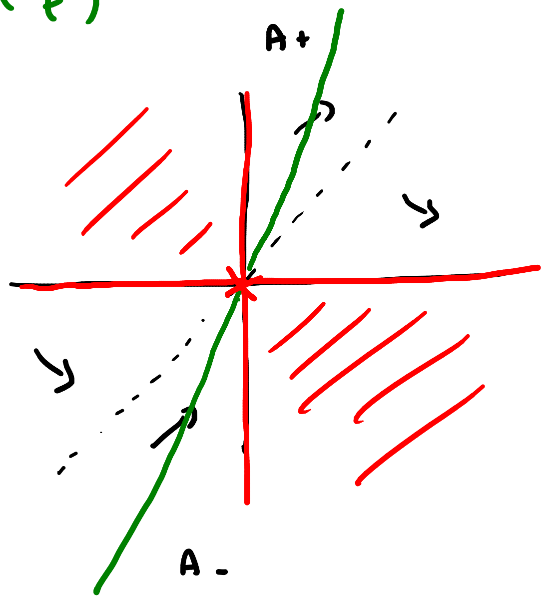
12 dicembre 2023

(1) $y' = \underbrace{\frac{y}{t}\ln\left(\frac{y}{t}\right)}_{=:f(t,y)} = g\left(\frac{y}{t}\right)$ $\qquad g(s) = s\ln(s)$



dom(f) : vedi →

monotonia : "

asintoti : no previsioni

$z := \frac{y}{t}$ $\quad z>0 \qquad y = tz$

$z + tz' = z\ln(z)$

$t z' = z(\ln(z) - 1)$,

(2) $z' = \frac{1}{t} z(\ln(z) - 1)$

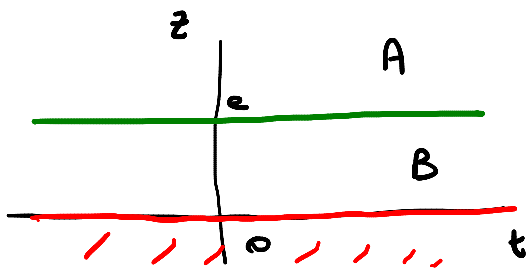
Sol. costante $z = e$ $\Big(\to y(t) = e\,t$ sol. di (1)

$t \in]-\infty, 0[\qquad t \in]0, +\infty[$

Cerco sol. non costanti di (2)

$H'(z) = \dfrac{1}{z(\ln(z)-1)}$, $\quad H(z) = \ln|\ln(z) - 1|$

$G'(t) = \frac{1}{t}$, $\quad G(t) = \ln|t|$



$z = z(t)$ sol. di (2) :

$\ln|\ln(z(t)) - 1| = \ln|t| + c \qquad c \in \mathbb{R}$

(*) $|\ln(z(t)) - 1| = c|t| \qquad c > 0$

Sol. in A : $z(t) > e \iff \frac{y(t)}{t} > e$

$t<0$: $y(t) < et$; $t>0$: $y(t) > et$

(*) $\iff$ $\ln(z(t)) - 1 = c|t|$

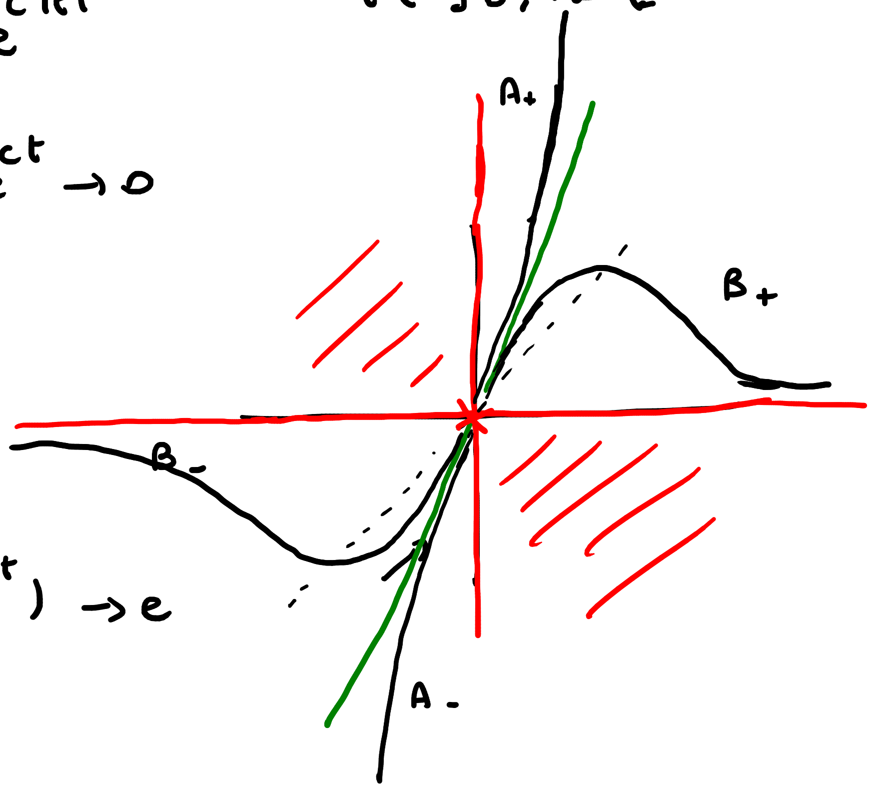
$\ln(z(t)) = 1 + c|t|$

$z(t) = e^{1+c|t|}$

A queste corrispondono sol. di (1) in A^+ e A^-

$y_c(t) = t e^{1+c|t|}$ — $t \in]-\infty, 0[$; $t \in]0, +\infty[$

$= et \cdot e^{c|t|}$



$t \to 0^+$: $y_c(t) = e t e^{ct} \to 0$

$y_c'(t) \to ??$

$y' = \frac{y}{t} \ln\left(\frac{y}{t}\right)$

$y_c'(t) = e^{1+ct} \ln(e^{1+ct}) \to e$

Sol. in B di (2)

(*) $\iff$ $-(\ln(z(t)) - 1) = c|t|$

$\ln(z(t)) - 1 = -c|t|$

$\ln(z(t)) = 1 - c|t|$

$z(t) = e^{1-c|t|}$

A queste corrispondono

$y_c(t) = t e^{1-c|t|}$ — $t \in]-\infty, 0[$; $t \in]0, +\infty[$

Per $t \to 0^+$: $\quad y_c(t) \to 0$

come prima: $y_c'(t) \to e$

$t \to +\infty$: $\quad y_c(t) \to 0$ $\square$

EQUAZIONI DI BERNOULLI

$$y' = \frac{2}{3} y - \frac{4}{9} \frac{t}{y^2} =: f(t,y) \qquad a = -2$$

$\mathrm{dom}(f) = \mathbb{R} \times \mathbb{R}^*$

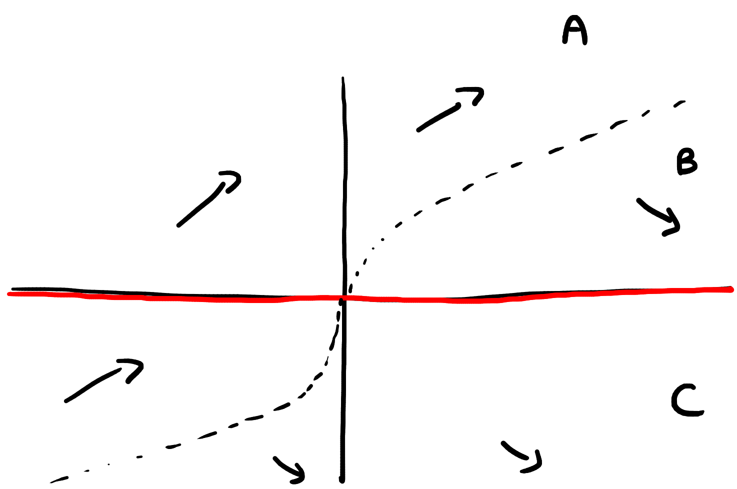
$f \in C^1 \Rightarrow$ TEUL applicabile $\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^*$

(TEUG: no!)

$$f(t,y) = \frac{2}{3}\left(y - \frac{2}{3}\frac{t}{y^2}\right) \overset{>}{=} 0$$

$$\Leftrightarrow \frac{1}{y^2}\left(y^3 - \frac{2}{3}t\right) \overset{>}{=} 0 \quad \Leftrightarrow \quad y^3 - \frac{2}{3}t \overset{>}{=} 0$$

$$\Leftrightarrow \quad y^3 \overset{>}{=} \frac{2}{3}t \quad \Leftrightarrow \quad y \overset{>}{=} \sqrt[3]{\frac{2t}{3}}$$ zero-clina



Asintoti ?

Suppongo $\varphi: [a, +\infty[\to \mathbb{R}$ sol.

$\overset{\text{monot.}}{\Rightarrow} \quad \exists \lim_{t \to +\infty} \varphi(t) = \alpha$

Se $\alpha \in \mathbb{R}$: $\quad \varphi'(t) = \underbrace{\frac{2}{3}\varphi(t)}_{\to \frac{2}{3}\alpha} - \frac{4}{9}\frac{t}{\varphi(t)^2} \quad \to +\infty$

Per qualsiasi $\alpha \in \mathbb{R}$ ho una contraddizione!

Quindi $\alpha \notin \mathbb{R}$

$\Rightarrow$ • le sol. in A o in C : divergono (se esistono)

• le sol. in B : non esistono!

Cioè : le sol. in B non possono essere definite in intervalli illimitati superiormente.

Idem per $t \to -\infty$.

Risolvo l'equazione :

$$y' = \frac{2}{3}y - \frac{4}{9}\frac{t}{y^2}$$

$$y^2 y' = \frac{2}{3}y^3 - \frac{4}{9}t \qquad\qquad z := y^3 \qquad z \neq 0$$

$$3y^2y' = 2y^3 - \frac{4}{3}t \qquad\qquad z' = 3y^2y'$$

$$\odot \quad z' = 2z - \frac{4}{3}t$$

Richiamo: $z' = a(t)\, z + b(t)$ $\qquad A' = a$

$$z(t) = e^{A(t)} \int e^{-A(t)}\, b(t)\, dt$$

$a(t) = 2 \quad \Rightarrow \quad A(t) = 2t$

$$z(t) = e^{2t} \int e^{-2t} \left(-\frac{4}{3}t\right) dt$$

$$= e^{2t} \int (-2)\, e^{-2t} \left(\frac{2}{3}t\right) dt$$

$$= e^{2t} \left[e^{-2t} \cdot \frac{2}{3}t - \int e^{-2t}\, \frac{2}{3}\, dt \right]$$

$$= e^{2t} \left[e^{-2t}\, \frac{2}{3}t + \frac{1}{3} e^{-2t} + c \right] \qquad c \in \mathbb{R}$$

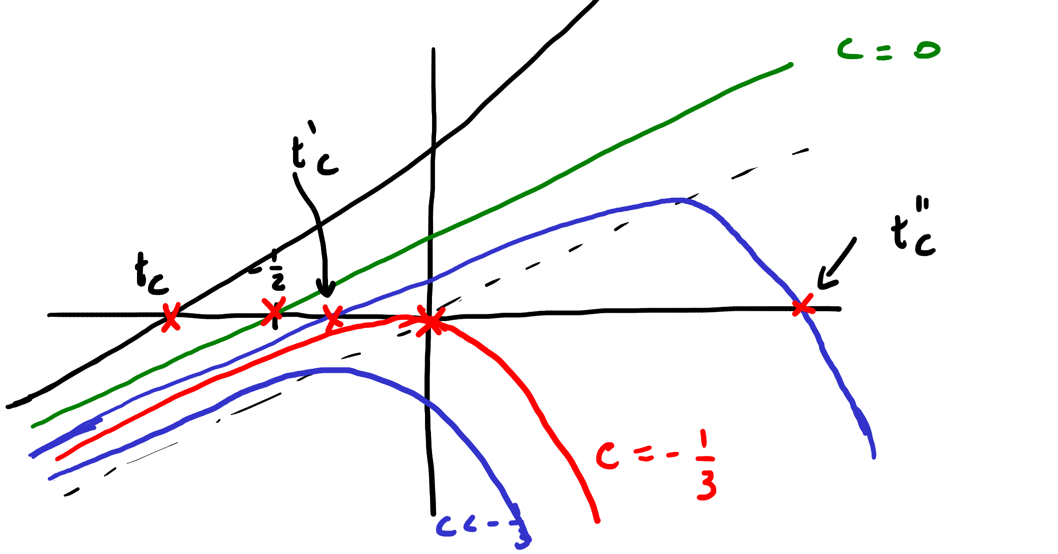
$$\Rightarrow \quad z_c(t) = \frac{2}{3}t + \frac{1}{3} + c\, e^{2t} \qquad c \in \mathbb{R}$$

Per l'eq. data:

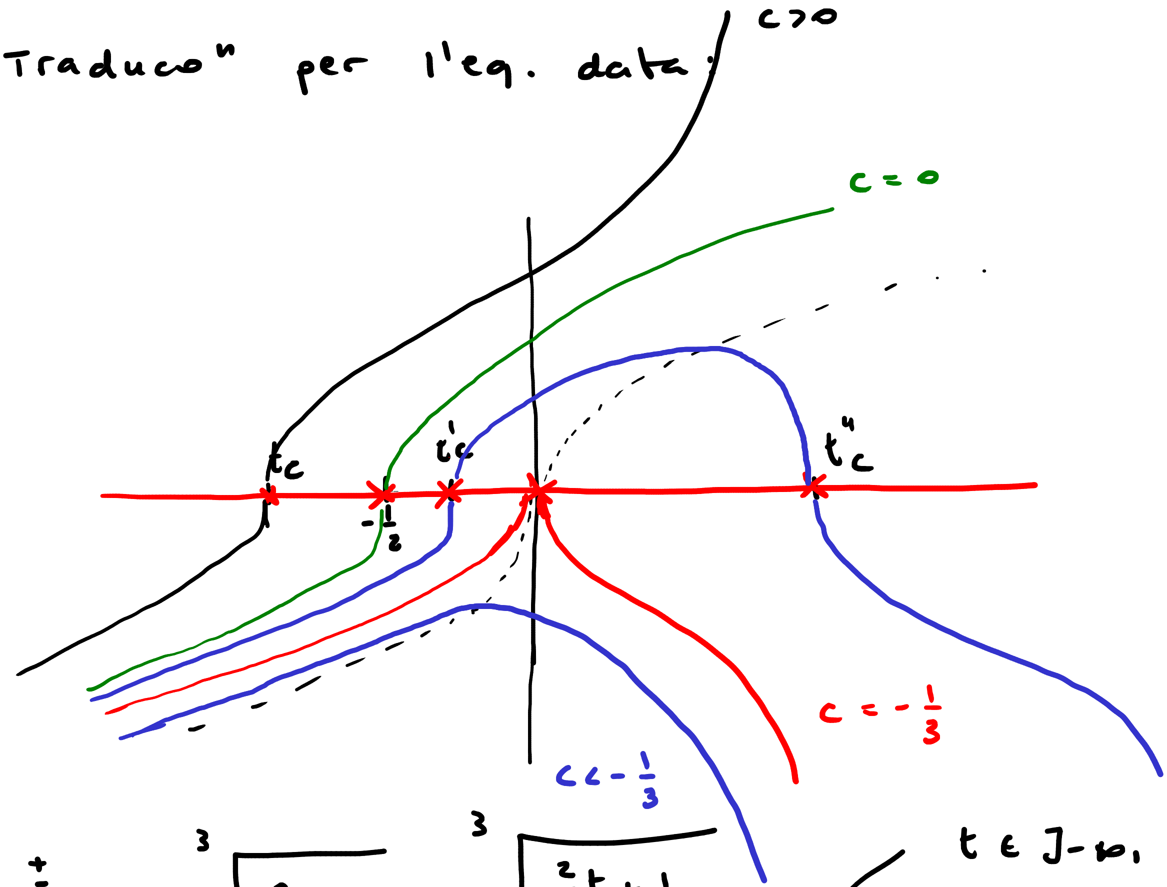
$$y_c(t) = \sqrt[3]{z_c(t)}$$

con condizione $\quad z_c(t) \neq 0$

Rappresento le sol. dell'eq. ausiliaria



"Traduco" per l'eq. data:



$$y_0^{\pm}(t) = \sqrt[3]{z_0(t)} = \sqrt[3]{\frac{2}{3}t + \frac{1}{3}}$$

$t \in]-\infty, -\frac{1}{2}[$

$t \in]-\frac{1}{2}, +\infty[$

$\vdots$

Cerco di capire l'andamento di $z_{-\frac{1}{3}}$ vicino a $t=0$:

$$z_{-\frac{1}{3}}(t) = \frac{2}{3}t + \frac{1}{3} - \frac{1}{3}e^{2t}$$

$$= \cancel{\frac{2t}{3}} + \cancel{\frac{1}{3}} - \frac{1}{3}\left(\cancel{1} + \cancel{2t} + \frac{4t^2}{2} + o(t^2)\right)$$

$$= -\frac{2}{3}t^2 + o(t^2)$$

$$\Rightarrow y_{-\frac{1}{3}}(t) \sim \sqrt[3]{-\frac{2}{3}t^2} \qquad t=0 \text{ punto cuspidale}$$

$$y' = \underbrace{3y - 2ty^{1/3}}_{=: f(t,y)} \qquad \alpha = \frac{1}{3}$$

$dom(f) = \mathbb{R} \times \mathbb{R}$

f continua in $\mathbb{R} \times \mathbb{R}$

f di classe C^1 in $\mathbb{R} \times \mathbb{R}^*$

$\Rightarrow$ TEUL applicabile a PdC con cond. iniziale $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^*$

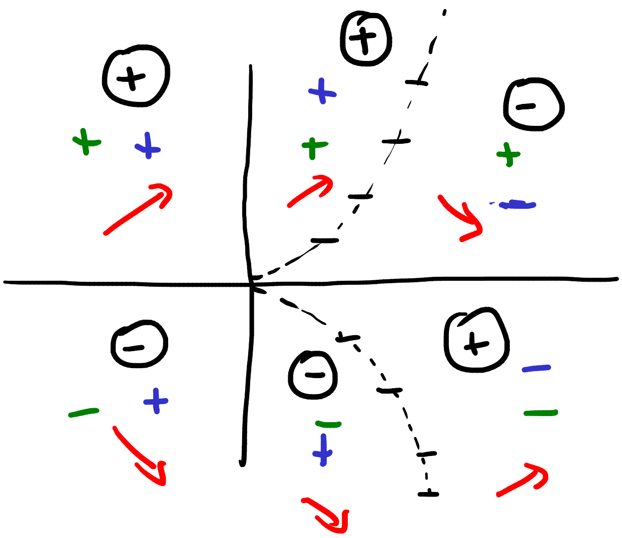
Per cond. iniziale $(t_0, 0)$ non è garantita l'unicità.

$$f(t,y) = 3y - 2ty^{1/3} = y^{1/3}(3y^{2/3} - 2t)$$

$f(t,y) = 0 \iff y = 0$ (sol. costante)

$$3y^{2/3} - 2t \overset{>}{=} 0 \iff y^{2/3} \overset{>}{=} \frac{2}{3}t$$

$$\iff y^2 \overset{>}{=} \left(\frac{2}{3}t\right)^3$$



$t \geq 0$: $y = \pm \left(\frac{2}{3}t\right)^{3/2}$

Disuguaglianza :

vera $\forall t < 0$

e per $t \geq 0$: $|y| > \left(\frac{2}{3}t\right)^{3/2}$

$y' = 3y - 2ty^{1/3}$ ....

Unico possibile asintoto orizzontale: $y = 0$.

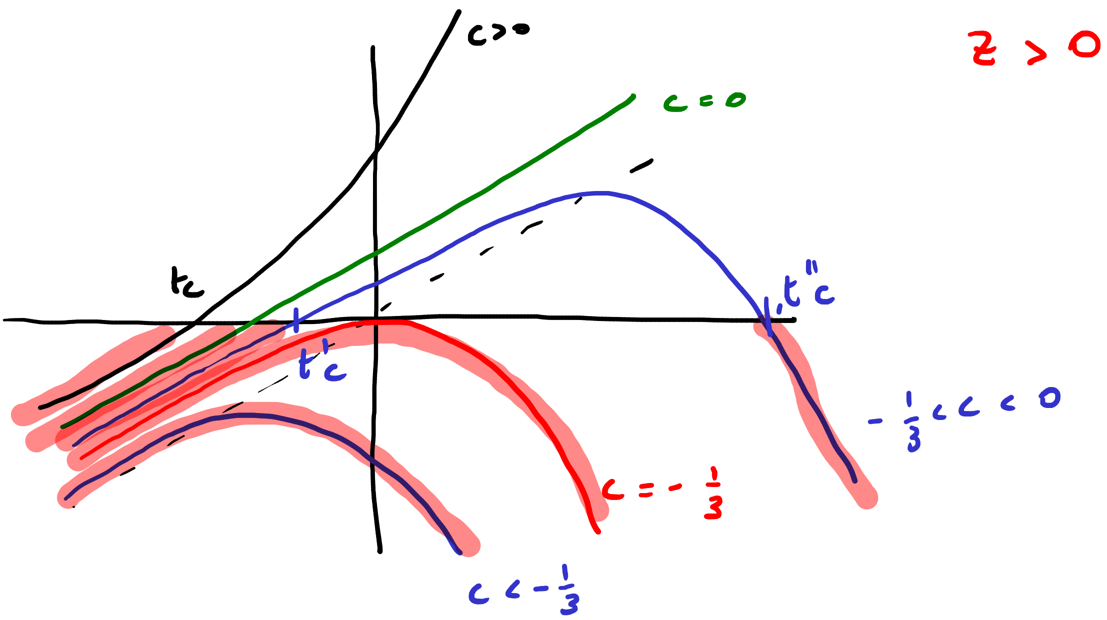
Risolvo l'equazione:

$$y^{-\frac{1}{3}}\, y' = 3\, y^{\frac{2}{3}} - 2t \qquad\qquad z = y^{\frac{2}{3}} \qquad z > 0$$

$$\frac{2}{3}\, y^{-\frac{1}{3}}\, y' = 2\, y^{\frac{2}{3}} - \frac{4}{3}t \qquad\qquad z' = \frac{2}{3}\, y^{-\frac{1}{3}}\, y'$$

$\odot \quad z' = 2z - \frac{4}{3}t$ la stessa eq. ausiliaria dell' es. precedente

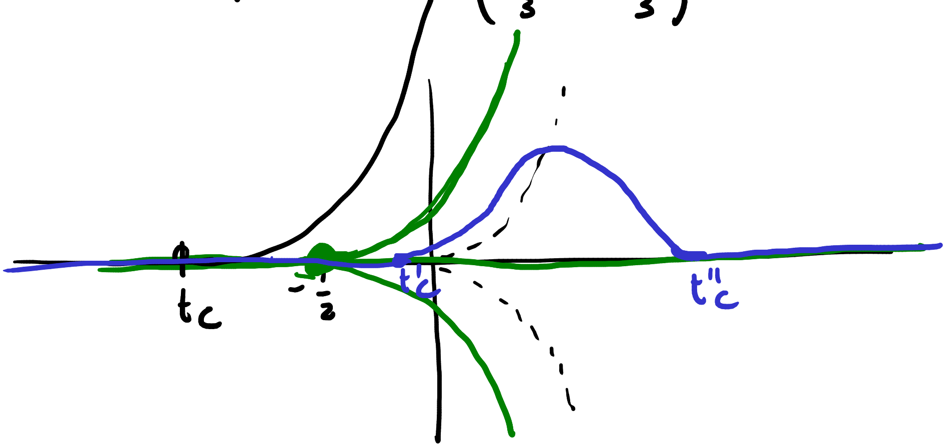
Ricordo le sol. di $\odot$



$$z = y^{\frac{2}{3}} \iff y^2 = z^3 \longrightarrow y = \pm z^{3/2}$$

Per $c \le -\frac{1}{3}$, z_c non "produce" soluzioni dell'eq. data.

$c = 0$: $\quad y_0^{\pm}(t) = \pm\left(\frac{2}{3}t + \frac{1}{3}\right)^{3/2} \qquad t \in \left]-\frac{1}{2}, +\infty\right[$



Osservo che $y_0^+(t) \to 0$ per $t \to -\frac{1}{2}^+$,
quindi posso "incollare" questa soluzione alla sol. costante di valore 0.

(Idem per y_0^-)

Per $c>0$: $y_c^\pm(t) = z_c(t)^{3/2}$, $t \in]t_c, +\infty[$

"incollabile" a $y \equiv 0$. a sin. di t_c

Per $-\frac{1}{3} < c < 0$: $y_c^\pm(t) = z_c(t)^{3/2}$ $t \in]t_c', t_c''[$

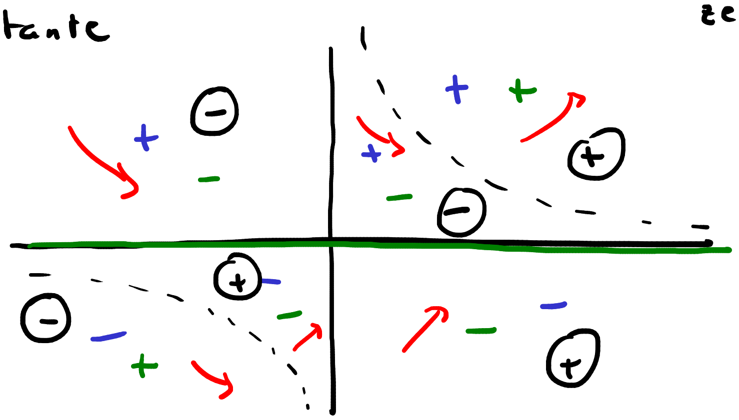
"incollabile" a $y = 0$ a sin. di t_c' e a destra di t_c''.

$$y' = -2y + \frac{4}{3} t y^2 =: f(t,y) \qquad \alpha = 2$$

$\left.\begin{array}{l} dom(f) = \mathbb{R} \times \mathbb{R} \\ f \in C^1 \end{array}\right\} \Rightarrow$ TEUL ok in $\mathbb{R} \times \mathbb{R}$

$$f(t,y) = 2y\left(-1 + \frac{2}{3} t y\right) = 0$$

$\Leftrightarrow \underbrace{y = 0}_{\text{sol. costante}}$ opp. $\frac{2}{3} t y = 1 \Leftrightarrow \underbrace{ty = \frac{3}{2}}_{\text{zero-clino}}$



$$y' = -2y + \frac{4}{3}t\,y^2$$

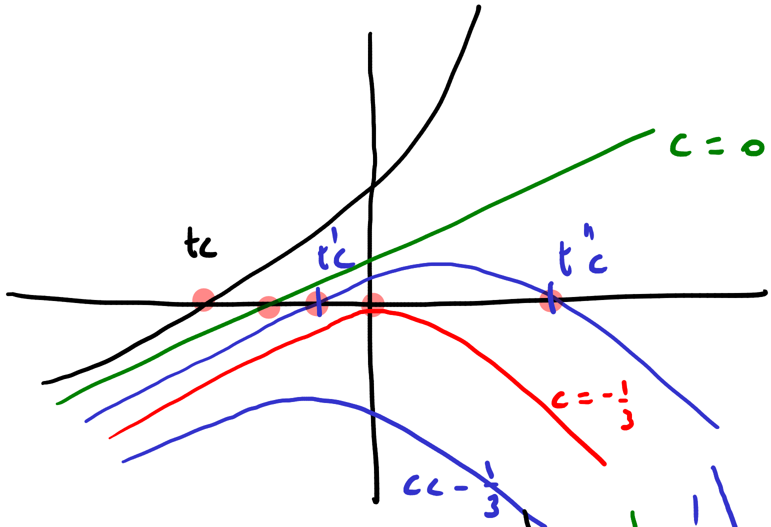
Unico poss. asintoto: $y = 0$

$$y^{-2}y' = -2y^{-1} + \frac{4}{3}t \qquad z = y^{-1} \qquad z \neq 0$$

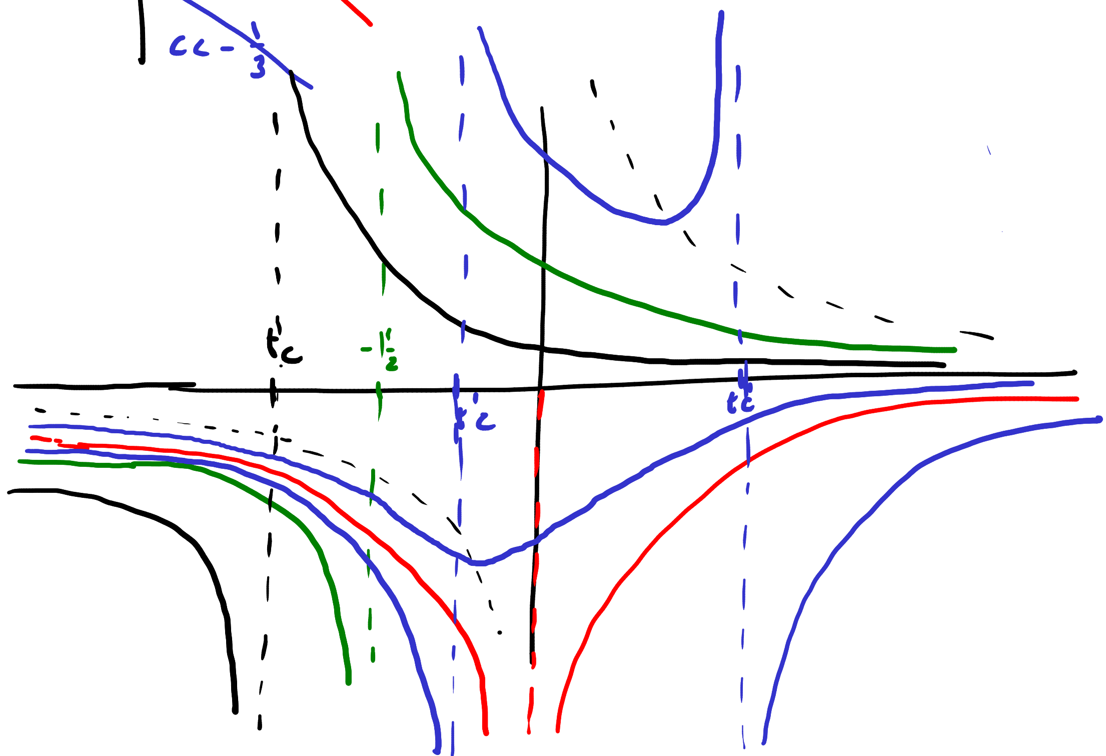
$$z' = -y^{-2}y'$$

$$z' = 2z - \frac{4}{3}t$$

Stessa eq. ausiliaria!



$$y_c(t) = \frac{1}{z_c(t)}$$



Per $c = 0$: due soluzioni con intervalli di def.

$$]-\infty, -\tfrac{1}{2}[, \qquad]-\tfrac{1}{2}, +\infty[$$

Per $c > 0$: due sol. definite in

$$]-\infty, t_c[, \qquad]t_c, +\infty[$$

Per $c = -\frac{1}{3}$: due sol. definite in
$]-\infty, 0[\,, \quad]0, +\infty[$

Per $c < -\frac{1}{3}$: una sol. def. in $\mathbb{R}$

Per $-\frac{1}{3} < c < 0$: tre sol. def. in
$]-\infty, t'_c[\,, \quad]t'_c, t''_c[\,, \quad]t''_c, +\infty[$

$$y' = t(\underbrace{y^3 - y}_{\text{a var. sep.}}) = -\underbrace{ty + t}_{\text{Bernoulli}}y^3 =: f(t,y)$$

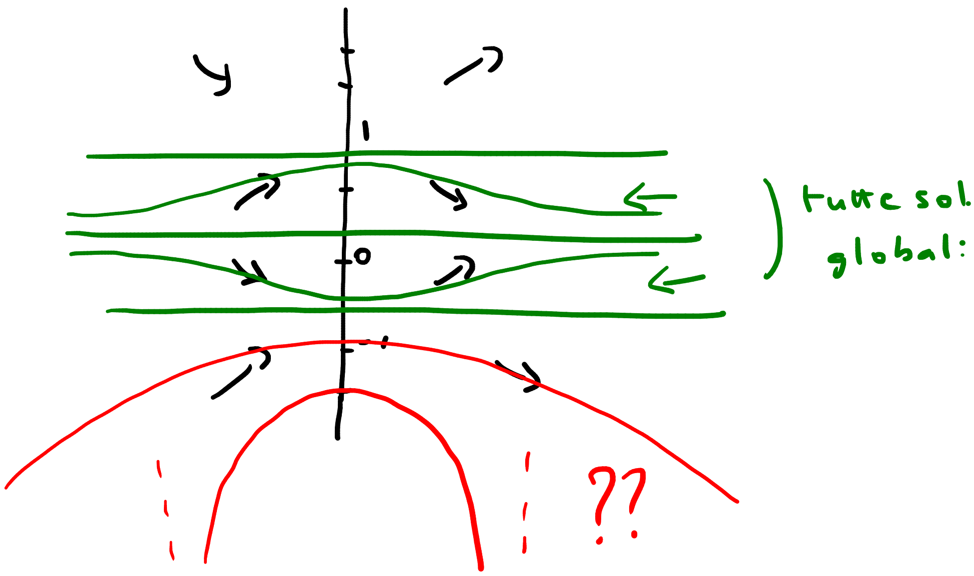
$\mathrm{dom}(f) = \mathbb{R}\times\mathbb{R}$, $\quad f \in C^1 \quad \Rightarrow$ TEUL ✓
(TEUG ✗)

$f(t,y) = 0 \iff t = 0$ zero-clina
$y^3 - y = 0 \iff y = 0,\ y = 1,\ y = -1$

$f(t,y) = ty(y^2-1)$

$y' = t(y^3 - y)$

Possibili asintoti:
$0,\ 1,\ -1$



$y' = -ty + ty^3$

$$y^{-3} y' = -t y^{-2} + t \qquad z = y^{-2} \qquad z > 0$$

$$z' = -2 y^{-3} z'$$

$$-2 y^{-3} y' = 2t y^{-2} - 2t$$

$$z' = 2tz - 2t \qquad a(t) = 2t \qquad A(t) = t^2$$

$$z(t) = e^{t^2} \int e^{-t^2} (-2t)\, dt$$

$$= e^{t^2} \left(e^{-t^2} + c \right)$$

$$z_c(t) = 1 + c e^{t^2} \qquad c \in \mathbb{R}$$

$$z = y^{-2} = \frac{1}{y^2} \qquad y^2 = \frac{1}{z} \qquad y = \pm \frac{1}{\sqrt{z}}$$