14 dicembre 2023

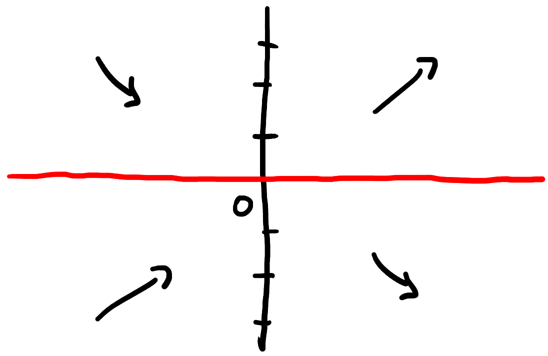
$$y' = ty + \frac{t^3}{y} =: f(t,y)$$

$\mathrm{dom}(f) = \mathbb{R} \times \mathbb{R}^*$

$f \in C^1 \Rightarrow$ TEUL ✓

TEUG × (dominio non di "tipo striscia")

$$f(t,y) = ty\,(\underbrace{1 + \frac{t^2}{y^2}}_{>0})$$



$f(t,y) = 0 \iff t = 0$ zero. clina

$f(t,y) > 0 \iff ty > 0$

Asintoti ?

$$y' = ty + \frac{t^3}{y}$$

diverge per $t \to \pm\infty$ se $y(t) \to \alpha \in \mathbb{R}$

$\Rightarrow$ eventuali soluz. definite in intervalli illimitati superiormente (o inferiormente) divergono

$$y' = ty + \frac{t^3}{y}$$

$$yy' = ty^2 + t^3 \qquad z = y^2 \qquad z > 0$$

$$2yy' = 2ty^2 + 2t^3 \qquad z' = 2yy'$$

$$z' = 2t\,z + 2t^3 = 2t(z + t^2)$$

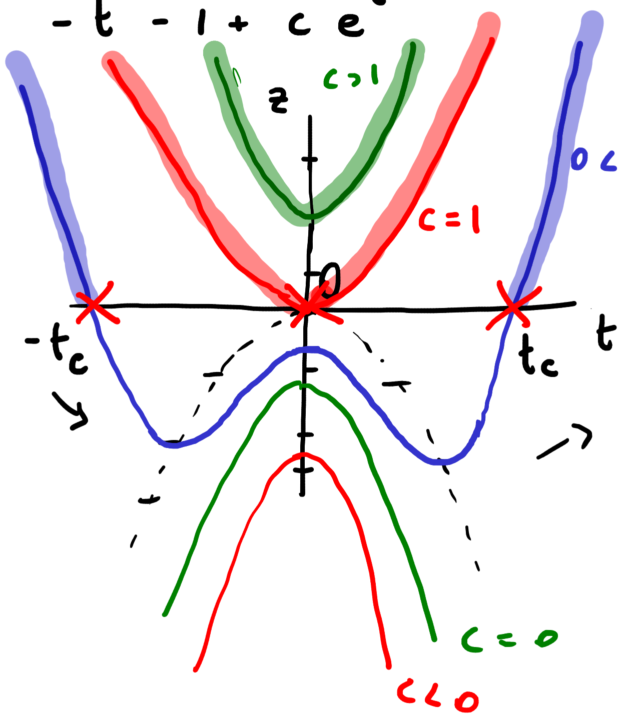
$$z(t) = e^{t^2} \int e^{-t^2}(2t^3)\,dt$$

$$\int e^{-t^2}(2t^3)\,dt = \int \underbrace{(-2t)\,e^{-t^2}}\,(-t^2)\,dt$$

$$= e^{-t^2}(-t^2) - \int e^{-t^2}(-2t)\,dt$$

$$= -t^2 e^{-t^2} - e^{-t^2} + c \qquad c \in \mathbb{R}$$

$$\Rightarrow \quad z_c(t) = -t^2 - 1 + c\,e^{t^2} \qquad c \in \mathbb{R}$$



$c>0$:

$z_c(t) \longrightarrow +\infty$ for $t \to \pm\infty$

$z_c(0) = 0$

$-1 + c = 0 \qquad c = 1$

$$z = y^2 \quad \Rightarrow \quad y = \pm\sqrt{z} \quad (\text{dove è lecito})$$

$c>1$: $\qquad \varphi_c^{\pm}(t) = \pm\sqrt{z_c(t)} \qquad t \in \mathbb{R}$

$c=1 \qquad \varphi_1^{\pm}(t) = \pm\sqrt{z_1(t)} \qquad t \in \,]0, +\infty[$

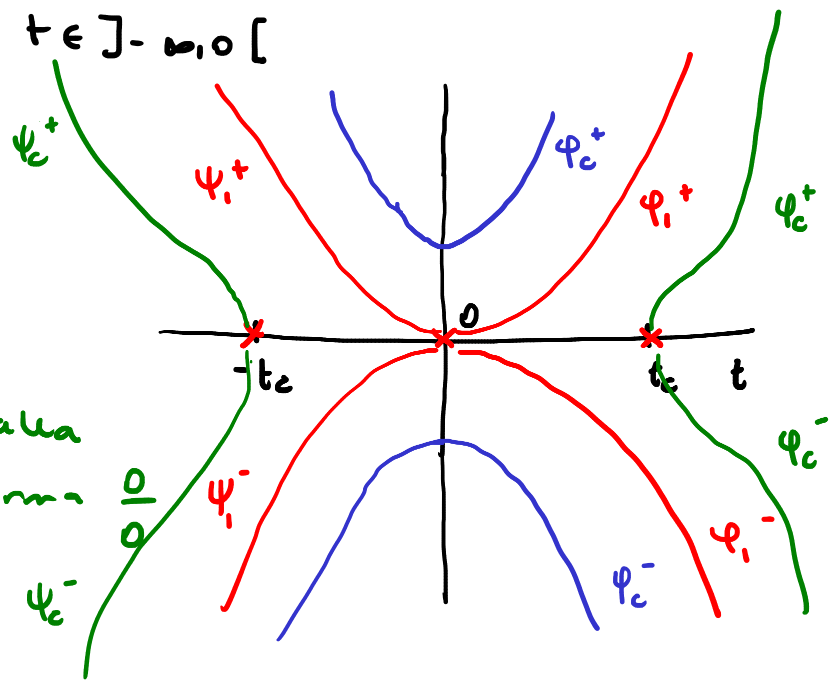
$\qquad\quad\ \varphi_1^{\pm}(t) = \pm\sqrt{z_1(t)} \qquad t \in \,]-\infty, 0[$

$$\varphi_1^{+}(t) \underset{t\to 0}{\longrightarrow} 0$$

Con quale pendenza?

Non posso deciderlo dalla eq. diff., che presenta forma $\frac{0}{0}$

$$z_1(t) = -t^2 - 1 + e^{t^2}$$



per $t \to 0$ $\quad = \cancel{-t^2} \cancel{-1} + \cancel{1} + \cancel{t^2} + \frac{t^4}{2} + o(t^2)$

$\Rightarrow$ per $t \to 0$: $\varphi_1(t) \sim \frac{t^2}{\sqrt{2}}$

$0 < c < 1$: $\quad \varphi_c^{\pm}(t) = \pm\sqrt{z_c(t)} \qquad t \in]t_c, +\infty[$

$\qquad\qquad\quad \varphi_c^{\pm}(t) = \pm\sqrt{z_c(t)} \qquad t \in]-\infty, t_c[$

Per $t \to t_c^+$: $\varphi_c^+(t) \to 0$

Dall'equaz. diff: $\quad t_c > 0$

$$\varphi_c^{+\prime}(t) = \underbrace{\underbrace{t}_{\downarrow\, t_c}\ \underbrace{\varphi_c^+(t)}_{\downarrow\, 0}}_{\downarrow\, 0} + \frac{t^3}{\underbrace{\varphi_c^+(t)}_{\to 0^+}} \longrightarrow +\infty$$

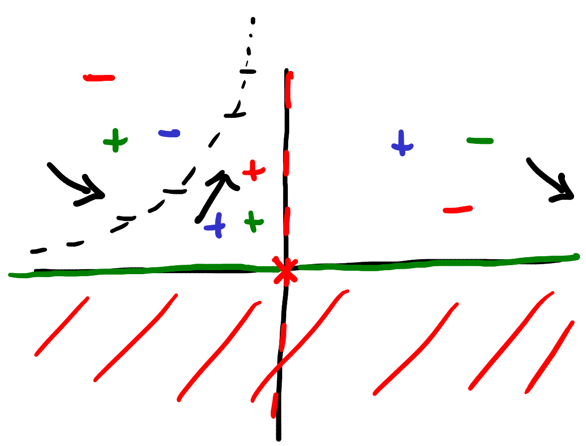
$$y' = -\frac{6}{t}y - 2t^2 y^{3/2} =: f(t,y)$$

$\mathrm{dom}(f) = \mathbb{R}^* \times [0, +\infty[$

$f \in C^1$ nel dominio

$\Rightarrow$ TEUL applicabile $\forall (t_0, x_0) \in \mathbb{R}^* \times]0, +\infty[$

$$f(t,y) = \left(-\frac{2y}{t}\right)\left(3 + t^3 y^{1/2}\right)$$



$f(t,y) = 0 \iff$

$y = 0$ (sol. cost.) due

oppure $\quad 3 + t^3 y^{1/2} = 0$

$t > 0: \quad 3 + t^3 y^{1/2} > 0$

$t < 0: \quad 3 + t^3 y^{1/2} \geq 0 \iff t^3 y^{1/2} \geq -3$

$\iff y^{1/2} \leq -\frac{3}{t^3}$

$\iff y \leq \frac{9}{t^6}$ $\qquad y = \frac{9}{t^6}, \ t<0$

zero-clina

Asintoti ?

$$y' = -\underbrace{\frac{6}{t}\,y}_{\alpha \in \mathbb{R}} - 2\underbrace{t^2}_{+\infty}\,\underbrace{y^{3/2}}_{\alpha^{3/2}}$$

0

$\alpha \neq 0$: y' diverge !!

$\Rightarrow \alpha = 0$ unico possibile valore per as. orizzontale

$$y^{-\frac{3}{2}} y' = -\frac{6}{t} y^{-\frac{1}{2}} - 2t^2$$

$z = y^{-\frac{1}{2}}$ $\qquad z > 0$

$z' = -\frac{1}{2} y^{-3/2} y'$

$$-\frac{1}{2} y^{-\frac{3}{2}} y' = \frac{3}{t} y^{-\frac{1}{2}} + t^2$$

$$z' = \frac{3}{t} z + t^2$$

$$z(t) = e^{3\ln|t|} \int e^{-3\ln|t|}\, t^2\, dt$$

$$= |t|^3 \int \frac{1}{|t|^3} t^2\, dt$$

$|t|^3 = t^3 \cdot (\mathrm{sign}(t))^3$

$= t^3\, \mathrm{sign}(t)$

$$= |t|^3 \int \frac{\mathrm{sign}(t)}{t^3}\, t^2\, dt$$

$$= |t|^3\, \mathrm{sign}(t) \int \frac{1}{t^3} t^2\, dt = t^3 \int \frac{1}{t}\, dt$$

$$= t^3 (\ln|t| + c) \qquad c \in \mathbb{R}$$

Condizione : $z_c(t) > 0$

$$\ln|t| + c > 0 \iff \ln|t| > -c \iff |t| > e^{-c}$$

		$-e^{c}$		0		e^{-c}	
t^3	$-$		$-$		$+$		$+$
$\ln\|t\|+c$	$+$		$-$		$-$		$+$
$z_c(t)$	$-$		$+$		$-$		$+$

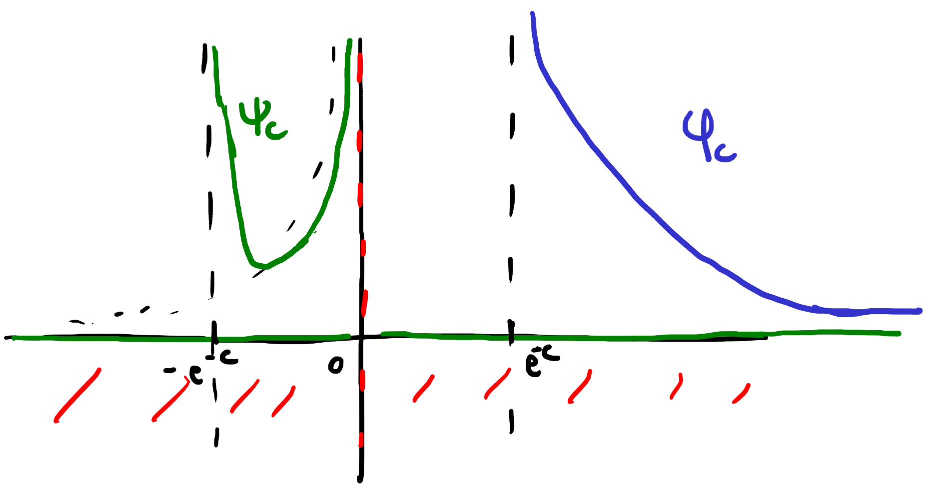
$$z = y^{-\frac{1}{2}} \qquad y = z^{-2}$$

$\forall c \in \mathbb{R}$: due soluzioni:

- $\varphi_c(t) = \dfrac{1}{(z_c(t))^2} = \dfrac{1}{t^6(\ln|t|+c)^2} \qquad t \in \,]e^{-c}, +\infty[$

- $\psi_c(t) = \dfrac{1}{(z_c(t))^2} \qquad t \in \,]-e^{-c}, 0[$

$$\psi_c(t) \xrightarrow[t \to 0^-]{} +\infty$$



$j \in \{1,\dots,n-1\}$, $\quad f_j(t,x_1,\dots,x_n) = x_{j+1}$

$$|f_j(t,\underbrace{x_1,\dots,x_n}_{x}) - f_j(t,\underbrace{y_1,\dots,y_n}_{y})| = |x_{j+1} - y_{j+1}|$$
$$\le \|x-y\|$$
$$= 1\cdot \|x-y\|$$

$\Rightarrow$ f_j è lipschitziana con $L = 1$

$f : I \times \mathbb{R}^n \to \mathbb{R}$ è sublineare se $\forall K \subset I$ compatto esistono $p, q_1, \dots, q_n \in \mathbb{R}_+$ t.c. $\forall (t, x_1, \dots, x_n) \in K \times \mathbb{R}^n$

$$|f(t,x_1,\dots,x_n)| \le p + q_1|x_1| + q_2|x_2| + \dots + q_n|x_n|$$

Oss: $\mathbf{f}$ è sublineare $\Leftrightarrow$ lo sono tutte le sue componenti

$(\|\mathbf{f}\| \le |f_1| + \dots + |f_n|)$

$f \in C^k \Rightarrow \mathbf{f} \in C^k \Rightarrow$ le sol. di (PV) sono di classe C^{k+1}

$$\underset{\text{PV}}{\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}} \longleftrightarrow \underset{\text{PS}}{\begin{pmatrix} \varphi \\ \varphi' \\ \vdots \\ \varphi^{(n-1)} \end{pmatrix}}$$

C^{k+1}

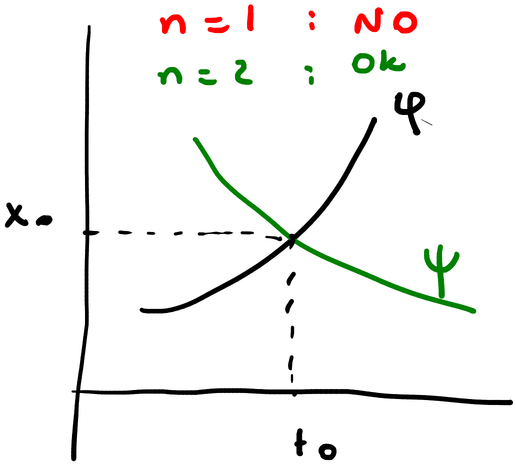
$$\varphi^{(n-1)} \in C^{k+1} \Rightarrow \varphi^{(n-2)} \in C^{k+2} \Rightarrow \varphi^{(n-3)} \in C^{k+3}$$

$$\cdots \Rightarrow \varphi^{(n-n)} \in C^{k+n}$$

$$\varphi^{(n-n)} = \varphi^{(0)} = \varphi$$

n = 1 : NO

n = 2 : ok



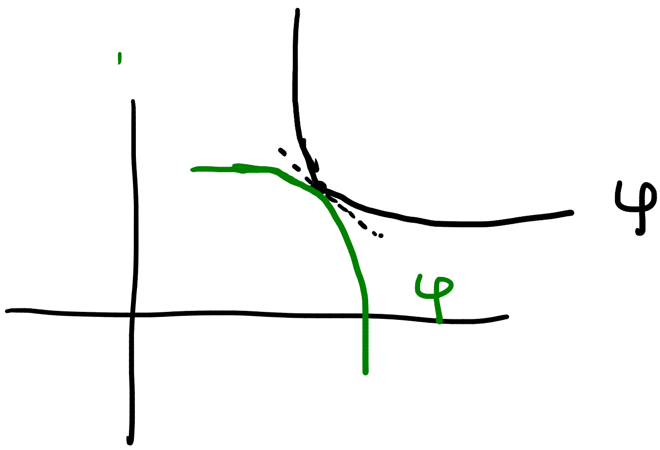
$$\begin{cases} y'' = f(t, y, y') \\ y(t_0) = x_0 \\ y'(t_0) = x_1 \end{cases}$$

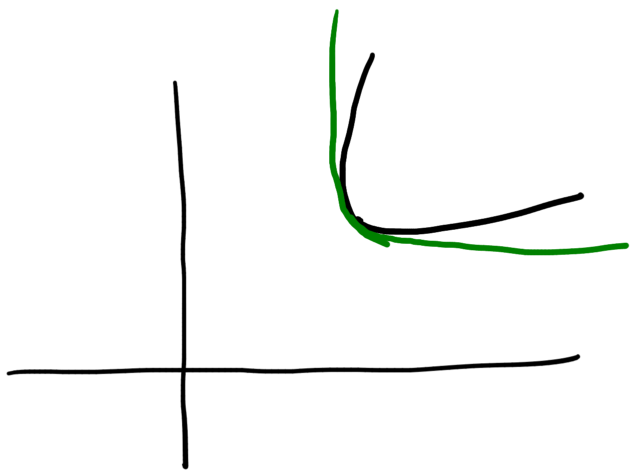
$\varphi(t_0) = x_0 \quad \psi(t_0)$

n = 2 : NO!

$\varphi'(t_0) > 0 \qquad \psi'(t_0) < 0$

$\neq$





$$y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \dots + a_1(t)\, y' + a_0(t)\, y = b(t)$$

$$y^{(n)} = b(t) - a_0(t)\, y - a_1(t)\, y' - \dots - a_{n-1}(t)\, y^{(n-1)}$$

$$f(t, x_1, \dots, x_n) := b(t) - a_0(t)\, x_1 - a_1(t)\, x_2 - \dots - a_{n-1}(t)\, x_n$$

Suppongo $a_0, a_1, \dots, a_{n-1}, b \in C(I, \mathbb{R})$

- $\mathrm{dom}(f) = I \times \mathbb{R}^n$
- $a_0, \dots, a_{n-1}, b$ continui $\Rightarrow$ f continua $(= C^0)$

  ($\overset{\text{regol.}}{\Rightarrow}$ soluz. di classe C^n)

- $\forall j \in \{1, \dots, n\}$:

$$\frac{\partial f}{\partial x_j}(t, x_1, \dots, x_n) = a_{j-1}(t)$$

continua in I (funz. della var. t)

$\Rightarrow$ continua in $I \times \mathbb{R}^n$ (funz. di $(t, x_1, \dots, x_n)$)

$\overset{\text{cond. suff.}}{\Rightarrow}$ f è loc. lipsc. rispetto a x unif. in t

- $\forall (t, x_1, \dots, x_n) \in I \times \mathbb{R}^n$:

$$|f(t, x_1, \dots, x_n)| = |b(t) - a_0(t)x_1 - \dots - a_{n-1}(t)x_n|$$

$$\le |b(t)| + |a_0(t)||x_1| + \dots + |a_{n-1}(t)||x_n|$$

Fisso $K \subseteq I$ compatto

$\forall (t, x_1, \dots, x_n) \in K \times \mathbb{R}^n$:

$$|f(t, x_1, \dots, x_n)| \le \underbrace{\max_{t \in K} |b(t)|}_{=: p} + \underbrace{\max_{t \in K} |a_0(t)|}_{=: q_1} |x_1| + \dots + \underbrace{\max_{t \in K} |a_{n-1}(t)|}_{=: q_n} |x_n|$$

esistono per teor. Weierstrass

$\Rightarrow$ f è sublineare

$\boxed{\square}$

Quanto segue riguarda il "seminario" sui sistemi differenziali lineari. E' stato improvvisato in aula; appena possibile lo inserirò nelle "slide".

$$y' = A(t)\,y + B(t)$$

Se $(\varphi_1, \dots, \varphi_n)$ sol. di $y' = A(t)\,y$

Int. gen. di omog:

$$c_1\varphi_1 + \dots + c_n\varphi_n \qquad c_1, \dots, c_n \in \mathbb{R}$$

$$= \begin{pmatrix} \varphi_1 & \cdots & \varphi_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = W(t)\,C \qquad C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^N$$

Cerco $\bar{\varphi}(t) = W(t)\,C(t)$ $\qquad C = C(t)$ funz. vettoriale

$$\boxed{\bar{\varphi}'(t) = W'(t)\,C(t) + W(t)\,C'(t)}$$

$$\begin{pmatrix} \varphi_{11} c_1 + \varphi_{21} c_2 + \dots + \varphi_{n1} c_n \\ \vdots \\ \vdots \end{pmatrix} = W(t)\,C(t)$$

$$\begin{pmatrix} \varphi_{11}' c_1 + \varphi_{11} c_1' + \varphi_{21}' c_2 + \varphi_{21} c_2' + \dots + \varphi_{n1}' c_n + \varphi_{n1} c_n' \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} \varphi_{11}' c_1 + \varphi_{21}' c_2 + \dots + \varphi_{n1}' c_n \\ \vdots \end{pmatrix} + \begin{pmatrix} \varphi_{11} c_1' + \varphi_{21} c_2' + \dots + \varphi_{n1} c_n' \end{pmatrix}$$

$w'(t)\,c(t) + w(t)\,c'(t)$

$$\underbrace{w'(t)\cancel{c(t)} + w(t)\,c'(t)}^{\bar{\varphi}'(t)} = \underbrace{A(t)\,\cancel{w(t)}c(t)}^{A(t)\,\bar{\varphi}(t)} + B(t)$$

$w'(t) = A(t)\,w(t)$ eq. matric.

$\varphi' = A(t)\,\varphi$ eq. vect.

$\bar{\varphi}$ ris. eq. completa $\iff$

$\Downarrow$ $\underbrace{w(t)}_{\text{invertibile}}\,c'(t) = B(t)$ $\iff$

$c'(t) = w(t)^{-1}\,B(t)$

$c(t) = \int w(t)^{-1}\,B(t)\,dt$ ...