22 ottobre 2024

Es.

- $f_n(x) = (-1)^n \dfrac{x^4}{n+x^4}$ $\quad x \in \mathbb{R}, \quad n \in \mathbb{N}^*$

$\forall n: \quad f_n(0) = 0 \quad \Rightarrow \quad \sum_n f_n(0)$ conv. (assol.)

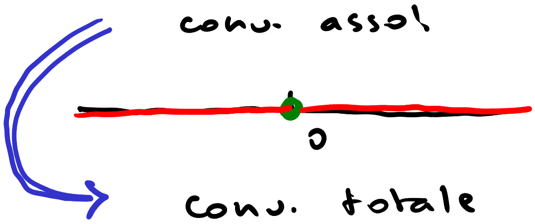
$x \neq 0: \quad |f_n(x)| = \dfrac{x^4}{n+x^4} \sim \dfrac{x^4}{n} = x^4 \cdot \dfrac{1}{n}$

$\sum_n \dfrac{1}{n}$ div. $\overset{\text{mult.}}{\Rightarrow} \sum_n x^4 \dfrac{1}{n}$ div.

$\overset{\text{confr. as.}}{\Rightarrow} \sum_n |f_n(x)|$ diverge

Quindi:

$\forall x \neq 0: \quad \sum_n f_n(x)$ non conv. assol.




Studio conv. punt. e unif. sfruttando il criterio di Leibniz (e le sue conseguenze)

$f_n(x) = (-1)^n \dfrac{x^4}{n+x^4} =: g_n(x)$

- $\forall x: \quad \forall n \quad g_n(x) \geq 0$
- $\forall x: \quad g_{n+1}(x) = \dfrac{x^4}{n+1+x^4} \leq \dfrac{x^4}{n+x^4} = g_n(x)$

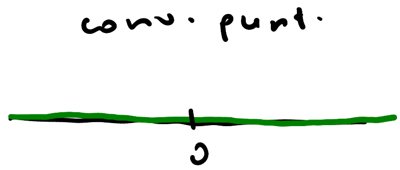
$\Rightarrow \forall x: \quad (g_n(x))_n$ decrescente

- $\forall x: \quad \lim_{n \to +\infty} g_n(x) = 0$

Quindi: $\forall x \in \mathbb{R}$ $f_n(x)$ soddisfa le ipotesi del criterio di Leibniz.

Pertanto:

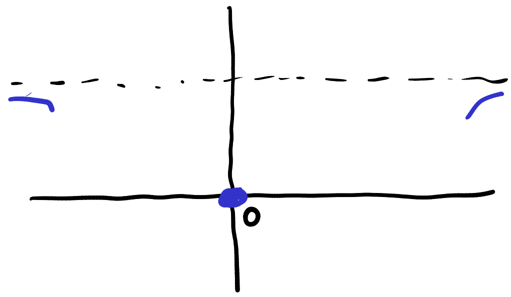
- $\forall x \in \mathbb{R}$: $\sum_n f_n(x)$ converge




- Per $\sum_n f_n$ la cond. necessaria per la conv. unif. è anche sufficiente

Calcolo $\sup_{\mathbb{R}} |f_n|$

$$|f_n(x)| = g_n(x) = \frac{x^4}{n+x^4}$$




Oss: $$\left.\begin{array}{l} |f_n| \leq 1 \quad \forall x \\ |f_n(x)| \xrightarrow[x \to \pm\infty]{} 1 \end{array}\right\} \Rightarrow \sup_{\mathbb{R}} |f_n| = 1 \quad (\forall n)$$

$$\Rightarrow \lim_n \sup_{\mathbb{R}} |f_n| \neq 0$$

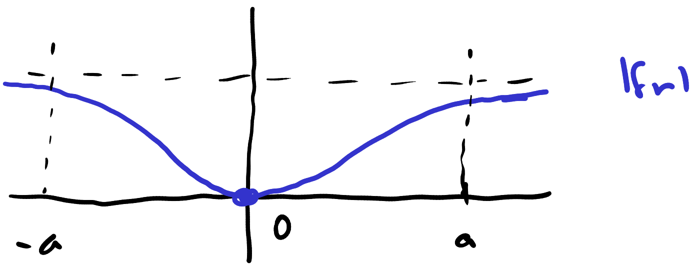
$\Rightarrow$ la serie <u>non</u> conv. unif. in $\mathbb{R}$

Provo a recuperare.

Determino la monotonia di $|f_n|$:

$$f_n(x) = \frac{x^4}{n+x^4} = \frac{x^4+n-n}{n+x^4} = \left(1 - \left(\frac{n}{n+x^4}\right)\right)$$

$\uparrow \quad \uparrow \quad \downarrow \quad \uparrow$ in $[0,+\infty)$




(alternativa: $g_n'(x) = \dots$ )

Fisso $a>0$

$\forall n$: $\sup_{[-a,a]} |f_n| = |f_n(a)|$ vera $\forall x \in \mathbb{R}$

$\lim_{n\to\infty} \sup_{[-a,a]} |f_n| = \lim_{n\to\infty} |f_n(a)| = 0$

Quindi: in $[-a,a]$ $(f_n)$ conv. unif. a $0$

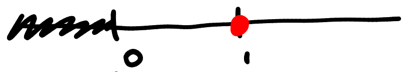
$\Rightarrow$ in $[-a,a]$ $\sum_n f_n$ conv. unif. $\square$

- $f_n(x) = \dfrac{\arctan(x^{2n})}{x^n}$ $x \in (0,+\infty)$

$>0$ $\forall x \in (0,+\infty)$

$x=1$: $f_n(1) = \dfrac{\arctan(1)}{1} = \dfrac{\pi}{4}$

$\Rightarrow \sum_n f_n(1)$ div. posit.



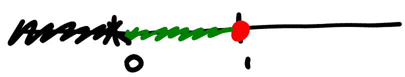
$t \to 0$: $\arctan(t) \sim t$

$0<x<1$: $f_n(x) = \dfrac{\arctan(x^{2n})}{x^n} \sim \dfrac{x^{2n}}{x^n} = x^n$

$\Downarrow$ $x^{2n} \to 0$

$\sum_n x^n$ conv. (geom. con ragione $x \in (0,1)$ )

confr. asint. $\Rightarrow$ $\sum_n f_n(x)$ conv.



$x>1$: $0 \le f_n(x) = \dfrac{\arctan(x^{2n})}{x^n} \le \dfrac{\pi/2}{x^n} = \dfrac{\pi}{2} \cdot \left(\dfrac{1}{x}\right)^n$

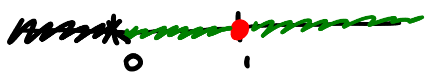
$\Downarrow$ $x^{2n} \to +\infty$, $x^n \to +\infty$

$x>1 \Rightarrow 0<\frac{1}{x}<1 \Rightarrow \sum_n \left(\frac{1}{x}\right)^n$ conv.

mult. $\Rightarrow \sum_n \frac{\pi}{2}\left(\frac{1}{x}\right)^n$ conv.

confr. $\Rightarrow \sum_n f_n(x)$ converge

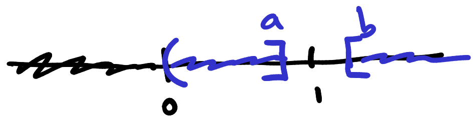
Conclusione:




la serie converge punt. ($\equiv$ assol.) in $(0,1) \cup (1,+\infty)$

Osservando che $f_n$ è continua in $(0,+\infty)$, quindi anche in $x=1$, per le note considerazioni deduco che la serie non può convergere unif. / totalm. in $(0,1)\cup(1,+\infty)$; per recuperare la convergenza devo escludere un intorno di $x=1$

Fisso $0<a<1<b$;




studio separatamente $(0,a]$ e $[b,+\infty)$

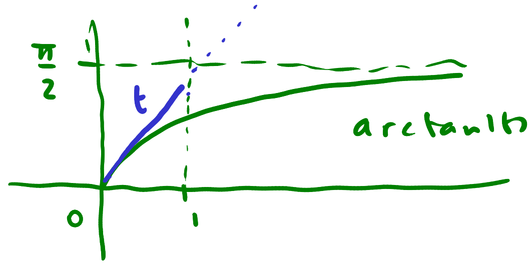
Comincio da $[b,+\infty)$:

$$\forall n: \quad \sup_{[b,+\infty)} |f_n(x)| = \sup_{[b,+\infty)} \frac{\arctan(x^{2n})}{x^n} \underset{x\geq b}{\leq} \frac{\pi/2}{b^n}$$

$$\sum_n \frac{\pi/2}{b^n} \text{ converge } (b>1) \Rightarrow \sum_n \sup_{[b,+\infty)} |f_n| \text{ converge}$$

Quindi: $\sum_n f_n$ conv. totalm. in $[b,+\infty)$

In $(0,a]$:



$$\forall x \in (0,a]: \quad 0 < f_n(x) = \frac{\arctan(x^{2n})}{x^n} \leq \frac{x^{2n}}{x^n} = x^n \leq \underbrace{a^n}_{=: M_n}$$

$$\sum_n a^n \quad \text{conv.} \quad (a<1)$$

$$\Rightarrow \quad \sum_n f_n \quad \text{conv. totalmente in } (0,a].$$

Conclusione: $\sum_n f_n$ conv. totalm. in

$$(0,a] \cup [b,+\infty) \quad \forall\, 0<a<1<b.$$

$\square$

Oss: $\sum_n f_n$ conv. punt. / unif. in $X \overset{def}{\iff}$

$(S_n)$ 〃 〃 〃 〃

$$\underbrace{S_n}:= f_0 + f_1 + \dots + f_n \qquad \underset{\text{unif.}}{S_n \to f}$$

$$\int_a^b f(x)\,dx \overset{TPLSSI}{=} \lim_n \int_a^b S_n(x)\,dx$$

$$= \lim_n \int_a^b (f_0(x) + \dots + f_n(x))\,dx$$

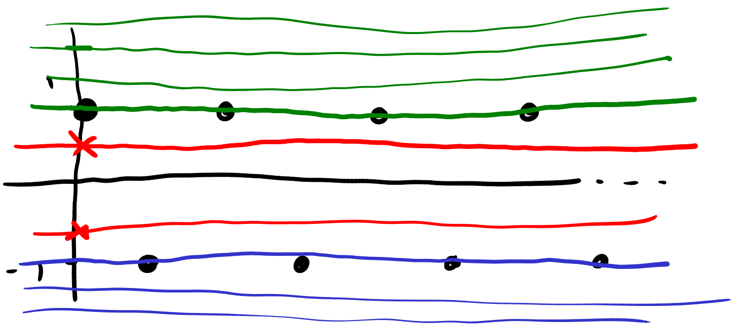
$$= \lim_n \left( \int_a^b f_0(x)\,dx + \dots + \int_a^b f_n(x)\,dx \right)$$

$$= \sum_{n=0}^{\infty} \int_a^b f_n(x)\,dx$$

$$S_n = f_0 + \dots + f_n \quad \Rightarrow \quad S_n' = f_0' + \dots + f_n'$$

Es. ( lim'' , lim' )

• $x_n = (-1)^n$

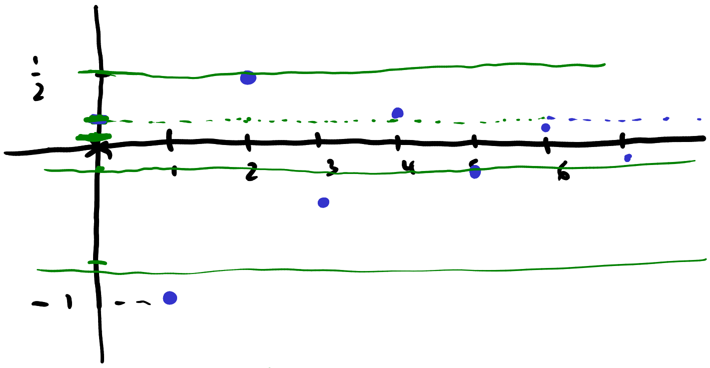



$M^* = [1, +\infty)$ $(\neq \emptyset)$ $\Rightarrow$ $\lim''_{n \to +\infty} (-1)^n = \inf M^* = 1$

$\lim'_{n \to +\infty} (-1)^n = -1$ $\neq$

....

• $x_n = \dfrac{(-1)^n}{n}$




$M^* = (0, +\infty)$

$\lim''_{n \to +\infty} \dfrac{(-1)^n}{n} = \inf (0, +\infty) = 0$

$=$

$M_* = (-\infty, 0)$ $\Rightarrow$ $\lim'_{n \to +\infty} \dfrac{(-1)^n}{n} = \sup (-\infty, 0) = 0$

Verifico che $\lim''_n \sqrt[n]{a_n} \leq \underbrace{\lim''_n \dfrac{a_{n+1}}{a_n}}_{=:\, L \ (\geq 0)}$

(con $a_n > 0$ $\forall n$).

Se $L = +\infty$, non ho nulla da dimostrare.

Suppongo $L \in [0, +\infty)$

Fisso $\varepsilon > 0$; per la caratterizzazione:

$\exists\, \nu \in \mathbb{N}$ t.c. $\forall n \geq \nu$: $\underbrace{\dfrac{a_{n+1}}{a_n} < L + \varepsilon}$

$a_n > 0$ $\Leftrightarrow$ $a_{n+1} < (L + \varepsilon) a_n$

In particolare:

$$a_{\nu+1} < (L+\varepsilon)\, a_\nu$$

$$a_{\nu+2} < (L+\varepsilon)\, a_{\nu+1} < (L+\varepsilon)^2 a_\nu$$

$$\vdots$$

$$\forall k \geq 1 \qquad a_{\nu+k} < (L+\varepsilon)^k a_\nu$$

$$\Leftrightarrow \quad \forall n \geq \nu+1 \qquad a_n < (L+\varepsilon)^{n-\nu} a_\nu = (L+\varepsilon)^n \frac{a_\nu}{(L+\varepsilon)^\nu}$$

$$\Rightarrow \quad \forall n \geq \nu+1 : \quad \sqrt[n]{a_n} < (L+\varepsilon) \sqrt[n]{\frac{a_\nu}{(L+\varepsilon)^\nu}}$$

$$\overset{(2)}{\Rightarrow} \quad \text{lim}''_n \sqrt[n]{a_n} \leq \text{lim}''_n \left( \underbrace{(L+\varepsilon)}_{\text{cost.}} \underbrace{\sqrt[n]{\frac{a_\nu}{(L+\varepsilon)^\nu}}}_{\to 1} \right) \quad \text{costante}$$

$$\Leftrightarrow \quad \text{lim}''_n \sqrt[n]{a_n} \leq L+\varepsilon$$

Ho provato che:

$$\forall \varepsilon > 0 : \qquad \text{lim}''_n \sqrt[n]{a_n} \leq L+\varepsilon$$

$$\overset{\varepsilon \to 0^+}{\Rightarrow} \quad \text{lim}''_n \sqrt[n]{a_n} \leq L \qquad \square$$

Dimostro il criterio della radice.

$$L := \text{lim}''_n \sqrt[n]{|x_n|}$$

- $L \in [0,1)$; scelgo $\varepsilon > 0$ t.c. $L+\varepsilon < 1$

Per la caratterizzazione:

$\exists \nu \in \mathbb{N}$ t.c. $\forall n \geq \nu$ : $\sqrt[n]{|x_n|} < L+\varepsilon$

$\Rightarrow$ $\forall n \geq \nu$ : $|x_n| < (L+\varepsilon)^n$

$\sum_n (L+\varepsilon)^n$ converge (geom. con ragione $L+\varepsilon \in (0,1)$ )

$\overset{\text{crit. confr.}}{\Rightarrow}$ $\sum_n |x_n|$ converge, cioè :

$\sum_n x_n$ conv. assolutamente $\square$

- $L \in (1,+\infty] := (1,+\infty) \cup \{+\infty\}$

Se $L \in (1,+\infty)$, scelgo $\varepsilon > 0$ t.c. $L-\varepsilon > 1$

Per la caratterizzazione :

per infiniti indici : $\sqrt[n]{|x_n|} > L-\varepsilon$
$L-\varepsilon > 1$

$\Rightarrow$ per infiniti indici : $\sqrt[n]{|x_n|} > 1$

Se $L = +\infty$ : l'insieme dei magg. definitivi di $(\sqrt[n]{|x_n|})$ è vuoto

Perciò : $1$ non è magg. definitivo di $(\sqrt[n]{|x_n|})$

cioè : per infiniti indici : $\sqrt[n]{|x_n|} > 1$

In entrambi i casi :

per infiniti indici : $\sqrt[n]{|x_n|} > 1$

$\Leftrightarrow$ " " " : $|x_n| > 1$

Pertanto : $(|x_n|)$ <u>non</u> può tendere a $0$, che equivale a : $(x_n)$ non tende a $0$

che implica che $\sum_n x_n$ <u>non</u> converge.

- $L = 1$ ?? $\quad x_n = \frac{1}{n}$, $\quad y_n = \frac{1}{n^2}$

$$\sqrt[n]{x_n} \to 1 \qquad \sqrt[n]{y_n} \to 1$$

Però: $\sum_n \frac{1}{n}$ div., $\quad \sum_n \frac{1}{n^2}$ conv.

$\square$