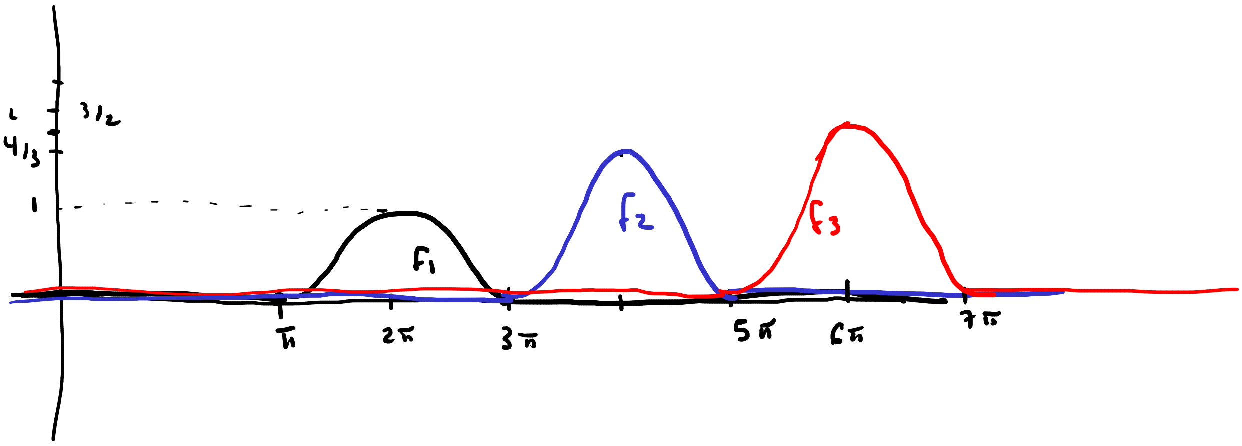
7 ottobre 2024

Es.
$$f_n(x) = \begin{cases} \dfrac{n(\cos(x)+1)}{n+1} & x \in [(2n-1)\pi, (2n+1)\pi] \\ 0 & \text{altrimenti} \end{cases}$$



Fisso $x \in \mathbb{R}$.

Se $x \leq \pi$: $f_n(x) = 0 \quad \forall n \implies \lim\limits_{n \to +\infty} f_n(x) = 0 =: f(x)$

Se $x > \pi$: definitivamente : $(2n-1)\pi > x$

$\implies$ dfnt : $x \notin [(2n-1)\pi, (2n+1)\pi]$

$\implies$ dfnt : $f_n(x) = 0$

$$\implies \lim_{n \to \infty} f_n(x) = 0 =: f(x)$$

Conclusione : (f_n) converge puntualm. in $\mathbb{R}$ a $f: \mathbb{R} \to \mathbb{R}$ t.c. $f(x) = 0 \quad \forall x \in \mathbb{R}$.

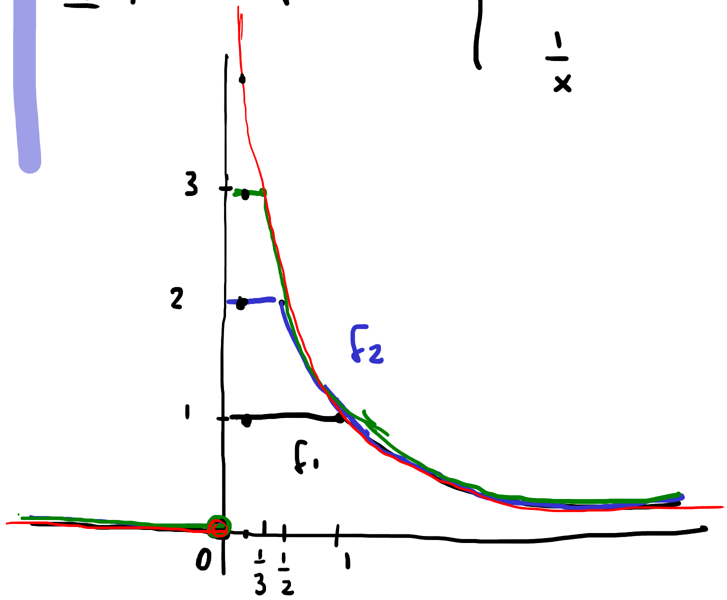
Verifico se la convergenza è uniforme.

Fisso n :
$$\sup_{x \in \mathbb{R}} |f_n(x) - \overbrace{f(x)}^{=0}| = \sup_{x \in \mathbb{R}} f_n(x) = f_n(2n\pi) = \frac{n}{n+1}\,2$$

$$\implies \lim_{n \to +\infty} \sup_{\mathbb{R}} |f_n - f| = \lim_{n \to +\infty} \frac{2n}{n+1} = 2 \neq 0$$

Conclusione: no conv. uniforme in $\mathbb{R}$. $\square$

Es:
$$f_n(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ n & x \in (0, \frac{1}{n}) \\ \frac{1}{x} & x \in [\frac{1}{n}, +\infty) \end{cases} \qquad n \geq 1$$

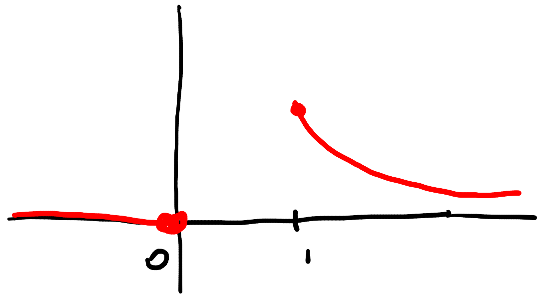


Fisso $x \in \mathbb{R}$.

Se $x \leq 0$: $f_n(x) = 0 \quad \forall n$

$$\Rightarrow \lim_n f_n(x) = 0 =: f(x)$$

Se $x \geq 1$: $x \in [\frac{1}{n}, +\infty) \quad \forall n$



$$\Rightarrow f_n(x) = \frac{1}{x} \quad \forall n$$

$$\Rightarrow \lim_n f_n(x) = \frac{1}{x} =: f(x)$$

Se $0 < x < 1$: ??

(Per esempio: $x = \frac{1}{2}$

$f_1(\frac{1}{2}) = 1$, $f_2(\frac{1}{2}) = 2$, $f_3(\frac{1}{2}) = \frac{1}{\frac{1}{2}} = 2$)

Formalizzo:

fissato $0 < x < 1$: d'fnt $\frac{1}{n} < x$

$\Rightarrow$ d'fnt: $x \in [\frac{1}{n}, +\infty)$

$\Rightarrow$ d'fnt: $f_n(x) = \frac{1}{x}$

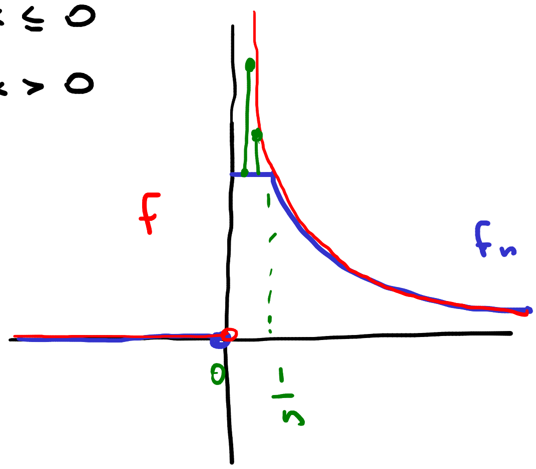
$$\Rightarrow \lim_n f_n(x) = \frac{1}{x} =: f(x)$$

Conclusione: (f_n) conv. punt. a $f: \mathbb{R} \to \mathbb{R}$ t.c.

$$f(x) = \begin{cases} 0 & x \le 0 \\ 1/x & 0 < x < 1 \\ 1/x & x \ge 1 \end{cases}$$

Scritto meglio:

$$f(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{x} & x > 0 \end{cases}$$



È limite uniforme?

Fisso n;

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = +\infty$$

$$\Rightarrow \lim_n \sup_{\mathbb{R}} |f_n - f| = +\infty \ \underline{\underline{\neq 0}}$$

Quindi: no conv. uniforme in $\mathbb{R}$. $\square$

Esamino la convergenza di $f_n(x) = x^n$, $x \in [0,1]$ "con ε e ν".

So che il limite puntuale è $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

Fisso $\varepsilon > 0$ e studio (= mi chiedo per quali n sia soddisfatta)

$$(*) \qquad |f_n(x) - f(x)| < \varepsilon$$

Se $x \in \{0,1\}$: $|f_n(x) - f(x)| = 0$

$\Rightarrow (*)$ è soddisfatta $\forall n$.

Se $x \in (0,1)$

$$(*) \iff |x^n - 0| < \varepsilon \iff \underbrace{x^n < \varepsilon}$$

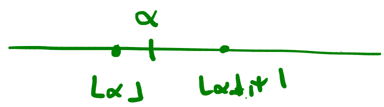
Se $\varepsilon \geq 1$: (*) è soddisfatta $\forall n$.

Se $\varepsilon < 1$:

$$(*) \iff \ln(x^n) < \ln(\varepsilon)$$

$$\iff n \overbrace{\ln(x)}^{<0} < \ln(\varepsilon)$$

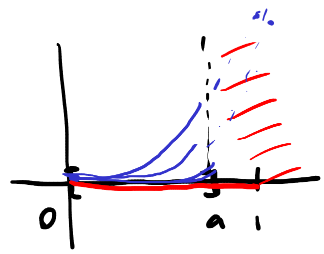
$$\iff n > \underbrace{\frac{\overbrace{\ln(\varepsilon)}^{<0}}{\underbrace{\ln(x)}_{<0}}}_{>0}$$



$$(*) \text{ vera } \iff n \geq \left\lfloor \frac{\ln(\varepsilon)}{\ln(x)} \right\rfloor + 1 =: \nu_{\varepsilon,x}$$

OSS: $$\lim_{x\to 1^-} \nu_{\varepsilon,x} = \lim_{x\to 1^-} \left\lfloor \underbrace{\frac{\overbrace{\ln(\varepsilon)}^{<0}}{\underbrace{\ln(x)}_{\to 0^-}}}_{\to +\infty} \right\rfloor + 1 = +\infty$$

Provo a "recuperare" la convergenza uniforme, escludendo un intorno (sinistro) di $x=1$ (il "colpevole"!)



Fisso $a \in (0,1)$ e considero $[0,a]$

Pongo $$\nu_\varepsilon := \left\lfloor \frac{\ln(\varepsilon)}{\ln(a)} \right\rfloor + 1$$

OSS: $$0 < x \leq a \implies \ln(x) \leq \ln(a) \implies \frac{1}{\ln(x)} \geq \frac{1}{\ln(a)}$$

$$\implies \frac{\ln(\varepsilon)}{\ln(x)} \leq \frac{\ln(\varepsilon)}{\ln(a)}$$

$$\implies \dots \implies \nu_{\varepsilon,x} \leq \nu_{\varepsilon,a}$$

Quindi:

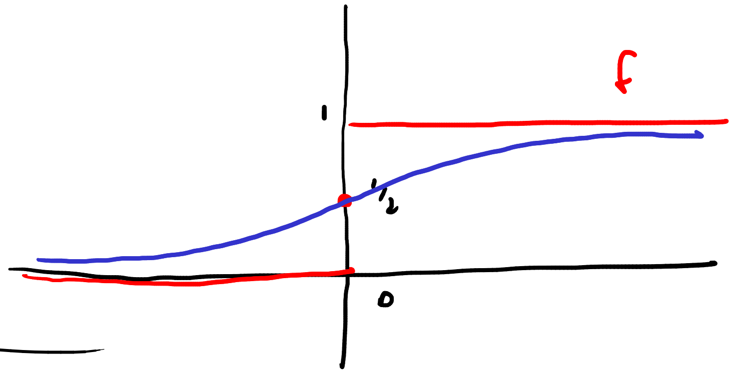
se $n \geq \nu_{\varepsilon,a} \implies \underbrace{n \geq \nu_{\varepsilon,x}}\ \forall x \in [0,a]$

$$\iff (*) \text{ è soddisfatta } \forall x \in [0,a]$$

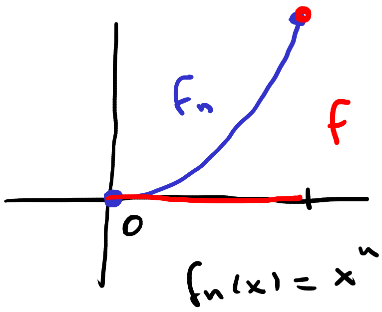
Perciò: la conv. è uniforme in $[0,a]$. $\square$

Es. (in cui la continuità / limitatezza del limite puntuale si verifica oppure no)

- $f_n(x) = \dfrac{e^{nx}}{e^{nx}+1}$



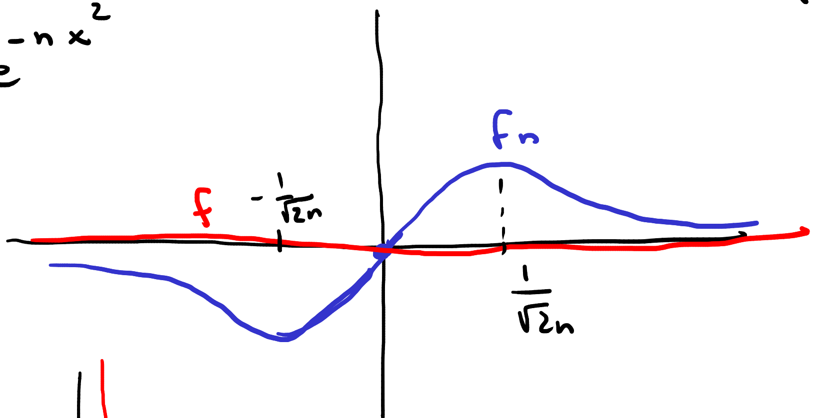
	f_n	f
limitatezza	sì	sì
continuità	sì	no



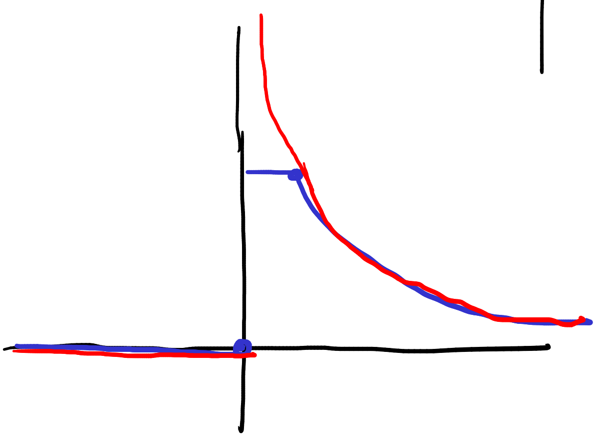
$f_n(x) = x^n$
$x \in [0,1]$

- $f_n(x) = n x e^{-nx^2}$

	f_n	f
lim.	✓	✓
cont.	✓	✓



- f_n:



f_n limitata

f non limitata

Es: "recupero" la conv. unif. per

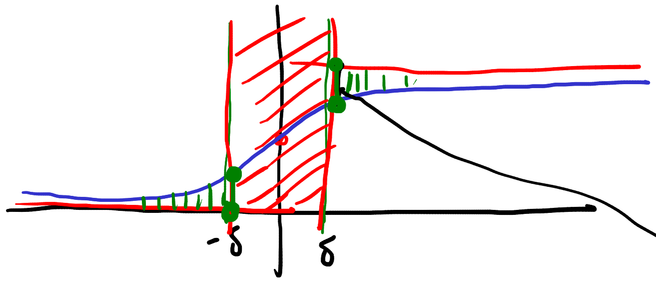
$$f_n(x) = \frac{e^{nx}}{e^{nx}+1} \qquad x \in \mathbb{R}$$

f (lim. punt.) non è continua in $x=0$,

posso escludere che f_n converga a f unif. in un intorno di $x=0$.

Provo a escludere un intorno di $x=0$:

Fisso $\delta>0$ e considero $X = \mathbb{R} \setminus (-\delta,\delta)$
$= (-\infty,-\delta] \cup [\delta,+\infty)$



$$\underbrace{\sup_X |f_n - f|}_{\to 0} = \max\{\underbrace{|f_n(\delta)-f(\delta)|}_{\to 0}, \underbrace{|f_n(-\delta)-f(-\delta)|}_{\to 0}\}$$
$$\Leftarrow$$

So che: $\forall x \in \mathbb{R}$: $f_n(x) \to f(x)$
$|f_n(x) - f(x)| \to 0$

$$\Rightarrow \begin{cases} |f_n(\delta) - f(\delta)| \to 0 \\ |f_n(-\delta) - f(-\delta)| \to 0 \end{cases}$$

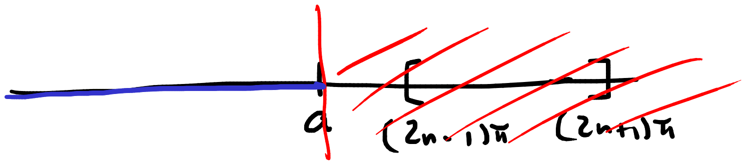
Analisi 1: $a_n \to 0$, $b_n \to 0$
$c_n := \max\{a_n, b_n\}$ $\Rightarrow c_n \to 0$

Es: "recupero" la conv. unif. per

$$f_n(x) = \begin{cases} \dfrac{n(\cos(x)+1)}{n+1} & x \in [(2n-1)\pi, (2n+1)\pi] \\ 0 & \text{altrimenti} \end{cases}$$

"Colpevole": $\forall n$: $2n\pi$
$2n\pi \xrightarrow[n\to+\infty]{} +\infty$ $\Rightarrow$ "colpevole" $\underline{+\infty}$

Fisso $a \in \mathbb{R}$.



Osservo che dfnt: $(2n-1)\pi > a$

$\Rightarrow$ dfnt: $f_n\big|_{(-\infty,a]} \equiv 0$

$\Rightarrow$ dfnt: $\sup\limits_{(-\infty,a]} |f_n - f| = 0$

$$\Rightarrow \lim_{n\to+\infty} \sup_{(-\infty,a]} |f_n - f| = 0$$

Conclusione: $f_n \to f$ unif. in $(-\infty, a]$ $(\forall a \in \mathbb{R})$

Dimostro il TPLSSI

Osservo che f è continua in quanto limite uniforme di una successione di funzioni continue; dunque: f è integrabile in $[a,b]$

Per ogni n:

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \overset{\text{linearità}}{=} \left| \int_a^b (f_n(x) - f(x))\,dx \right|$$

$$\underset{\substack{\text{disug. triang.}\\ \text{per integrali}}}{\leq} \int_a^b \underbrace{|f_n(x) - f(x)|}_{\leq \sup\limits_{t\in[a,b]} |f_n(t) - f(t)|}\,dx \overset{\text{monoti}}{\leq} \int_a^b \underbrace{\sup_{[a,b]} |f_n - f|}_{\substack{\text{non dipende}\\ \text{da } x}}\,dx$$

$$= \sup_{[a,b]} |f_n - f| \, (b-a)$$

Ricapitolando, $\forall n$:

$$\underset{\downarrow\, 0}{0} \leq \left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \leq \overbrace{\underbrace{\sup_{[a,b]} |f_n - f|}_{\xrightarrow{n\to+\infty} 0} (b-a)}^{\to 0}$$

$\overset{TCO}{\Rightarrow}$ $\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \xrightarrow[n\to+\infty]{} 0$

cioè: $\int_a^b f_n(x)\,dx \xrightarrow[n\to+\infty]{} \int_a^b f(x)\,dx$ $\square$

Es. (ruolo della conv. unif. nel TPLSSI)

$f_n(x) = n^p x (1-x^2)^n \qquad x \in [0,1] \qquad (p \in \mathbb{R})$

Determino il limite puntuale di (f_n)

Se $x = 0$: $f_n(0) = 0 \quad \forall n \Rightarrow f_n(0) \to 0 =: f(0)$

Se $x = 1$: $f_n(1) = 0 \quad \forall n \Rightarrow f_n(1) \to 0 =: f(1)$

Se $x \in (0,1)$:

$$f_n(x) = n^p \underbrace{x}_{\text{cost.}} \overbrace{(1-x^2)^n}^{\to 0}$$

prog. geom. con $q = 1-x^2 \in (0,1)$

$0 < x < 1$
$0 < x^2 < 1$
$0 < 1-x^2 < 1$

$n^p \to \begin{cases} 0 & p<0 \\ 1 & p=0 \\ +\infty & p>0 \end{cases}$

Se $p \le 0$: $f_n(x) = \underbrace{n^p}_{\to 0} \underbrace{x}_{\text{fisso}} \underbrace{(1-x^2)^n}_{\to 0} \longrightarrow 0 =: f(x)$

Se $p > 0$: $f_n(x) = x \underbrace{n^p}_{+\infty} \underbrace{(1-x^2)^n}_{\to 0} \longrightarrow 0 =: f(x)$

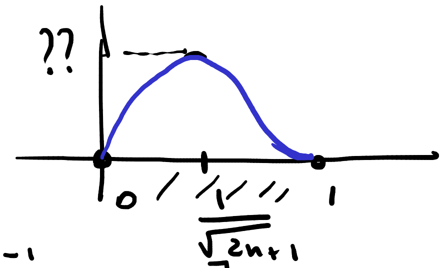
AMI: $\underset{0<a<1}{n^p a^n} = \frac{\overbrace{n^p}^{\to+\infty}}{\underbrace{b^n}_{\to+\infty}} \to 0 \qquad b>1$

Conclusione: $\forall p \in \mathbb{R}$: $f_n(x) \to f(x) \equiv 0 \quad \forall x \in \mathbb{R}$

Verifico se il limite è uniforme.

? $\sup_{x\in[0,1]} |f_n(x) - \underbrace{f(x)}_{\equiv 0}| = \sup_{x\in[0,1]} f_n(x)$

$f_n(x) = n^p x (1-x^2)^n$



Calcolo:

$$f_n'(x) = n^p \left[(1-x^2)^n + x\, n (1-x^2)^{n-1} (-2x) \right]$$

$$= n^p (1-x^2)^{n-1} \left[1 - x^2 - 2n x^2 \right]$$

$$= n^p \underbrace{(1-x^2)^{n-1}}_{\geq 0} \underbrace{\left[1 - (2n+1)x^2 \right]}_{\geq 0} \iff x^2 \leq \frac{1}{2n+1}$$

$$\overset{[0,1]}{\iff} \quad x \leq \frac{1}{\sqrt{2n+1}}$$

Quindi:

$$\sup_{[0,1]} |f_n - f| = f_n\left(\frac{1}{\sqrt{2n+1}}\right) = n^p \frac{1}{\sqrt{2n+1}} \left(1 - \frac{1}{2n+1}\right)^n$$

$$= n^p \frac{1}{\sqrt{2n+1}} \underbrace{\left(\frac{2n}{2n+1}\right)^n}_{\to e^{-\frac{1}{2}}}$$

$$\left(\frac{2n+1}{2n}\right)^n = \Bigg[\underbrace{\left(1 + \frac{1}{2n}\right)^{2n}}_{\to e} \Bigg]^{\frac{1}{2}} \to e^{1/2}$$

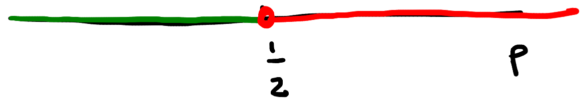
$$\Rightarrow \sup_{[0,1]} |f_n - f| \sim \frac{n^p}{n^{1/2}\sqrt{2}} \cdot \frac{1}{\sqrt{e}} = \frac{1}{\sqrt{2e}} \cdot \frac{n^p}{n^{1/2}}$$

$$\xrightarrow[n \to +\infty]{} \begin{cases} 0 \ ✓ & p < \frac{1}{2} \\ \frac{1}{\sqrt{2e}} \ ✗ & p = \frac{1}{2} \\ +\infty \ ✗ & p > \frac{1}{2} \end{cases}$$

Conclusione:

$f_n \to f$ unif. in $[0,1]$ $\iff$ $p < \frac{1}{2}$

CONV. UNIF.



Calcolo $\displaystyle\int_0^1 f_n(x)\,dx = \int_0^1 n^p x(1-x^2)^n\,dx$

$$= n^p\left(-\frac{1}{2}\right)\int_0^1 (1-x^2)^n(-2x)\,dx \qquad s = 1-x^2 \quad ds = (-2x)\,dx$$

$$= -\frac{n^p}{2}\int_1^0 s^n\,ds = +\frac{n^p}{2}\left[\frac{s^{n+1}}{n+1}\right]_0^1$$

$$= \frac{n^p}{2}\,\frac{1}{n+1} \sim \frac{1}{2}\,n^{p-1} \longrightarrow \begin{cases} 0 & p<1 \\ \frac{1}{2} & p=1 \\ +\infty & p>1 \end{cases}$$

$$\int_0^1 f(x)\,dx = 0$$

$$\lim_n \int_0^1 f_n(x)\,dx \overset{?}{=} \int_0^1 f(x)\,dx$$