|$$

Suppongo $R \in (0, +\infty)$, cioè: $R = \frac{1}{\alpha}$, con $\alpha \in (0, +\infty)$

$$\lim_n \sqrt[n]{|c_n x^n|} = \dots = \underbrace{\alpha |x|}_{L}$$

crit. radice

$\alpha |x| < 1 \Rightarrow$ la serie conv. assol.

$\alpha |x| > 1 \Rightarrow$ la serie non conv.

$\Leftrightarrow$ se $|x| < \frac{1}{\alpha} = R$ la serie conv. assol.

se $|x| > \frac{1}{\alpha} = R$ la serie non conv. $\square$

Se $R = +\infty$, che equivale a $\alpha = 0$:

$$\forall x \in \mathbb{R}: \quad \lim_n \sqrt[n]{|c_n x^n|} = \underbrace{\alpha}_{=0} |x| = 0 < 1$$

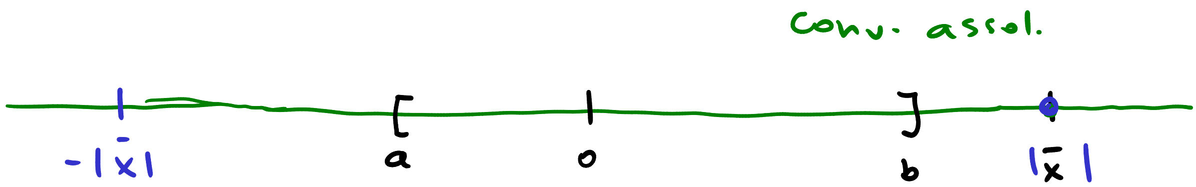
$\Rightarrow$ la serie conv. assol.

Se $R = 0$, che equivale a $\alpha = +\infty$:

$$\forall x \in \mathbb{R}^*: \quad \lim_n \sqrt[n]{|c_n x^n|} = \underbrace{\alpha}_{=+\infty} \underbrace{|x|}_{>0} = +\infty$$

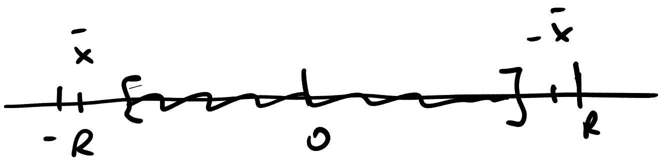
$\Rightarrow$ la serie non converge. $\square$

Oss: $R = +\infty$



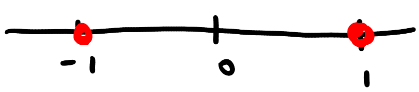
$$\sup_{x\in\mathbb{R}} |\underbrace{c_n x^n}_{f_n(x)}| = +\infty \Rightarrow f_n \not\to 0 \text{ unif. in } \mathbb{R}$$

$$\Rightarrow \sum_n f_n \text{ non conv. unif.}$$

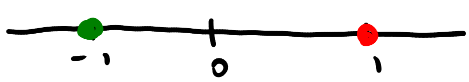


Es:

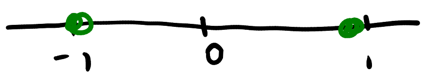
- $\sum_n x^n$ $\quad R = 1$



- $\sum_n \frac{x^n}{n}$ $\quad R = 1$



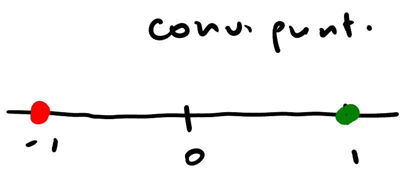
- $\sum_n \frac{x^n}{n^2}$ $\quad R = 1$



$$|c_n R^n| = |c_n| R^n$$
$$\parallel$$
$$|c_n (-R)^n| = |c_n|\,|-R|^n = |c_n| R^n$$

Es: $\sum_n (-1)^n \frac{x^n}{n}$

Già noto: $R = 1$



$\Rightarrow$ conv. punt. $(-1, 1]$

conv. assol. $(-1, 1)$

conv. unif. $[-\rho, 1]$ $\quad 0 < \rho < 1$

conv. totale $[-\rho, \rho]$ $\quad 0 < \rho < 1$

Es. $\sum_n (2^n + 3^n) x^n$

$\forall n: \quad c_n = 2^n + 3^n$

$$\sqrt[n]{c_n} = \sqrt[n]{2^n + 3^n} = \sqrt[n]{3^n\left(1 + \left(\tfrac{2}{3}\right)^n\right)}$$

$$= 3 \sqrt[n]{\underbrace{1 + \left(\tfrac{2}{3}\right)^n}_{\to 1}} \longrightarrow 3 =: \alpha$$

$\Rightarrow R = \frac{1}{3}$

Per $x = \frac{1}{3}$: $\quad \sum_n \underbrace{(2^n + 3^n)\frac{1}{3^n}}_{\not\to 0}$ $\quad = \frac{2^n + 3^n}{3^n} \sim \frac{3^n}{3^n} = 1$

$\Rightarrow$ la serie non conv.

Per $x = -\frac{1}{3}$: $\quad \sum_n \underbrace{(2^n + 3^n)\frac{(-1)^n}{3^n}}_{\not\to 0}$

$\Rightarrow$ la serie non conv.

Conclusioni:

la serie conv. punt/assol. in $\left(-\frac{1}{3}, \frac{1}{3}\right)$

conv. unif. /total. in $[-\rho, \rho]$, $0 < \rho < \frac{1}{3}$.