3 ottobre 2024

Verifico che l'applicazione $x = (x_1, \dots, x_n)$

$$x \in \mathbb{R}^n \longmapsto \|x\|_1 := \sum_{i=1}^{n} |x_i|$$

è una norma.

(N1): ✓

$$\|x\|_1 = 0 \overset{def}{\iff} \sum_{i=1}^{n} \underbrace{|x_i|}_{\geq 0} = 0$$

$$\iff \forall i \in \{1, \dots, n\} : \quad |x_i| = 0$$

$$\iff \forall i \in \{1, \dots, n\} : \quad x_i = 0 \iff x = 0$$

(N2): ✓

$\lambda x = (\lambda x_1, \dots, \lambda x_n)$

$$\|\lambda x\|_1 = \sum_{i=1}^{n} |\lambda x_i| = \sum_{i=1}^{n} |\lambda| \, |x_i|$$

$$= |\lambda| \sum_{i=1}^{n} |x_i| = |\lambda| \, \|x\|_1$$

(N3): ✓

$x + y = (x_1 + y_1, \dots, x_n + y_n)$

$$\|x + y\|_1 = \sum_{i=1}^{n} \underbrace{|x_i + y_i|}_{\leq |x_i| + |y_i|}$$

$$\leq \sum_{i=1}^{n} (|x_i| + |y_i|) = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$$

$$= \|x\|_1 + \|y\|_1 \qquad \square$$

Verifico che l'applicazione

$$x \in \mathbb{R}^n \longmapsto \|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$$

è una norma.

(N1) ✓

$$\|x\|_\infty = 0 \overset{def}{\iff} \max_{1 \leq i \leq n} |x_i| = 0 \iff \forall i \in \{1, \dots, n\} : \cancel{|}x_i\cancel{|} = 0$$

$$\iff x = 0$$

(N2) $\|\lambda x\|_\infty \overset{def}{=} \max_{1\le i\le n} |\lambda x_i| = \max_{1\le i\le n} \underbrace{|\lambda|}_{\geq 0} |x_i|$

$= |\lambda| \max_{1\le i\le n} |x_i| = |\lambda| \, \|x\|_\infty$

$|x_i| \le \max_{1\le j\le n} |x_j|$

(N3) $\forall i \in \{1,\dots,n\}: \quad |x_i+y_i| \le |x_i| + |y_i| \le \|x\|_\infty + \|y\|_\infty$

$\Rightarrow \quad \max_{1\le i\le n} |x_i+y_i| \le \|x\|_\infty + \|y\|_\infty$

$\Leftrightarrow \quad \|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$ $\square$

$\|\cdot\|_*$, $\|\cdot\|_\square$ siano norme su uno stesso spazio vettoriale X

Si dicono equivalenti se esistono $a, b \in \mathbb{R}_+^*$ t.c.

$\forall x \in X: \quad a\|x\|_* \le \|x\|_\square \le b\|x\|_*$

Considero le tre norme definite in $\mathbb{R}^n$.

Ricordo che $\forall x \in \mathbb{R}^n$:

$|x_k| \le \|x\|_{\mathbb{R}^n} \le \sum_{i=1}^n |x_i|$

$\forall k$

$\forall x \in \mathbb{R}^n$:

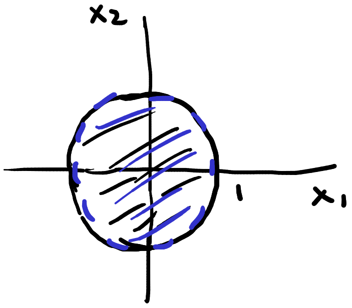
$\|x\|_\infty \le \|x\|_{\mathbb{R}^n} \le \|x\|_1 \le n\|x\|_\infty \le n\|x\|_{\mathbb{R}^n} \le n\|x\|_1$

$\le \dots$

Rappresento in $\mathbb{R}^2$ l'intorno sferico di centro $(0,0)$ e raggio 1 rispetto alle distanze indotte dalle tre norme.

$\|\cdot\|_{\mathbb{R}^2}$ $\quad B_1(0,0) = \{(x_1,x_2) \mid \|(x_1,x_2)\|_{\mathbb{R}^2} < 1\}$

$= \{(x_1,x_2) \mid \sqrt{x_1^2 + x_2^2} < 1\}$



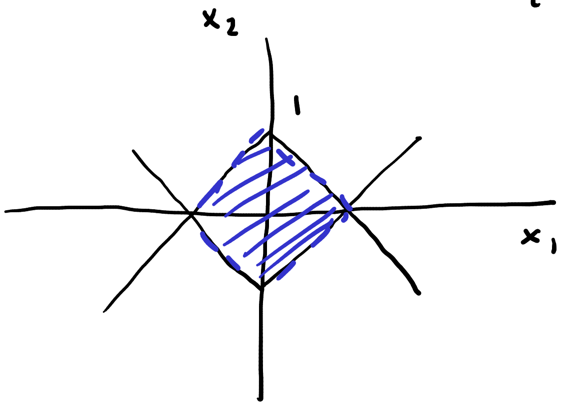
$$= \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$$

$\|\cdot\|_1$ $\qquad B_1(0,0) = \{(x_1, x_2) \mid \|(x_1, x_2)\|_1 < 1\}$

$$= \{(x_1, x_2) \mid |x_1| + |x_2| < 1\}$$

$$|x_2| < 1 - |x_1|$$

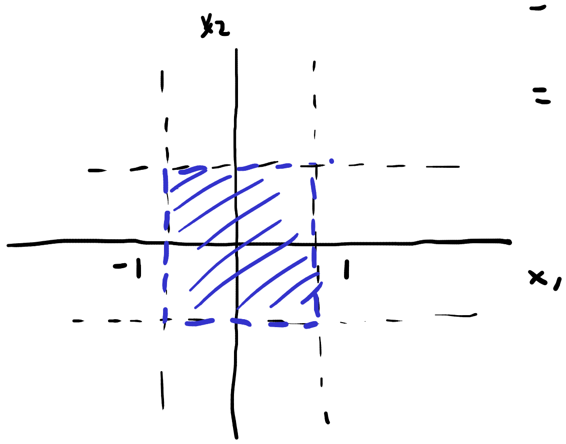
$$-(1-|x_1|) < x_2 < 1 - |x_1|$$

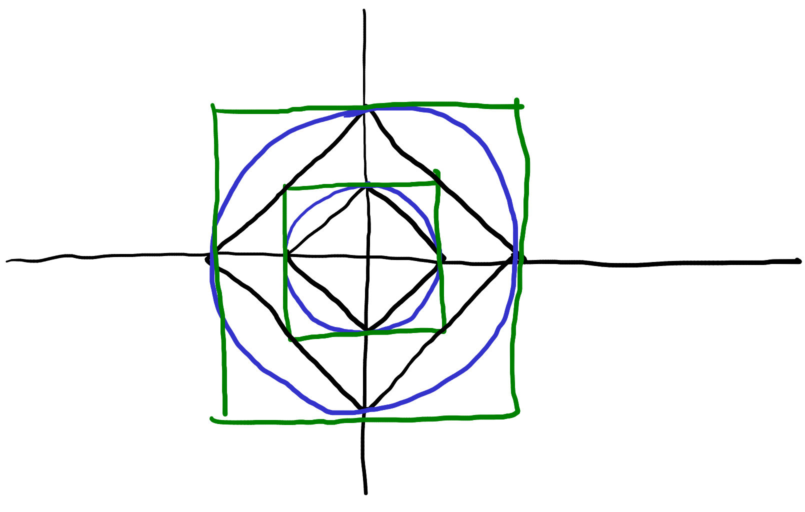


$\|\cdot\|_\infty$ $\qquad B_1(0,0) = \{(x_1, x_2) \mid \|(x_1, x_2)\|_\infty < 1\}$

$$= \{(x_1, x_2) \mid \max\{|x_1|, |x_2|\} < 1\}$$

$$= \{(x_1, x_2) \mid |x_1| < 1 \text{ e } |x_2| < 1\}$$





Illustro lo "scioglilingua":

Banach $(Y, \|\cdot\|_Y) \rightsquigarrow (Y, d_Y)$ completo (metrica indotta da $\|\cdot\|_Y$)

$\forall\, y_1, y_2 \in Y$:

$$d_Y(y_1, y_2) \overset{def}{=} \|y_1 - y_2\|_Y$$

norma del sup associata a $\|\cdot\|_Y$:

$(B(X,Y), \|\cdot\|_\infty)$ Banach

$\forall\, h \in B(X,Y)$:

$$\|h\|_\infty \overset{def}{=} \sup_{x \in X} \|h(x)\|_Y$$

metrica del sup. associata a d_Y:

$(B(X,Y), d_\infty)$ completo

$\forall\, f, g \in B(X,Y)$,

$$d_\infty(f,g) \overset{def}{=} \sup_{x \in X} d_Y(f(x), g(x)) = \sup_{x \in X} \|f(x) - g(x)\|_Y$$

$(B(X,Y), \tilde{d})$ metrica indotta da $\|\cdot\|_\infty$

$\forall\, f, g \in B(X,Y)$:

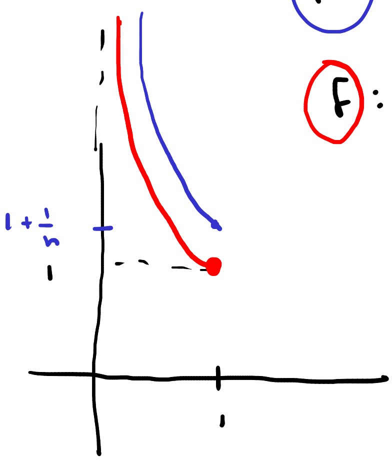
$$\tilde{d}(f,g) \overset{def}{=} \|f - g\|_\infty = \sup_{x \in X} \|(f-g)(x)\|_Y$$

$$\sup_{x \in X} \|f(x) - g(x)\|_Y = \sup_{x \in X} \|(f-g)(x)\|_Y$$

Es:

$\forall n \geq 1$: $f_n : (0,1] \to \mathbb{R}$ t.c. $f_n(x) = \dfrac{1}{x} + \dfrac{1}{n}$

$f : (0,1] \to \mathbb{R}$ t.c. $f(x) = \dfrac{1}{x}$



f_n, f divergono positivamente per $x \to 0^+$

$\Rightarrow$ non sono limitate

Però: $\sup_{x\in(0,1]} |f_n(x) - f(x)| = \sup_{x\in(0,1]} \left| \cancel{\frac{1}{x}} + \frac{1}{n} - \cancel{\frac{1}{x}} \right| = \frac{1}{n}$

$$\Rightarrow \lim_{n\to+\infty} \sup_{x\in(0,1]} |f_n(x) - f(x)| = \lim_{n\to+\infty} \frac{1}{n} = 0$$

Esempi (successioni che convergono puntualmente e non uniformemente)

- $f_n(x) = x^n \quad x \in [0,1] \quad (n \in \mathbb{N}^*)$

Fisso $x \in [0,1]$:

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} x^n = \begin{cases} 0 & x\in[0,1) \\ 1 & x = 1 \end{cases}$$

La funzione $f: [0,1] \to \mathbb{R}$ t.c.

$$f(x) = \begin{cases} 0 & x\in[0,1) \\ 1 & x = 1 \end{cases}$$

è la funzione limite puntuale di (f_n).

Verifico se f è anche limite uniforme.

Fisso n e calcolo

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \max\left\{ \sup_{x\in[0,1)} |f_n(x)-f(x)|, \underbrace{|\overset{1}{f_n(1)} - \overset{1}{f(1)}|}_{=0} \right\}$$

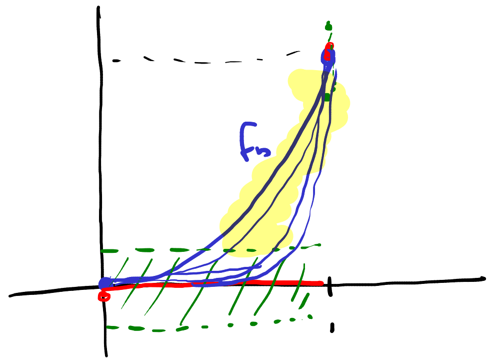
$$= \sup_{x\in[0,1)} |f_n(x) - \underbrace{f(x)}_{\equiv 0}| = \sup_{x\in[0,1)} |f_n(x)|$$

$$= \sup_{x\in[0,1)} x^n = 1$$

Quindi: $\forall n \quad \sup_{x\in[0,1]} |f_n(x) - f(x)| = 1$

$$\Rightarrow \lim_{n\to+\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| = 1 \neq 0$$

Conclusione: (f_n) non converge a f uniformemente in $[0,1]$.



- $f_n(x) = \dfrac{e^{nx}}{e^{nx}+1}$ $\quad x \in \mathbb{R} \quad (n \in \mathbb{N}^*)$

oss: $$\lim_{n\to+\infty} e^{nx} = \begin{cases} 0 & x<0 \\ 1 & x=0 \\ +\infty & x>0 \end{cases}$$

Fissato $x \in \mathbb{R}$:

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{e^{nx}}{e^{nx}+1} = \begin{cases} 0 & x<0 \\ \frac{1}{2} & x=0 \\ 1 & x>0 \end{cases}$$

Dunque:

(f_n) converge (in $\mathbb{R}$) alla funzione $f: \mathbb{R} \to \mathbb{R}$ t.c.

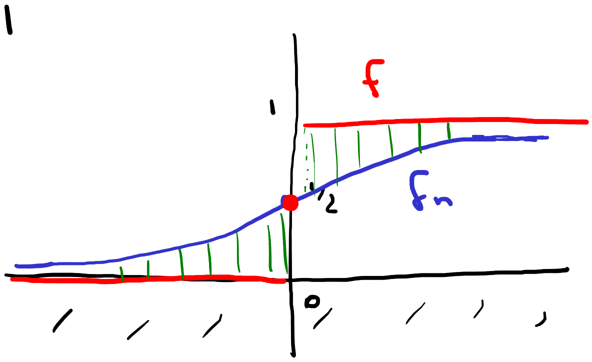
$$f(x) = \begin{cases} 0 & x<0 \\ 1/2 & x=0 \\ 1 & x>0 \end{cases}$$

Verifico se (f_n) converge a f uniformemente in $\mathbb{R}$.
Fisso $n \in \mathbb{N}^*$ e calcolo

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)|$$

Traccio il grafico di f_n

$f_n(x) > 0 \quad \forall x$



$f_n(0) = 1/2$

$$\lim_{x\to+\infty} f_n(x) = \lim_{x\to+\infty} \frac{e^{nx}}{e^{nx}+1} = 1$$

$$\lim_{x\to-\infty} f_n(x) = \lim_{x\to-\infty} \frac{e^{nx}}{e^{nx}+1} = 0$$

Osservo che:

$$f_n(x) = \frac{e^{nx}+1-1}{e^{nx}+1} = 1 - \underbrace{\frac{1}{\underbrace{e^{nx}}_{\text{cresc.}}+1}}_{\text{decresc.}} \quad \text{cresc.}$$

crescente!

Dai grafici:

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \frac{1}{2} \qquad \forall n$$

$$\Rightarrow \lim_{n\to+\infty} \sup_{\mathbb{R}} |f_n - f| = \frac{1}{2} \neq 0$$

Dunque: (f_n) non conv. unif. a f in $\mathbb{R}$.

- $f_n(x) = n x e^{-nx^2} \qquad x \in \mathbb{R} \qquad (n \geq 1)$

$\forall n$: $\quad f_n(0) = 0 \quad \Rightarrow \quad \lim_{n\to+\infty} f_n(0) = 0 \quad (=: f(0))$

Se $x \neq 0$: $(x^2 > 0)$

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \underbrace{n}_{\to+\infty} x \underbrace{e^{\overbrace{-nx^2}^{\to-\infty}}}_{\to 0} = 0 \quad (=: f(x))$$

Quindi: (f_n) conv. punt. in $\mathbb{R}$ alla funzione $f: \mathbb{R}\to\mathbb{R}$ t.c. $f(x) = 0 \quad \forall x \in \mathbb{R}$

(è continua)

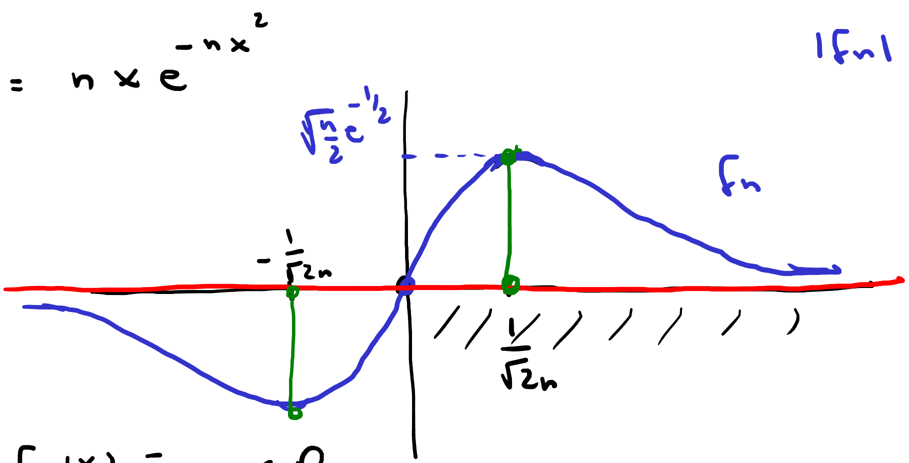
Verifico se la convergenza è uniforme.

Fisso $n \geq 1$. Calcolo

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \sup_{x \in \mathbb{R}} |f_n(x)| = \sup_{x \in \mathbb{R}_+} f_n(x)$$

$\uparrow$ $|f_n|$ è pari

$f_n(x) = n x e^{-nx^2}$



$\lim_{x \to +\infty} f_n(x) = \ldots = 0$

$$f_n'(x) = n \left(e^{-nx^2} + x e^{-nx^2} (-2nx) \right)$$

$$= \underbrace{n e^{-nx^2}}_{>0} (1 - 2nx^2) \geq 0$$

$$\iff x^2 \leq \frac{1}{2n} \iff |x| \leq \frac{1}{\sqrt{2n}}$$

Quindi:

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \ldots = \sup_{x \in \mathbb{R}_+} f_n(x) = f_n\left(\frac{1}{\sqrt{2n}}\right)$$

$$\Rightarrow \lim_{n \to +\infty} \sup_{\mathbb{R}} |f_n - f| = \lim_{n \to +\infty} f_n\left(\frac{1}{\sqrt{2n}}\right) =$$

$$= \lim_{n \to +\infty} n \cdot \frac{1}{\sqrt{2n}} e^{-\cancel{n} \cdot \frac{1}{2\cancel{n}}} = +\infty \neq 0$$